

**SOLUTIONS MANUAL**

to accompany

**Digital Signal Processing:  
A Computer-Based Approach**

**Second Edition**

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Prepared by

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**Errata List**  
of  
**"Digital Signal Processing: A Computer-Based Approach"**, Second Edition

**Chapter 2**

1. Page 48, Eq. (2.17): Replace " $y[n]$ " with " $x_u[n]$ ".
2. Page, 51: Eq. (2.24a): Delete " $\frac{1}{2}(x[n] + x^*[N - n])$ ". Eq. (2.24b): Delete " $\frac{1}{2}(x[n] - x^*[N - n])$ ".
3. Page 59, Line 5 from top and line 2 from bottom: Replace " $-\cos((\omega_1 + \omega_2 - \pi)n)$ " with " $\cos((2\pi - \omega_1 - \omega_2)n)$ ".
4. Page 61, Eq. (2.52): Replace " $A \cos((\Omega_o + k \Omega_T)t + \phi)$ " with " $A \cos(\pm(\Omega_o t + \phi) + k \Omega_T t)$ ".
5. Page 62, line 11 from bottom: Replace " $\Omega_T > 2\Omega_o$ " with " $\Omega_T > 2|\Omega_o|$ ".
6. Page 62, line 8 from bottom: Replace " $2\pi\Omega_o / \omega_T$ " with " $2\pi\Omega_o / \Omega_T$ ".
7. Page 65, Program 2\_4, line 7: Replace " $x = s + d$ ;" with " $x = s + d$ ";".
8. Page 79, line 5 below Eq. (2.76): Replace " $\sum_{n=0}^{\infty} |\alpha^n|$ " with " $\sum_{n=0}^{\infty} |\alpha|^n$ ".
9. Page 81, Eq. (2.88): Replace " $\alpha_{L+1}\lambda_2^n + \alpha_N\lambda_{N-L}^n$ " with " $\alpha_{L+1}\lambda_2^n + \mathbf{L} + \alpha_N\lambda_{N-L}^n$ ".
10. Page 93, Eq. (2.116): Replace the lower limit " $n=-M+1$ " on all summation signs with " $n=0$ ".
11. Page 100, line below Eq. (2.140) and caption of Figure 2.38: Replace " $\omega_o = 0.03$ " with " $\omega_o = 0.06\pi$ ".
12. Page 110, Problem 2.44: Replace " $\{y[n]\} = \{-1, -1, 11, -3, -10, 20, -16\}$ " with " $\{y[n]\} = \{-1, -1, 11, -3, 30, 28, 48\}$ ", and " $\{y[n]\} = \{-14 - j5, -3 - j17, -2 + j5, -26 + j22, 9 + j12\}$ " with " $\{y[n]\} = \{-14 - j5, -3 - j17, -2 + j5, -9.73 + j12.5, 5.8 + j5.67\}$ ".
13. Page 116, Exercise M2.15: Replace "randn" with "rand".

**Chapter 3**

1. Page 118, line 10 below Eq. (3.4): Replace "real" with "even".

2. Page 121, Line 5 below Eq. (3.9): Replace " $\sum_{n=0}^{\infty} |\alpha^n|$ " with " $\sum_{n=0}^{\infty} |\alpha|^n$ ".
3. Page 125, Eq. (3.16): Delete the stray  $\alpha$ .
4. Page 138, line 2 below Eq. (3.48): Replace "frequency response" with "discrete-time Fourier transform".
5. Page 139, Eq. (3.53): Replace " $x(n + mN)$ " with " $x[n + mN]$ ".
6. Page 139, line 2, Example 3.14: Replace " $x[n] = \{0 \ 1 \ 2 \ 3 \ 4 \ 5\}$ " with " $\{x[n]\} = \{0 \ 1 \ 2 \ 3 \ 4 \ 5\}$ ".
7. Page 139, line 3, Example 3.14: Replace " $x[n]$ " with " $\{x[n]\}$ ", and " $\pi k/4$ " with " $2\pi k/4$ ".
8. Page 139, line 6 from bottom: Replace " $y[n] = \{4 \ 6 \ 2 \ 3 \ 4 \ 6\}$ " with " $\{y[n]\} = \{4 \ 6 \ 2 \ 3\}$ ".
9. Page 141, Table 3.5: Replace " $N[g < -k >_N]$ " with " $N g[< -k >_N]$ ".
10. Page 142, Table 3.7: Replace " $\arg X[< -k >_N]$ " with " $-\arg X[< -k >_N]$ ".
11. Page 147, Eq. (3.86): Replace " $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$ " with " $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$ ".
12. Page 158, Eq.(3.112): Replace " $\sum_{n=-\infty}^{-1} \alpha^n z^{-n}$ " with " $-\sum_{n=-\infty}^{-1} \alpha^n z^{-n}$ ".
13. Page 165, line 4 above Eq. (3.125); Replace "0.0667" with "0.6667".
14. Page 165, line 3 above Eq. (3.125): Replace "10.0000" with "1.000", and "20.0000" with "2.0000".
15. Page 165, line above Eq. (3.125): Replace "0.0667" with "0.6667", "10.0" with "1.0", and "20.0" with "2.0".
16. Page 165, Eq. (3.125): Replace "0.667" with "0.6667".
17. Page 168, line below Eq. (3.132): Replace " $z > |\lambda_1|$ " with " $|z| > |\lambda_1|$ ".
18. Page 176, line below Eq. (3.143): Replace " $R_h$ " with " $1/R_h$ ".
19. Page 182, Problem 3.18: Replace " $X(e^{-j\omega/2})$ " with " $X(-e^{j\omega/2})$ ".

20. Page 186, Problem 3.42, Part (e): Replace " $\arg X[\langle -k \rangle_N]$ " with " $-\arg X[\langle -k \rangle_N]$ ".
21. Page 187, Problem 3.53: Replace "N-point DFT" with "MN-point DFT", replace " $0 \leq k \leq N-1$ " with " $0 \leq k \leq MN-1$ ", and replace " $x[\langle n \rangle_M]$ " with " $x[\langle n \rangle_N]$ ".
22. Page 191, Problem 3.83: Replace " $\lim_{n \rightarrow \infty}$ " with " $\lim_{z \rightarrow \infty}$ ".
23. Page 193, Problem 3.100: Replace " $\frac{P(z)}{D'(z)}$ " with " $-\lambda_1 \frac{P(z)}{D'(z)}$ ".
24. Page 194, Problem 3.106, Parts (b) and (d): Replace " $|z| < |\alpha|$ " with " $|z| > 1/|\alpha|$ ".
25. page 199, Problem 3.128: Replace " $(0.6)^\mu[n]$ " with " $(0.6)^n \mu[n]$ ", and replace " $(0.8)^\mu[n]$ " with " $(0.8)^n \mu[n]$ ".
26. Page 199, Exercise M3.5: Delete "following".

## Chapter 4

1. Page 217, first line: Replace " $\xi_N$ " with " $\xi_M$ ".
2. Page 230, line 2 below Eq. (4.88): Replace " $\theta_g(\omega)$ " with " $\theta(\omega)$ ".
3. Page 236, line 2 below Eq. (4.109): Replace "decreases" with "increases".
4. Page 246, line 4 below Eq. (4.132): Replace " $\theta_c(e^{j\omega})$ " with " $\theta_c(\omega)$ ".
5. Page 265, Eq. (4.202): Replace "1,2,K,3" with "1,2,3".
6. Page 279, Problem 4.18: Replace " $|H(e^{j0})|$ " with " $|H(e^{j\pi/4})|$ ".
7. Page 286, Problem 4.71: Replace " $z_3 = -j0.3$ " with " $z_3 = -0.3$ ".
8. Page 291, Problem 4.102: Replace
 
$$H(z) = \frac{0.4 + 0.5z^{-1} + 1.2z^{-2} + 1.2z^{-3} + 0.5z^{-4} + 0.4z^{-5}}{1 + 0.9z^{-2} + 0.2z^{-4}}$$
 with
 
$$H(z) = \frac{0.1 + 0.5z^{-1} + 0.45z^{-2} + 0.45z^{-3} + 0.5z^{-4} + 0.1z^{-5}}{1 + 0.9z^{-2} + 0.2z^{-4}}.$$
9. Page 295, Problem 4.125: Insert a comma ", " before "the autocorrelation".

## Chapter 5

1. Page 302, line 7 below Eq. (5.9): Replace "response" with "spectrum".

2. Page 309, Example 5.2, line 4: Replace "10 Hz to 20 Hz" with "5 Hz to 10 Hz". Line 6: Replace " $5k + 15$ " with " $5k + 5$ ". Line 7: Replace " $10k + 6$ " with " $5k + 3$ ", and replace " $10k - 6$ " with " $5(k+1) - 3$ ".
3. Page 311, Eq. (5.24): Replace " $G_a(j\Omega - 2k(\Delta\Omega))$ " with " $G_a(j(\Omega - 2k(\Delta\Omega)))$ ".
4. Page 318, Eq. (5.40): Replace " $H(s)$ " with " $H_a(s)$ ", and replace "" with "".
5. Page 321, Eq. (5.54): Replace "" with "".
6. Page 333, first line: Replace " $\Omega_{p1}$ " with " $\hat{\Omega}_{p1}$ ", and " $\Omega_{p2}$ " with " $\hat{\Omega}_{p2}$ ".
7. Page 349, line 9 from bottom: Replace " $1/T$ " with " $2\pi/T$ ".
8. Page 354, Problem 5.8: Interchange " $\Omega_1$ " and " $\Omega_2$ ".
9. Page 355, Problem 5.23: Replace "1 Hz" in the first line with "0.2 Hz".
10. Page 355, Problem 5.24: Replace "1 Hz" in the first line with "0.16 Hz".

## Chapter 6

1. Page 394, line 4 from bottom: Replace "alpha1" with "flipplr(alpha1)".
2. Page 413, Problem 6.16: Replace
 
$$"H(z) = b_0 + b_1 \left( z^{-1} + b_2 z^{-1} \left( z^{-1} + b_3 z^{-1} \left( 1 + \mathbf{L} + b_{N-1} z^{-1} (1 + b_N z^{-1}) \right) \right) \right)"$$
 with
 
$$"H(z) = b_0 + b_1 z^{-1} \left( 1 + b_2 z^{-1} \left( z^{-1} + b_3 z^{-1} \left( 1 + \mathbf{L} + b_{N-1} z^{-1} (1 + b_N z^{-1}) \right) \right) \right)".$$
3. Page 415, Problem 6.27: Replace " $H(z) = \frac{3z^2 + 18.5z + 17.5}{(2z + 1)(z + 2)}$ " with
 
$$"H(z) = \frac{3z^2 + 18.5z + 17.5}{(z + 0.5)(z + 2)}".$$
4. Page 415, Problem 6.28: Replace the multiplier value "0.4" in Figure P6.12 with "-9.75".
5. Page 421, Exercise M6.1: Replace " $-7.6185z^{-3}$ " with " $-71.6185z^{-3}$ ".
6. Page 422, Exercise M6.4: Replace "Program 6\_3" with "Program 6\_4".
7. Page 422, Exercise M6.5: Replace "Program 6\_3" with "Program 6\_4".
8. Page 422, Exercise M6.6: Replace "Program 6\_4" with "Program 6\_6".

## Chapter 7

1. Page 426, Eq. (7.11): Replace " $h[n - N]$ " with " $h[N - n]$ ".
2. Page 436, line 14 from top: Replace "(5.32b)" with "(5.32a)".
3. Page 438, line 17 from bottom: Replace "(5.60)" with "(5.59)".
4. Page 439, line 7 from bottom: Replace "50" with "40".
5. Page 442, line below Eq. (7.42): Replace " $F^{-1}(\hat{z})$ " with " $1 / F(\hat{z})$ ".
6. Page 442, line above Eq. (7.43): Replace " $F^{-1}(\hat{z})$ " with " $F(\hat{z})$ ".
7. Page 442, Eq. (7.43): Replace it with " $F(\hat{z}) = \pm \prod_{l=1}^L \left( \frac{\hat{z} - \alpha_l}{1 - \alpha_l^* \hat{z}} \right)$ ".
8. Page 442, line below Eq. (7.43): Replace "where  $|\alpha_1|$ " with "where  $\alpha_1$ ".
9. Page 446, Eq. (7.51): Replace " $\beta(1 - \alpha)$ " with " $\beta(1 + \alpha)$ ".
10. Page 448, Eq. (7.58): Replace " $\omega_c < \omega \leq \pi$ " with " $\omega_c < |\omega| \leq \pi$ ".
11. Page 453, line 6 from bottom: Replace " $\omega_p - \omega_s$ " with " $\omega_s - \omega_p$ ".
12. Page 457, line 8 from bottom: Replace "length" with "order".
13. Page 465, line 5 from top: Add "at  $\omega = \omega_i$ " before "or in".
14. Page 500, Problem 7.15: Replace "2 kHz" in the second line with "0.5 kHz".
15. Page 502, Problem 7.22: Replace Eq. (7.158) with " $H_a(s) = \frac{Bs}{s^2 + Bs + \Omega_0^2}$ ".
16. Page 502, Problem 7.25: Replace "7.2" with "7.1".
17. Page 504, Problem 7.41: Replace " $H_{\text{int}}(e^{j\omega}) = e^{-j\omega}$ " in Eq. (7.161) with " $H_{\text{int}}(e^{j\omega}) = \frac{1}{j\omega}$ ".
18. Page 505, Problem 7.46: Replace "16" with "9" in the third line from bottom.
19. Page 505, Problem 7.49: Replace "16" with "9" in the second line.
20. Page 510, Exercise M7.3: Replace "Program 7\_5" with "Program 7\_3".
21. Page 510, Exercise M7.4: Replace "Program 7\_7" with "M-file impinvar".
22. Page 510, Exercise M7.6: Replace "Program 7\_4" with "Program 7\_2".
23. Page 511, Exercise M7.16: Replace "length" with "order".

24. Page 512, Exercise M7.24: Replace "length" with "order".

## Chapter 8

1. Page 518, line 4 below Eq. (6.7): Delete "set" before "digital".

2. Page 540, line 3 above Eq. (8.39): Replace " $G[k]$ " with " $X_0[k]$ " and " $H[k]$ " with " $X_1[k]$ ".

## Chapter 9

1. Page 595, line 2 below Eq. (9.30c): Replace "this vector has" with "these vectors have".

2. Page 601, line 2 below Eq. (9.63): Replace " $2^b$ " with " $2^{-b}$ ".

3. Page 651, Problem 9.10, line 2 from bottom: Replace " $\left(\frac{a_k z + 1}{1 + a_k z}\right)$ " with " $\left(\frac{a_k z + 1}{z + a_k}\right)$ ".

4. Page 653, Problem 9.15, line 7: Replace "two cascade" with "four cascade".

5. Page 653, Problem 9.17: Replace " $A_2(z) = \frac{d_1 d_2 + d_1 z^{-1} + z^{-2}}{1 + d_1 z^{-1} + d_1 d_2 z^{-2}}$ " with

$$"A_2(z) = \frac{d_2 + d_1 z^{-1} + z^{-2}}{1 + d_1 z^{-1} + d_2 z^{-2}}."$$

6. Page 654, Problem 9.27: Replace "structure" with "structures".

7. Page 658, Exercise M9.9: Replace "*alpha*" with " $\alpha$ ".

## Chapter 10

1. Page 692, Eq. (10.57b): Replace " $P_0(\alpha_1) = 0.2469$ " with " $P_0(\alpha_1) = 0.7407$ ".

2. Page 693, Eq. (10.58b): Replace " $P_0(\alpha_2) = -0.4321$ " with " $P_0(\alpha_2) = -1.2963$ ".

3. Page 694, Figure 10.38(c): Replace " $P_{-2}(\alpha_0)$ " with " $P_1(\alpha_0)$ ", " $P_{-1}(\alpha_0)$ " with " $P_0(\alpha_0)$ ", " $P_0(\alpha_0)$ " with " $P_{-1}(\alpha_0)$ ", " $P_1(\alpha_0)$ " with " $P_{-2}(\alpha_0)$ ", " $P_{-2}(\alpha_1)$ " with " $P_1(\alpha_1)$ ", " $P_{-1}(\alpha_1)$ " with " $P_0(\alpha_1)$ ", " $P_0(\alpha_1)$ " with " $P_{-1}(\alpha_1)$ ", " $P_1(\alpha_1)$ " with " $P_{-2}(\alpha_1)$ ", " $P_{-2}(\alpha_2)$ " with " $P_1(\alpha_2)$ ", " $P_{-1}(\alpha_2)$ " with " $P_0(\alpha_2)$ ", " $P_0(\alpha_2)$ " with " $P_{-1}(\alpha_2)$ ", and " $P_{-1}(\alpha_2)$ " with " $P_{-2}(\alpha_2)$ ".

4. Page 741, Problem 10.13: Replace "2.5 kHz" with "1.25 kHz".

5. Page 741, Problem 10.20: Replace " $\sum_{i=0}^N z^i$ " with " $\sum_{i=0}^{N-1} z^i$ ".

6. Page 743, Problem 10.28: Replace "half-band filter" with a "lowpass half-band filter with a zero at  $z = -1$ ".
7. Page 747, Problem 10.50: Interchange " $\mathbf{Y}_k$ " and "the output sequence  $y[n]$ ".
8. Page 747, Problem 10.51: Replace the unit delays " $z^{-1}$ " on the right-hand side of the structure of Figure P10.8 with unit advance operators " $z$ ".
9. Page 749, Eq. (10.215): Replace " $3H^2(z) - 2H^2(z)$ " with " $z^{-2}[3H^2(z) - 2H^2(z)]$ ".
10. Page 751, Exercise M10.9: Replace "60" with "61".
11. Page 751, Exercise M10.10: Replace the problem statement with "Design a fifth-order IIR half-band Butterworth lowpass filter and realize it with 2 multipliers".
12. Page 751, Exercise M10.11: Replace the problem statement with "Design a seventh-order IIR half-band Butterworth lowpass filter and realize it with 3 multipliers".

## Chapter 11

1. Page 758, line 4 below Figure 11.2 caption: Replace "grid" with "grid;".
2. Page 830, Problem 11.5: Insert " $g_a(t) = \cos(200\pi t)$ " after "signal" and delete " $= \cos(200\pi n)$ ".
3. Page 831, Problem 11.11: Replace "has to be a power-of-2" with " $= 2^{\mathbf{l}}$ , where  $\mathbf{l}$  is an integer".



## Chapter 2 (2e)

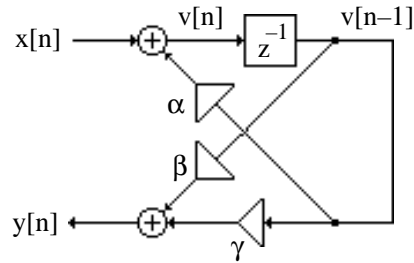
**2.1 (a)**  $u[n] = x[n] + y[n] = \{3 \ 5 \ 1 \ -2 \ 8 \ 14 \ 0\}$

**(b)**  $v[n] = x[n] \cdot w[n] = \{-15 \ -8 \ 0 \ 6 \ -20 \ 0 \ 2\}$

**(c)**  $s[n] = y[n] - w[n] = \{5 \ 3 \ -2 \ -9 \ 9 \ 9 \ -3\}$

**(d)**  $r[n] = 4.5y[n] = \{0 \ 31.5 \ 4.5 \ -13.5 \ 18 \ 40.5 \ -9\}$

**2.2 (a)** From the figure shown below we obtain



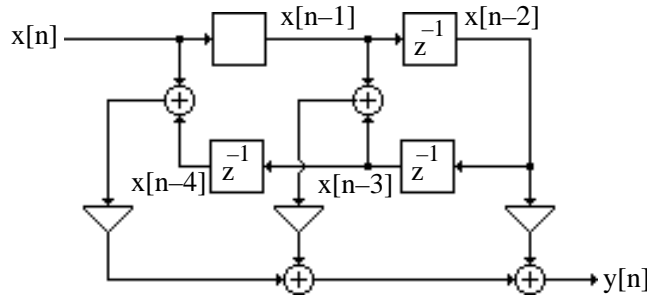
$v[n] = x[n] + \alpha v[n-1]$  and  $y[n] = \beta v[n-1] + \gamma v[n-1] = (\beta + \gamma)v[n-1]$ . Hence,

$v[n-1] = x[n-1] + \alpha v[n-2]$  and  $y[n-1] = (\beta + \gamma)v[n-2]$ . Therefore,

$$y[n] = (\beta + \gamma)v[n-1] = (\beta + \gamma)x[n-1] + \alpha(\beta + \gamma)v[n-2] = (\beta + \gamma)x[n-1] + \alpha(\beta + \gamma) \frac{y[n-1]}{(\beta + \gamma)}$$

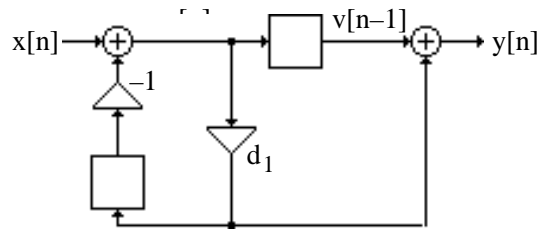
$$= (\beta + \gamma)x[n-1] + \alpha y[n-1].$$

**(b)** From the figure shown below we obtain



$$y[n] = \gamma x[n-2] + \beta(x[n-1] + x[n-3]) + \alpha(x[n] + x[n-4]).$$

**(c)** From the figure shown below we obtain



$v[n] = x[n] - d_1 v[n-1]$  and  $y[n] = d_1 v[n] + v[n-1]$ . Therefore we can rewrite the second equation as  $y[n] = d_1(x[n] - d_1 v[n-1]) + v[n-1] = d_1 x[n] + (1 - d_1^2)v[n-1]$  (1)

$$= d_1 x[n] + (1 - d_1^2)(x[n-1] - d_1 v[n-2]) = d_1 x[n] + (1 - d_1^2)x[n-1] - d_1(1 - d_1^2)v[n-2]$$

From Eq. (1),  $y[n-1] = d_1 x[n-1] + (1 - d_1^2)v[n-2]$ , or equivalently,

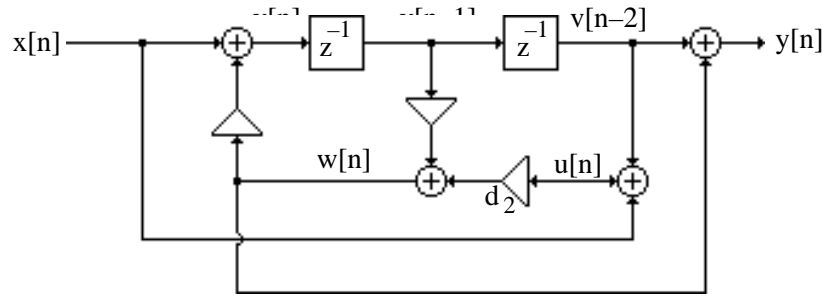
$$d_1 y[n-1] = d_1^2 x[n-1] + d_1(1 - d_1^2)v[n-2].$$

Therefore,

$$y[n] + d_1 y[n-1] = d_1 x[n] + (1 - d_1^2)x[n-1] - d_1(1 - d_1^2)v[n-2] + d_1^2 x[n-1] + d_1(1 - d_1^2)v[n-2]$$

$$= d_1 x[n] + x[n-1], \text{ or } y[n] = d_1 x[n] + x[n-1] - d_1 y[n-1].$$

(d) From the figure shown below we obtain



$v[n] = x[n] - w[n]$ ,  $w[n] = d_1 v[n-1] + d_2 u[n]$ , and  $u[n] = v[n-2] + x[n]$ . From these equations we get  $w[n] = d_2 x[n] + d_1 x[n-1] + d_2 x[n-2] - d_1 w[n-1] - d_2 w[n-2]$ . From the figure we also obtain  $y[n] = v[n-2] + w[n] = x[n-2] + w[n] - w[n-2]$ , which yields

$$d_1 y[n-1] = d_1 x[n-3] + d_1 w[n-1] - d_1 w[n-3], \text{ and}$$

$$d_2 y[n-2] = d_2 x[n-4] + d_2 w[n-2] - d_2 w[n-4].$$

Therefore,

$$y[n] + d_1 y[n-1] + d_2 y[n-2] = x[n-2] + d_1 x[n-3] + d_2 x[n-4]$$

$$+ (w[n] + d_1 w[n-1] + d_2 w[n-2]) - (w[n-2] + d_1 w[n-3] + d_2 w[n-4])$$

$$= x[n-2] + d_2 x[n] + d_1 x[n-1] \text{ or equivalently,}$$

$$y[n] = d_2 x[n] + d_1 x[n-1] + x[n-2] - d_1 y[n-1] - d_2 y[n-2].$$

2.3 (a)  $x[n] = \{3 \ -2 \ 0 \ 1 \ 4 \ 5 \ 2\}$ , Hence,  $x[-n] = \{2 \ 5 \ 4 \ 1 \ 0 \ -2 \ 3\}$ ,  $-3 \leq n \leq 3$ .

Thus,  $x_{ev}[n] = \frac{1}{2}(x[n] + x[-n]) = \{5/2 \ 3/2 \ 2 \ 1 \ 2 \ 3/2 \ 5/2\}$ ,  $-3 \leq n \leq 3$ , and

$x_{od}[n] = \frac{1}{2}(x[n] - x[-n]) = \{1/2 \ -7/2 \ -2 \ 0 \ 2 \ 7/2 \ -1/2\}$ ,  $-3 \leq n \leq 3$ .

(b)  $y[n] = \{0 \ 7 \ 1 \ -3 \ 4 \ 9 \ -2\}$ . Hence,  $y[-n] = \{-2 \ 9 \ 4 \ -3 \ 1 \ 7 \ 0\}$ ,  $-3 \leq n \leq 3$ .

Thus,  $y_{ev}[n] = \frac{1}{2}(y[n] + y[-n]) = \{-1 \ 8 \ 5/2 \ -3 \ 5/2 \ 8 \ -1\}$ ,  $-3 \leq n \leq 3$ , and

$y_{od}[n] = \frac{1}{2}(y[n] - y[-n]) = \{1 \ -1 \ -3/2 \ 0 \ 1 \ 3/2 \ -1\}$ ,  $-3 \leq n \leq 3$ .

(c)  $w[n] = \{-5 \ 4 \ 3 \ 6 \ -5 \ 0 \ 1\}$ , Hence,  $w[-n] = \{1 \ 0 \ -5 \ 6 \ 3 \ 4 \ -5\}$ ,  $-3 \leq n \leq 3$ .  
 Thus,  $w_{ev}[n] = \frac{1}{2}(w[n] + w[-n]) = \{-2 \ 2 \ -1 \ 6 \ -1 \ 2 \ -2\}$ ,  $-3 \leq n \leq 3$ , and  
 $w_{od}[n] = \frac{1}{2}(w[n] - w[-n]) = \{-3 \ 2 \ 4 \ 0 \ -4 \ -2 \ 3\}$ ,  $-3 \leq n \leq 3$ ,

**2.4 (a)**  $x[n] = g[n]g[n]$ . Hence  $x[-n] = g[-n]g[-n]$ . Since  $g[n]$  is even, hence  $g[-n] = g[n]$ .  
 Therefore  $x[-n] = g[-n]g[-n] = g[n]g[n] = x[n]$ . Hence  $x[n]$  is even.

(b)  $u[n] = g[n]h[n]$ . Hence,  $u[-n] = g[-n]h[-n] = g[n](-h[n]) = -g[n]h[n] = -u[n]$ . Hence  $u[n]$  is odd.

(c)  $v[n] = h[n]h[n]$ . Hence,  $v[-n] = h[n]h[n] = (-h[n])(-h[n]) = h[n]h[n] = v[n]$ . Hence  $v[n]$  is even.

**2.5** Yes, a linear combination of any set of a periodic sequences is also a periodic sequence and the period of the new sequence is given by the least common multiple (lcm) of all periods. For our example, the period =  $\text{lcm}(N_1, N_2, N_3)$ . For example, if  $N_1 = 3$ ,  $N_2 = 6$ , and  $N_3 = 12$ , then  $N = \text{lcm}(3, 5, 12) = 60$ .

**2.6 (a)**  $x_{pcs}[n] = \frac{1}{2}\{x[n] + x^*[-n]\} = \frac{1}{2}\{A\alpha^n + A^*(\alpha^*)^{-n}\}$ , and  
 $x_{pca}[n] = \frac{1}{2}\{x[n] - x^*[-n]\} = \frac{1}{2}\{A\alpha^n - A^*(\alpha^*)^{-n}\}$ ,  $-N \leq n \leq N$ .

(b)  $h[n] = \{-2 + j5 \ 4 - j3 \ 5 + j6 \ 3 + j \ -7 + j2\}$   $-2 \leq n \leq 2$ , and hence,  
 $h^*[-n] = \{-7 - j2 \ 3 - j \ 5 - j6 \ 4 + j3 \ -2 - j5\}$ ,  $-2 \leq n \leq 2$ . Therefore,  
 $h_{pcs}[n] = \frac{1}{2}\{h[n] + h^*[-n]\} = \{-4.5 + j1.5 \ 3.5 - j2 \ 5 \ 3.5 + j2 \ -4.5 - j1.5\}$  and  
 $h_{pca}[n] = \frac{1}{2}\{h[n] - h^*[-n]\} = \{2.5 + j3.5 \ 0.5 - j \ j6 \ -0.5 - j \ -2.5 + j3.5\}$   $-2 \leq n \leq 2$ .

**2.7 (a)**  $\{x[n]\} = \{A\alpha^n\}$  where  $A$  and  $\alpha$  are complex numbers, with  $|\alpha| < 1$ .

Since for  $n < 0$ ,  $|\alpha|^n$  can become arbitrarily large hence  $\{x[n]\}$  is not a bounded sequence.

(b)  $\{y[n]\} = A\alpha^n \mu[n]$  where  $A$  and  $\alpha$  are complex numbers, with  $|\alpha| < 1$ .

In this case  $|y[n]| \leq |A| \ \forall n$  hence  $\{y[n]\}$  is a bounded sequence.

(c)  $\{h[n]\} = C\beta^n \mu[n]$  where  $C$  and  $\beta$  are complex numbers, with  $|\beta| > 1$ .

Since  $|\beta|^n$  becomes arbitrarily large as  $n$  increases hence  $\{h[n]\}$  is not a bounded sequence.

(d)  $\{g[n]\} = 4 \sin(\omega_a n)$ . Since  $-4 \leq g[n] \leq 4$  for all values of  $n$ ,  $\{g[n]\}$  is a bounded sequence.

(e)  $\{v[n]\} = 3 \cos^2(\omega_b n^2)$ . Since  $-3 \leq v[n] \leq 3$  for all values of  $n$ ,  $\{v[n]\}$  is a bounded sequence.

**2.8 (a)** Recall,  $x_{ev}[n] = \frac{1}{2}(x[n] + x[-n])$ .

Since  $x[n]$  is a causal sequence, thus  $x[-n] = 0 \quad \forall n > 0$ . Hence,  
 $x[n] = x_{ev}[n] + x_{ev}[-n] = 2x_{ev}[n], \quad \forall n > 0$ . For  $n = 0$ ,  $x[0] = x_{ev}[0]$ .

Thus  $x[n]$  can be completely recovered from its even part.

Likewise,  $x_{od}[n] = \frac{1}{2}(x[n] - x[-n]) = \begin{cases} \frac{1}{2}x[n], & n > 0, \\ 0, & n = 0. \end{cases}$

Thus  $x[n]$  can be recovered from its odd part  $\forall n$  except  $n = 0$ .

(b)  $2y_{ca}[n] = y[n] - y^*[-n]$ . Since  $y[n]$  is a causal sequence  $y[n] = 2y_{ca}[n] \quad \forall n > 0$ .

For  $n = 0$ ,  $\text{Im}\{y[0]\} = y_{ca}[0]$ . Hence real part of  $y[0]$  cannot be fully recovered from  $y_{ca}[n]$ .

Therefore  $y[n]$  cannot be fully recovered from  $y_{ca}[n]$ .

$2y_{cs}[n] = y[n] + y^*[-n]$ . Hence,  $y[n] = 2y_{cs}[n] \quad \forall n > 0$ .

For  $n = 0$ ,  $\text{Re}\{y[0]\} = y_{cs}[0]$ . Hence imaginary part of  $y[0]$  cannot be recovered from  $y_{cs}[n]$ .

Therefore  $y[n]$  cannot be fully recovered from  $y_{cs}[n]$ .

**2.9**  $x_{ev}[n] = \frac{1}{2}(x[n] + x[-n])$ . This implies,  $x_{ev}[-n] = \frac{1}{2}(x[-n] + x[n]) = x_{ev}[n]$ .

Hence even part of a real sequence is even.

$x_{od}[n] = \frac{1}{2}(x[n] - x[-n])$ . This implies,  $x_{od}[-n] = \frac{1}{2}(x[-n] - x[n]) = -x_{od}[n]$ .

Hence the odd part of a real sequence is odd.

**2.10** RHS of Eq. (2.176a) is  $x_{cs}[n] + x_{cs}[n-N] = \frac{1}{2}(x[n] + x^*[-n]) + \frac{1}{2}(x[n-N] + x^*[N-n])$ .

Since  $x[n] = 0 \quad \forall n < 0$ , Hence

$x_{cs}[n] + x_{cs}[n-N] = \frac{1}{2}(x[n] + x^*[N-n]) = x_{pcs}[n], \quad 0 \leq n \leq N-1$ .

RHS of Eq. (2.176b) is

$x_{ca}[n] + x_{ca}[n-N] = \frac{1}{2}(x[n] - x^*[-n]) + \frac{1}{2}(x[n-N] - x^*[n-N])$   
 $= \frac{1}{2}(x[n] - x^*[N-n]) = x_{pca}[n], \quad 0 \leq n \leq N-1$ .

**2.11**  $x_{\text{pcs}}[n] = \frac{1}{2}(x[n] + x^*[\langle -n \rangle_N])$  for  $0 \leq n \leq N-1$ , Since,  $x[\langle -n \rangle_N] = x[N-n]$ , it follows that  $x_{\text{pcs}}[n] = \frac{1}{2}(x[n] + x^*[N-n])$ ,  $1 \leq n \leq N-1$ .

For  $n = 0$ ,  $x_{\text{pcs}}[0] = \frac{1}{2}(x[0] + x^*[0]) = \text{Re}\{x[0]\}$ .

Similarly  $x_{\text{pca}}[n] = \frac{1}{2}(x[n] - x^*[\langle -n \rangle_N]) = \frac{1}{2}(x[n] - x^*[N-n])$ ,  $1 \leq n \leq N-1$ . Hence,

for  $n = 0$ ,  $x_{\text{pca}}[0] = \frac{1}{2}(x[0] - x^*[0]) = j\text{Im}\{x[0]\}$ .

**2.12 (a)** Given  $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ . Therefore, by Schwartz inequality,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 \leq \left( \sum_{n=-\infty}^{\infty} |x[n]| \right) \left( \sum_{n=-\infty}^{\infty} |x[n]| \right) < \infty.$$

**(b)** Consider  $x[n] = \begin{cases} 1/n, & n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$  The convergence of an infinite series can be shown

via the integral test. Let  $a_n = f(x)$ , where  $f(x)$  is a continuous, positive and decreasing function for all  $x \geq 1$ . Then the series  $\sum_{n=1}^{\infty} a_n$  and the integral  $\int_1^{\infty} f(x) dx$  both converge or

both diverge. For  $a_n = 1/n$ ,  $f(x) = 1/x$ . But  $\int_1^{\infty} \frac{1}{x} dx = (\ln x)|_1^{\infty} = \infty - 0 = \infty$ . Hence,

$\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=1}^{\infty} \frac{1}{n}$  does not converge, and as a result,  $x[n]$  is not absolutely

summable. To show  $\{x[n]\}$  is square-summable, we observe here  $a_n = \frac{1}{n^2}$ , and thus,

$f(x) = \frac{1}{x^2}$ . Now,  $\int_1^{\infty} \frac{1}{x^2} dx = \left( -\frac{1}{x} \right) \Big|_1^{\infty} = -\frac{1}{\infty} + \frac{1}{1} = 1$ . Hence,  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges, or in other

words,  $x[n] = 1/n$  is square-summable.

**2.13** See Problem 2.12, Part (b) solution.

**2.14**  $x_2[n] = \frac{\cos \omega_c n}{\pi n}$ ,  $1 \leq n \leq \infty$ . Now,  $\sum_{n=1}^{\infty} \left( \frac{\cos \omega_c n}{\pi n} \right)^2 \leq \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2}$ . Since,  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ ,

$\sum_{n=1}^{\infty} \left( \frac{\cos \omega_c n}{\pi n} \right)^2 \leq \frac{1}{6}$ . Therefore  $x_2[n]$  is square-summable.

Using integral test we now show that  $x_2[n]$  is not absolutely summable.

$$\int_1^{\infty} \left| \frac{\cos \omega_c x}{\pi x} \right| dx = \frac{1}{\pi} \cdot \frac{|\cos \omega_c x|}{\cos \omega_c x} x \cdot \cos \text{int}(\omega_c x) \Big|_1^{\infty} \quad \text{where } \cos \text{int} \text{ is the cosine integral function.}$$

Since  $\int_1^{\infty} \left| \frac{\cos \omega_c x}{\pi x} \right| dx$  diverges,  $\sum_{n=1}^{\infty} \left| \frac{\cos \omega_c n}{\pi n} \right|$  also diverges.

**2.15** 
$$\sum_{n=-\infty}^{\infty} x^2[n] = \sum_{n=-\infty}^{\infty} (x_{ev}[n] + x_{od}[n])^2$$

$$= \sum_{n=-\infty}^{\infty} x_{ev}^2[n] + \sum_{n=-\infty}^{\infty} x_{od}^2[n] + 2 \sum_{n=-\infty}^{\infty} x_{ev}[n]x_{od}[n] = \sum_{n=-\infty}^{\infty} x_{ev}^2[n] + \sum_{n=-\infty}^{\infty} x_{od}^2[n]$$
as  $\sum_{n=-\infty}^{\infty} x_{ev}[n]x_{od}[n] = 0$  since  $x_{ev}[n]x_{od}[n]$  is an odd sequence.

**2.16**  $x[n] = \cos(2\pi kn/N)$ ,  $0 \leq n \leq N-1$ . Hence,

$$E_x = \sum_{n=0}^{N-1} \cos^2(2\pi kn/N) = \frac{1}{2} \sum_{n=0}^{N-1} (1 + \cos(4\pi kn/N)) = \frac{N}{2} + \frac{1}{2} \sum_{n=0}^{N-1} \cos(4\pi kn/N).$$

Let  $C = \sum_{n=0}^{N-1} \cos(4\pi kn/N)$ , and  $S = \sum_{n=0}^{N-1} \sin(4\pi kn/N)$ .

Therefore  $C + jS = \sum_{n=0}^{N-1} e^{j4\pi kn/N} = \frac{e^{j4\pi k} - 1}{e^{j4\pi k/N} - 1} = 0$ . Thus,  $C = \text{Re}\{C + jS\} = 0$ .

As  $C = \text{Re}\{C + jS\} = 0$ , it follows that  $E_x = \frac{N}{2}$ .

**2.17 (a)**  $x_1[n] = \mu[n]$ . Energy  $= \sum_{n=-\infty}^{\infty} \mu^2[n] = \sum_{n=-\infty}^{\infty} 1^2 = \infty$ .

Average power  $= \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=-K}^K (\mu[n])^2 = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=0}^K 1^2 = \lim_{K \rightarrow \infty} \frac{K}{2K+1} = \frac{1}{2}$ .

**(b)**  $x_2[n] = n\mu[n]$ . Energy  $= \sum_{n=-\infty}^{\infty} (n\mu[n])^2 = \sum_{n=-\infty}^{\infty} n^2 = \infty$ .

Average power  $= \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=-K}^K (n\mu[n])^2 = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=0}^K n^2 = \infty$ .

**(c)**  $x_3[n] = A_0 e^{j\omega_0 n}$ . Energy  $= \int_{n=-\infty}^{\infty} |A_0 e^{j\omega_0 n}|^2 = \sum_{n=-\infty}^{\infty} |A_0|^2 = \infty$ . Average power  $=$

$$\lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=-K}^K |A_0 e^{j\omega_0 n}|^2 = \lim_{K \rightarrow \infty} \frac{1}{2K+1} \sum_{n=-K}^K |A_0|^2 = \lim_{K \rightarrow \infty} \frac{2KA_0^2}{2K+1} = A_0^2.$$

**(d)**  $x[n] = A \sin\left(\frac{2\pi n}{M} + \phi\right) = A_0 e^{j\omega_0 n} + A_1 e^{-j\omega_0 n}$ , where  $\omega_0 = \frac{2\pi}{M}$ ,  $A_0 = -\frac{A}{2} e^{j\phi}$  and  $A_1 = \frac{A}{2} e^{-j\phi}$ . From Part (c), energy  $= \infty$  and average power  $= A_0^2 + A_1^2 + 4A_0^2 A_1^2 = \frac{3}{4} A^2$ .

**2.18** Now,  $\mu[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$  Hence,  $\mu[-n-1] = \begin{cases} 1, & n < 0, \\ 0, & n \geq 0. \end{cases}$  Thus,  $x[n] = \mu[n] + \mu[-n-1]$ .

**2.19 (a)** Consider a sequence defined by  $x[n] = \sum_{k=-\infty}^n \delta[k]$ .

If  $n < 0$  then  $k = 0$  is not included in the sum and hence  $x[n] = 0$ , whereas for  $n \geq 0$ ,  $k = 0$  is included in the sum hence  $x[n] = 1 \forall n \geq 0$ . Thus  $x[n] = \sum_{k=-\infty}^n \delta[k] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases} = \mu[n]$ .

(b) Since  $\mu[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0, \end{cases}$  it follows that  $\mu[n-1] = \begin{cases} 1, & n \geq 1, \\ 0, & n \leq 0. \end{cases}$

$$\text{Hence, } \mu[n] - \mu[n-1] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0, \end{cases} = \delta[n].$$

**2.20** Now  $x[n] = A \sin(\omega_0 n + \phi)$ .

(a) Given  $x[n] = \{0 \quad -\sqrt{2} \quad -2 \quad -\sqrt{2} \quad 0 \quad \sqrt{2} \quad 2 \quad \sqrt{2}\}$ . The fundamental period is  $N = 4$ , hence  $\omega_0 = 2\pi/8 = \pi/4$ . Next from  $x[0] = A \sin(\phi) = 0$  we get  $\phi = 0$ , and solving  $x[1] = A \sin(\frac{\pi}{4} + \phi) = A \sin(\pi/4) = -\sqrt{2}$  we get  $A = -2$ .

(b) Given  $x[n] = \{\sqrt{2} \quad \sqrt{2} \quad -\sqrt{2} \quad -\sqrt{2}\}$ . The fundamental period is  $N = 4$ , hence  $\omega_0 = 2\pi/4 = \pi/2$ . Next from  $x[0] = A \sin(\phi) = \sqrt{2}$  and  $x[1] = A \sin(\pi/2 + \phi) = A \cos(\phi) = \sqrt{2}$  it can be seen that  $A = 2$  and  $\phi = \pi/4$  is one solution satisfying the above equations.

(c)  $x[n] = \{3 \quad -3\}$ . Here the fundamental period is  $N = 2$ , hence  $\omega_0 = \pi$ . Next from  $x[0] = A \sin(\phi) = 3$  and  $x[1] = A \sin(\phi + \pi) = -A \sin(\phi) = -3$  observe that  $A = 3$  and  $\phi = \pi/2$  that  $A = 3$  and  $\phi = \pi/2$  is one solution satisfying these two equations.

(d) Given  $x[n] = \{0 \quad 1.5 \quad 0 \quad -1.5\}$ , it follows that the fundamental period of  $x[n]$  is  $N = 4$ . Hence  $\omega_0 = 2\pi/4 = \pi/2$ . Solving  $x[0] = A \sin(\phi) = 0$  we get  $\phi = 0$ , and solving  $x[1] = A \sin(\pi/2) = 1.5$ , we get  $A = 1.5$ .

**2.21** (a)  $\tilde{x}_1[n] = e^{-j0.4\pi n}$ . Here,  $\omega_0 = 0.4\pi$ . From Eq. (2.44a), we thus get

$$N = \frac{2\pi r}{\omega_0} = \frac{2\pi r}{0.4\pi} = 5r = 5 \text{ for } r = 1.$$

(b)  $\tilde{x}_2[n] = \sin(0.6\pi n + 0.6\pi)$ . Here,  $\omega_0 = 0.6\pi$ . From Eq. (2.44a), we thus get

$$N = \frac{2\pi r}{\omega_0} = \frac{2\pi r}{0.6\pi} = \frac{10}{3}r = 10 \text{ for } r = 3.$$

(c)  $\tilde{x}_3[n] = 2 \cos(1.1\pi n - 0.5\pi) + 2 \sin(0.7\pi n)$ . Here,  $\omega_1 = 1.1\pi$  and  $\omega_2 = 0.7\pi$ . From Eq. (2.44a), we thus get  $N_1 = \frac{2\pi r_1}{\omega_1} = \frac{2\pi r_1}{1.1\pi} = \frac{20}{11}r_1$  and  $N_2 = \frac{2\pi r_2}{\omega_2} = \frac{2\pi r_2}{0.7\pi} = \frac{20}{7}r_2$ . To be periodic

we must have  $N_1 = N_2$ . This implies,  $\frac{20}{11} r_1 = \frac{20}{7} r_2$ . This equality holds for  $r_1 = 11$  and  $r_2 = 7$ , and hence  $N = N_1 = N_2 = 20$ .

(d)  $N_1 = \frac{2\pi r_1}{1.3\pi} = \frac{20}{13} r_1$  and  $N_2 = \frac{2\pi r_2}{0.3\pi} = \frac{20}{3} r_2$ . It follows from the results of Part (c),  $N = 20$  with  $r_1 = 13$  and  $r_2 = 3$ .

(e)  $N_1 = \frac{2\pi r_1}{1.2\pi} = \frac{5}{3} r_1$ ,  $N_2 = \frac{2\pi r_2}{0.8\pi} = \frac{5}{2} r_2$  and  $N_3 = N_2$ . Therefore,  $N = N_1 = N_2 = N_3 = 5$  for  $r_1 = 3$  and  $r_2 = 2$ .

(f)  $\tilde{x}_6[n] = n \text{ modulo } 6$ . Since  $\tilde{x}_6[n+6] = (n+6) \text{ modulo } 6 = n \text{ modulo } 6 = \tilde{x}_6[n]$ .

Hence  $N = 6$  is the fundamental period of the sequence  $\tilde{x}_6[n]$ .

2.22 (a)  $\omega_o = 0.14\pi$ . Therefore,  $N = \frac{2\pi r}{\omega_o} = \frac{2\pi r}{0.14\pi} = \frac{100}{7} r = 100$  for  $r = 7$ .

(b)  $\omega_o = 0.24\pi$ . Therefore,  $N = \frac{2\pi r}{\omega_o} = \frac{2\pi r}{0.24\pi} = \frac{25}{3} r = 25$  for  $r = 3$ .

(c)  $\omega_o = 0.34\pi$ . Therefore,  $N = \frac{2\pi r}{\omega_o} = \frac{2\pi r}{0.34\pi} = \frac{100}{17} r = 100$  for  $r = 17$ .

(d)  $\omega_o = 0.75\pi$ . Therefore,  $N = \frac{2\pi r}{\omega_o} = \frac{2\pi r}{0.75\pi} = \frac{8}{3} r = 8$  for  $r = 3$ .

2.23  $x[n] = x_a(nT) = \cos(\Omega_o nT)$  is a periodic sequence for all values of  $T$  satisfying  $\Omega_o T \cdot N = 2\pi r$  for  $r$  and  $N$  taking integer values. Since,  $\Omega_o T = 2\pi r / N$  and  $r/N$  is a rational number,  $\Omega_o T$  must also be rational number. For  $\Omega_o = 18$  and  $T = \pi/6$ , we get  $N = 2r/3$ . Hence, the smallest value of  $N = 3$  for  $r = 3$ .

2.24 (a)  $x[n] = 3\delta[n+3] - 2\delta[n+2] + \delta[n] + 4\delta[n-1] + 5\delta[n-2] + 2\delta[n-3]$

(b)  $y[n] = 7\delta[n+2] + \delta[n+1] - 3\delta[n] + 4\delta[n-1] + 9\delta[n-2] - 2\delta[n-3]$

(c)  $w[n] = -5\delta[n+2] + 4\delta[n+2] + 3\delta[n+1] + 6\delta[n] - 5\delta[n-1] + \delta[n-3]$

2.25 (a) For an input  $x_i[n]$ ,  $i = 1, 2$ , the output is

$$y_i[n] = \alpha_1 x_i[n] + \alpha_2 x_i[n-1] + \alpha_3 x_i[n-2] + \alpha_4 x_i[n-4], \text{ for } i = 1, 2. \text{ Then, for an input } x[n] = A x_1[n] + B x_2[n], \text{ the output is } y[n] = \alpha_1 (A x_1[n] + B x_2[n]) + \alpha_2 (A x_1[n-1] + B x_2[n-1]) + \alpha_3 (A x_1[n-2] + B x_2[n-2]) + \alpha_4 (A x_1[n-4] + B x_2[n-4]) = A(\alpha_1 x_1[n] + \alpha_2 x_1[n-1] + \alpha_3 x_1[n-2] + \alpha_4 x_1[n-4]) + B(\alpha_1 x_2[n] + \alpha_2 x_2[n-1] + \alpha_3 x_2[n-2] + \alpha_4 x_2[n-4]) = A y_1[n] + B y_2[n].$$

Hence, the system is linear.



**(b)** For an input  $x_i[n]$ ,  $i = 1, 2$ , the output is

$y_i[n] = b_0x_i[n] + b_1x_i[n-1] + b_2x_i[n-2] + a_1y_i[n-1] + a_2y_i[n-2]$ ,  $i = 1, 2$ . Then, for an input  $x[n] = Ax_1[n] + Bx_2[n]$ , the output is

$$y[n] = A(b_0x_1[n] + b_1x_1[n-1] + b_2x_1[n-2] + a_1y_1[n-1] + a_2y_1[n-2]) \\ + B(b_0x_2[n] + b_1x_2[n-1] + b_2x_2[n-2] + a_1y_2[n-1] + a_2y_2[n-2]) = Ay_1[n] + By_2[n].$$

Hence, the system is linear.

**(c)** For an input  $x_i[n]$ ,  $i = 1, 2$ , the output is  $y_i[n] = \begin{cases} x_i[n/L], & n = 0, \pm L, \pm 2L, \mathbf{K} \\ 0, & \text{otherwise,} \end{cases}$

Consider the input  $x[n] = Ax_1[n] + Bx_2[n]$ , Then the output  $y[n]$  for  $n = 0, \pm L, \pm 2L, \mathbf{K}$  is given by  $y[n] = x[n/L] = Ax_1[n/L] + Bx_2[n/L] = Ay_1[n] + By_2[n]$ . For all other values of  $n$ ,  $y[n] = A \cdot 0 + B \cdot 0 = 0$ . Hence, the system is linear.

**(d)** For an input  $x_i[n]$ ,  $i = 1, 2$ , the output is  $y_i[n] = x_i[n/M]$ . Consider the input  $x[n] = Ax_1[n] + Bx_2[n]$ , Then the output  $y[n] = Ax_1[n/M] + Bx_2[n/M] = Ay_1[n] + By_2[n]$ . Hence, the system is linear.

**(e)** For an input  $x_i[n]$ ,  $i = 1, 2$ , the output is  $y_i[n] = \frac{1}{M} \sum_{k=0}^{M-1} x_i[n-k]$ . Then, for an

input  $x[n] = Ax_1[n] + Bx_2[n]$ , the output is  $y_i[n] = \frac{1}{M} \sum_{k=0}^{M-1} (Ax_1[n-k] + Bx_2[n-k])$

$$= A \left( \frac{1}{M} \sum_{k=0}^{M-1} x_1[n-k] \right) + B \left( \frac{1}{M} \sum_{k=0}^{M-1} x_2[n-k] \right) = Ay_1[n] + By_2[n].$$

Hence, the system is linear.

**(f)** The first term on the RHS of Eq. (2.58) is the output of an up-sampler. The second term on the RHS of Eq. (2.58) is simply the output of a unit delay followed by an up-sampler, whereas, the third term is the output of a unit advance operator followed by an up-sampler. We have shown in Part (c) that the up-sampler is a linear system. Moreover, the delay and the advance operators are linear systems. A cascade of two linear systems is linear and a linear combination of linear systems is also linear. Hence, the factor-of-2 interpolator is a linear system.

**(g)** Following the arguments given in Part (f), it can be shown that the factor-of-3 interpolator is a linear system.

**2.26 (a)**  $y[n] = n^2x[n]$ .

For an input  $x_i[n]$  the output is  $y_i[n] = n^2x_i[n]$ ,  $i = 1, 2$ . Thus, for an input  $x_3[n] = Ax_1[n] + Bx_2[n]$ , the output  $y_3[n]$  is given by  $y_3[n] = n^2(Ax_1[n] + Bx_2[n]) = Ay_1[n] + By_2[n]$ .

Hence the system is linear.

Since there is no output before the input hence the system is causal. However,  $|y[n]|$  being proportional to  $n$ , it is possible that a bounded input can result in an unbounded output. Let  $x[n] = 1 \forall n$ , then  $y[n] = n^2$ . Hence  $y[n] \rightarrow \infty$  as  $n \rightarrow \infty$ , hence not BIBO stable.

Let  $y[n]$  be the output for an input  $x[n]$ , and let  $y_1[n]$  be the output for an input  $x_1[n]$ . If  $x_1[n] = x[n - n_0]$  then  $y_1[n] = n^2x_1[n] = n^2x[n - n_0]$ . However,  $y[n - n_0] = (n - n_0)^2x[n - n_0]$ . Since  $y_1[n] \neq y[n - n_0]$ , the system is not time-invariant.

**(b)**  $y[n] = x^4[n]$ .

For an input  $x_1[n]$  the output is  $y_1[n] = x_1^4[n]$ ,  $i = 1, 2$ . Thus, for an input  $x_3[n] = Ax_1[n] + Bx_2[n]$ , the output  $y_3[n]$  is given by  $y_3[n] = (Ax_1[n] + Bx_2[n])^4 \neq A^4x_1^4[n] + A^4x_2^4[n]$

Hence the system is not linear.

Since there is no output before the input hence the system is causal.

Here, a bounded input produces bounded output hence the system is BIBO stable too.

Let  $y[n]$  be the output for an input  $x[n]$ , and let  $y_1[n]$  be the output for an input  $x_1[n]$ . If  $x_1[n] = x[n - n_0]$  then  $y_1[n] = x_1^4[n] = x^4[n - n_0] = y[n - n_0]$ . Hence, the system is time-invariant.

**(c)**  $y[n] = \beta + \sum_{l=0}^3 x[n-l]$ .

For an input  $x_i[n]$  the output is  $y_i[n] = \beta + \sum_{l=0}^3 x_i[n-l]$ ,  $i = 1, 2$ . Thus, for an input  $x_3[n] =$

$Ax_1[n] + Bx_2[n]$ , the output  $y_3[n]$  is given by

$$y[n] = \beta + \sum_{l=0}^3 (Ax_1[n-l] + Bx_2[n-l]) = \beta + \sum_{l=0}^3 Ax_1[n-l] + \sum_{l=0}^3 Bx_2[n-l]$$

$\neq Ay_1[n] + By_2[n]$ . Since  $\beta \neq 0$  hence the system is not linear.

Since there is no output before the input hence the system is causal.

Here, a bounded input produces bounded output hence the system is BIBO stable too.

Also following an analysis similar to that in part (a) it is easy to show that the system is time-invariant.

**(d)**  $y[n] = \beta + \sum_{l=-3}^3 x[n-l]$

For an input  $x_i[n]$  the output is  $y_i[n] = \beta + \sum_{l=-3}^3 x_i[n-l]$ ,  $i = 1, 2$ . Thus, for an input  $x_3[n] =$

$Ax_1[n] + Bx_2[n]$ , the output  $y_3[n]$  is given by

$$y[n] = \beta + \sum_{l=-3}^3 (Ax_1[n-l] + Bx_2[n-l]) = \beta + \sum_{l=-3}^3 Ax_1[n-l] + \sum_{l=-3}^3 Bx_2[n-l]$$

$\neq Ay_1[n] + By_2[n]$ . Since  $\beta \neq 0$  hence the system is not linear.

Since there is output before the input hence the system is non-causal.

Here, a bounded input produces bounded output hence the system is BIBO stable.

Let  $y[n]$  be the output for an input  $x[n]$ , and let  $y_1[n]$  be the output for an input  $x_1[n]$ . If  $x_1[n] = x[n - n_0]$  then  $y_1[n] = \beta + \sum_{l=-3}^3 x_1[n-1] = \beta + \sum_{l=-3}^3 x_1[n - n_0 - 1] = y[n - n_0]$ . Hence the system is time-invariant.

**(e)**  $y[n] = \alpha x[-n]$

The system is linear, stable, non causal. Let  $y[n]$  be the output for an input  $x[n]$  and  $y_1[n]$  be the output for an input  $x_1[n]$ . Then  $y[n] = \alpha x[-n]$  and  $y_1[n] = \alpha x_1[-n]$ .

Let  $x_1[n] = x[n - n_0]$ , then  $y_1[n] = \alpha x_1[-n] = \alpha x[-n - n_0]$ , whereas  $y[n - n_0] = \alpha x[n_0 - n]$ .

Hence the system is time-varying.

**(f)**  $y[n] = x[n - 5]$

The given system is linear, causal, stable and time-invariant.

**2.27**  $y[n] = x[n + 1] - 2x[n] + x[n - 1]$ .

Let  $y_1[n]$  be the output for an input  $x_1[n]$  and  $y_2[n]$  be the output for an input  $x_2[n]$ . Then for an input  $x_3[n] = \alpha x_1[n] + \beta x_2[n]$  the output  $y_3[n]$  is given by

$$\begin{aligned} y_3[n] &= x_3[n + 1] - 2x_3[n] + x_3[n - 1] \\ &= \alpha x_1[n + 1] - 2\alpha x_1[n] + \alpha x_1[n - 1] + \beta x_2[n + 1] - 2\beta x_2[n] + \beta x_2[n - 1] \\ &= \alpha y_1[n] + \beta y_2[n]. \end{aligned}$$

Hence the system is linear. If  $x_1[n] = x[n - n_0]$  then  $y_1[n] = y[n - n_0]$ . Hence the system is time-invariant. Also the system's impulse response is given by

$$h[n] = \begin{cases} -2, & n = 0, \\ 1, & n = 1, -1, \\ 0, & \text{elsewhere.} \end{cases}$$

Since  $h[n] \neq 0 \ \forall n < 0$  the system is non-causal.

**2.28** Median filtering is a nonlinear operation. Consider the following sequences as the input to a median filter:  $x_1[n] = \{3, 4, 5\}$  and  $x_2[n] = \{2, -2, -2\}$ . The corresponding outputs of the median filter are  $y_1[n] = 4$  and  $y_2[n] = -2$ . Now consider another input sequence  $x_3[n] = x_1[n] + x_2[n]$ . Then the corresponding output is  $y_3[n] = 3$ . On the other hand,

$y_1[n] + y_2[n] = 2$ . Hence median filtering is not a linear operation. To test the time-invariance property, let  $x[n]$  and  $x_1[n]$  be the two inputs to the system with corresponding outputs  $y[n]$  and  $y_1[n]$ . If  $x_1[n] = x[n - n_0]$  then

$$\begin{aligned} y_1[n] &= \text{median}\{x_1[n - k], \dots, x_1[n], \dots, x_1[n + k]\} \\ &= \text{median}\{x[n - k - n_0], \dots, x[n - n_0], \dots, x[n + k - n_0]\} = y[n - n_0]. \end{aligned}$$

Hence the system is time invariant.

$$2.29 \quad y[n] = \frac{1}{2} \left( y[n-1] + \frac{x[n]}{y[n-1]} \right)$$

Now for an input  $x[n] = \alpha \mu[n]$ , the output  $y[n]$  converges to some constant  $K$  as  $n \rightarrow \infty$ .

The above difference equation as  $n \rightarrow \infty$  reduces to  $K = \frac{1}{2} \left( K + \frac{\alpha}{K} \right)$  which is equivalent to  $K^2 = \alpha$  or in other words,  $K = \sqrt{\alpha}$ .

It is easy to show that the system is non-linear. Now assume  $y_1[n]$  be the output for an

$$\text{input } x_1[n]. \text{ Then } y_1[n] = \frac{1}{2} \left( y_1[n-1] + \frac{x_1[n]}{y_1[n-1]} \right)$$

$$\text{If } x_1[n] = x[n - n_0]. \text{ Then, } y_1[n] = \frac{1}{2} \left( y_1[n-1] + \frac{x[n - n_0]}{y_1[n-1]} \right).$$

Thus  $y_1[n] = y[n - n_0]$ . Hence the above system is time invariant.

**2.30** For an input  $x_i[n]$ ,  $i = 1, 2$ , the input-output relation is given by

$y_i[n] = x_i[n] - y_i^2[n-1] + y_i[n-1]$ . Thus, for an input  $Ax_1[n] + Bx_2[n]$ , if the output is  $Ay_1[n] + By_2[n]$ , then the input-output relation is given by  $Ay_1[n] + By_2[n] =$

$$\begin{aligned} & Ax_1[n] + Bx_2[n] - (Ay_1[n-1] + By_2[n-1])^2 + Ay_1[n-1] + By_2[n-1] = Ax_1[n] + Bx_2[n] \\ & - A^2 y_1^2[n-1] - B^2 y_2^2[n-1] + 2AB y_1[n-1] y_2[n-1] + Ay_1[n-1] + By_2[n-1] \\ & \neq Ax_1[n] - A^2 y_1^2[n-1] + Ay_1[n-1] + Bx_2[n] - B^2 y_2^2[n-1] + By_2[n-1]. \end{aligned}$$

Hence, the system is nonlinear.

Let  $y[n]$  be the output for an input  $x[n]$  which is related to  $x[n]$  through

$y[n] = x[n] - y^2[n-1] + y[n-1]$ . Then the input-output relation for an input  $x[n - n_0]$  is given by  $y[n - n_0] = x[n - n_0] - y^2[n - n_0 - 1] + y[n - n_0 - 1]$ , or in other words, the system is time-invariant.

Now for an input  $x[n] = \alpha \mu[n]$ , the output  $y[n]$  converges to some constant  $K$  as  $n \rightarrow \infty$ .

The above difference equation as  $n \rightarrow \infty$  reduces to  $K = \alpha - K^2 + K$ , or  $K^2 = \alpha$ , i.e.

$$K = \sqrt{\alpha}.$$

**2.31** As  $\delta[n] = \mu[n] - \mu[n-1]$ ,  $T\{\delta[n]\} = T\{\mu[n]\} - T\{\mu[n-1]\} \Rightarrow h[n] = s[n] - s[n-1]$

For a discrete LTI system

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} (s[k] - s[k-1])x[n-k] = \sum_{k=-\infty}^{\infty} s[k]x[n-k] - \sum_{k=-\infty}^{\infty} s[k-1]x[n-k]$$

**2.32**  $y[n] = \sum_{m=-\infty}^{\infty} h[m]\tilde{x}[n-m]$ . Hence,  $y[n+kN] = \sum_{m=-\infty}^{\infty} h[m]\tilde{x}[n+kN-m]$   
 $= \sum_{m=-\infty}^{\infty} h[m]\tilde{x}[n-m] = y[n]$ . Thus,  $y[n]$  is also a periodic sequence with a period  $N$ .

**2.33** Now  $\delta[n-r] \otimes \delta[n-s] = \sum_{m=-\infty}^{\infty} \delta[m-r]\delta[n-s-m] = \delta[n-r-s]$

$$\begin{aligned} \text{(a)} \quad y_1[n] &= x_1[n] \otimes h_1[n] = (2\delta[n-1] - 0.5\delta[n-3]) \otimes (2\delta[n] + \delta[n-1] - 3\delta[n-3]) \\ &= 4\delta[n-1] \otimes \delta[n] - \delta[n-3] \otimes \delta[n] + 2\delta[n-1] \otimes \delta[n-1] - 0.5\delta[n-3] \otimes \delta[n-1] \\ &\quad - 6\delta[n-1] \otimes \delta[n-3] + 1.5\delta[n-3] \otimes \delta[n-3] = 4\delta[n-1] - \delta[n-3] + 2\delta[n-1] \\ &\quad - 0.5\delta[n-4] - 6\delta[n-4] + 1.5\delta[n-6] \\ &= 4\delta[n-1] + 2\delta[n-1] - \delta[n-3] - 6.5\delta[n-4] + 1.5\delta[n-6] \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad y_2[n] &= x_2[n] \otimes h_2[n] = (-3\delta[n-1] + \delta[n+2]) \otimes (-\delta[n-2] - 0.5\delta[n-1] + 3\delta[n-3]) \\ &= -0.5\delta[n+1] - \delta[n] + 3\delta[n-1] + 1.5\delta[n-2] + 3\delta[n-3] - 9\delta[n-4] \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad y_3[n] &= x_1[n] \otimes h_2[n] = \\ &(2\delta[n-1] - 0.5\delta[n-3]) \otimes (-\delta[n-2] - 0.5\delta[n-1] + 3\delta[n-3]) = \\ &- \delta[n-2] - 2\delta[n-3] - 6.25\delta[n-4] + 0.5\delta[n-5] - 1.5\delta[n-6] \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad y_4[n] &= x_2[n] \otimes h_1[n] = (-3\delta[n-1] + \delta[n+2]) \otimes (2\delta[n] + \delta[n-1] - 3\delta[n-3]) = \\ &2\delta[n+2] + \delta[n+1] - \delta[n-1] - 3\delta[n-2] - 9\delta[n-4] \end{aligned}$$

**2.34**  $y[n] = \sum_{m=N_1}^{N_2} g[m]h[n-m]$ . Now,  $h[n-m]$  is defined for  $M_1 \leq n-m \leq M_2$ . Thus, for  $m=N_1$ ,  $h[n-m]$  is defined for  $M_1 \leq n-N_1 \leq M_2$ , or equivalently, for  $M_1+N_1 \leq n \leq M_2+N_1$ . Likewise, for  $m=N_2$ ,  $h[n-m]$  is defined for  $M_1 \leq n-N_2 \leq M_2$ , or equivalently, for  $M_1+N_2 \leq n \leq M_2+N_2$ .

**(a)** The length of  $y[n]$  is  $M_2 + N_2 - M_1 - N_1 + 1$ .

**(b)** The range of  $n$  for  $y[n] \neq 0$  is  $\min(M_1 + N_1, M_1 + N_2) \leq n \leq \max(M_2 + N_1, M_2 + N_2)$ , i.e.,  $M_1 + N_1 \leq n \leq M_2 + N_2$ .

**2.35**  $y[n] = x_1[n] \otimes x_2[n] = \sum_{k=-\infty}^{\infty} x_1[n-k]x_2[k]$ .

Now,  $v[n] = x_1[n-N_1] \otimes x_2[n-N_2] = \sum_{k=-\infty}^{\infty} x_1[n-N_1-k]x_2[k-N_2]$ . Let

$k-N_2 = m$ . Then  $v[n] = \sum_{m=-\infty}^{\infty} x_1[n-N_1-N_2-m]x_2[m] = y[n-N_1-N_2]$ ,

**2.36**  $g[n] = x_1[n] \otimes x_2[n] \otimes x_3[n] = y[n] \otimes x_3[n]$  where  $y[n] = x_1[n] \otimes x_2[n]$ . Define  $v[n] = x_1[n-N_1] \otimes x_2[n-N_2]$ . Then,  $h[n] = v[n] \otimes x_3[n-N_3]$ . From the results of Problem 2.32,  $v[n] = y[n-N_1-N_2]$ . Hence,  $h[n] = y[n-N_1-N_2] \otimes x_3[n-N_3]$ . Therefore, using the results of Problem 2.32 again we get  $h[n] = g[n-N_1-N_2-N_3]$ .

**2.37**  $y[n] = x[n] \otimes h[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$ . Substituting  $k$  by  $n-m$  in the above expression, we

get  $y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = h[n] \otimes x[n]$ . Hence convolution operation is commutative.

Let  $y[n] = x[n] \otimes (h_1[n] + h_2[n]) = \sum_{k=-\infty}^{\infty} x[n-k](h_1[k] + h_2[k])$   
 $= \sum_{k=-\infty}^{\infty} x[n-k]h_1[k] + \sum_{k=-\infty}^{\infty} x[n-k]h_2[k] = x[n] \otimes h_1[n] + x[n] \otimes h_2[n]$ . Hence convolution is distributive.

**2.38**  $x_3[n] \otimes x_2[n] \otimes x_1[n] = x_3[n] \otimes (x_2[n] \otimes x_1[n])$

As  $x_2[n] \otimes x_1[n]$  is an unbounded sequence hence the result of this convolution cannot be determined. But  $x_2[n] \otimes x_3[n] \otimes x_1[n] = x_2[n] \otimes (x_3[n] \otimes x_1[n])$ . Now  $x_3[n] \otimes x_1[n] = 0$  for all values of  $n$  hence the overall result is zero. Hence for the given sequences  $x_3[n] \otimes x_2[n] \otimes x_1[n] \neq x_2[n] \otimes x_3[n] \otimes x_1[n]$ .

**2.39**  $w[n] = x[n] \otimes h[n] \otimes g[n]$ . Define  $y[n] = x[n] \otimes h[n] = \sum_k x[k]h[n-k]$  and  $f[n] =$

$h[n] \otimes g[n] = \sum_k g[k]h[n-k]$ . Consider  $w_1[n] = (x[n] \otimes h[n]) \otimes g[n] = y[n] \otimes g[n]$   
 $= \sum_m g[m] \sum_k x[k]h[n-m-k]$ . Next consider  $w_2[n] = x[n] \otimes (h[n] \otimes g[n]) = x[n] \otimes f[n]$   
 $= \sum_k x[k] \sum_m g[m]h[n-k-m]$ . Difference between the expressions for  $w_1[n]$  and  $w_2[n]$  is

that the order of the summations is changed.

A) Assumptions:  $h[n]$  and  $g[n]$  are causal filters, and  $x[n] = 0$  for  $n < 0$ . This implies

$$y[m] = \begin{cases} 0, & \text{for } m < 0, \\ \sum_{k=0}^m x[k]h[m-k], & \text{for } m \geq 0. \end{cases}$$

Thus,  $w[n] = \sum_{m=0}^n g[m]y[n-m] = \sum_{m=0}^n g[m] \sum_{k=0}^{n-m} x[k]h[n-m-k]$ .

All sums have only a finite number of terms. Hence, interchange of the order of summations is justified and will give correct results.

B) Assumptions:  $h[n]$  and  $g[n]$  are stable filters, and  $x[n]$  is a bounded sequence with

$|x[n]| \leq X$ . Here,  $y[m] = \sum_{k=-\infty}^{\infty} h[k]x[m-k] = \sum_{k=k_1}^{k_2} h[k]x[m-k] + \epsilon_{k_1, k_2}[m]$  with

$$|\epsilon_{k_1, k_2}[m]| \leq \epsilon_n X.$$

In this case all sums have effectively only a finite number of terms and the error can be reduced by choosing  $k_1$  and  $k_2$  sufficiently large. Hence in this case the problem is again effectively reduced to that of the one-sided sequences. Here, again an interchange of summations is justified and will give correct results.

Hence for the convolution to be associative, it is sufficient that the sequences be stable and single-sided.

**2.40**  $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$ . Since  $h[k]$  is of length  $M$  and defined for  $0 \leq k \leq M-1$ , the convolution sum reduces to  $y[n] = \sum_{k=0}^{(M-1)} x[n-k]h[k]$ .  $y[n]$  will be non-zero for all those values of  $n$  and  $k$  for which  $n-k$  satisfies  $0 \leq n-k \leq N-1$ .

Minimum value of  $n-k = 0$  and occurs for lowest  $n$  at  $n = 0$  and  $k = 0$ . Maximum value of  $n-k = N-1$  and occurs for maximum value of  $k$  at  $M-1$ . Thus  $n-k = M-1 \Rightarrow n = N+M-2$ . Hence the total number of non-zero samples =  $N+M-1$ .

**2.41**  $y[n] = x[n] \otimes x[n] = \sum_{k=-\infty}^{\infty} x[n-k]x[k]$ .

Since  $x[n-k] = 0$  if  $n-k < 0$  and  $x[k] = 0$  if  $k < 0$  hence the above summation reduces to

$$y[n] = \sum_{k=n}^{N-1} x[n-k]x[k] = \begin{cases} n+1, & 0 \leq n \leq N-1, \\ 2N-n, & N \leq n \leq 2N-2. \end{cases}$$

Hence the output is a triangular sequence with a maximum value of  $N$ . Locations of the output samples with values  $\frac{N}{4}$  are  $n = \frac{N}{4} - 1$  and  $\frac{7N}{4} - 1$ . Locations of the output samples with values  $\frac{N}{2}$  are  $n = \frac{N}{2} - 1$  and  $\frac{3N}{2} - 1$ . Note: It is tacitly assumed that  $N$  is divisible by 4 otherwise  $\frac{N}{4}$  is not an integer.

**2.42**  $y[n] = \sum_{k=0}^{N-1} h[k]x[n-k]$ . The maximum value of  $y[n]$  occurs at  $n = N-1$  when all terms in the convolution sum are non-zero and is given by

$$y[N-1] = \sum_{k=0}^{N-1} h[k] = \sum_{k=1}^N k = \frac{N(N+1)}{2}.$$

**2.43 (a)**  $y[n] = g_{ev}[n] \otimes h_{ev}[n] = \sum_{k=-\infty}^{\infty} h_{ev}[n-k]g_{ev}[k]$ . Hence,  $y[-n] = \sum_{k=-\infty}^{\infty} h_{ev}[-n-k]g_{ev}[k]$

Replace  $k$  by  $-k$ . Then above summation becomes

$$\begin{aligned} y[-n] &= \sum_{k=-\infty}^{\infty} h_{ev}[-n+k]g_{ev}[-k] = \sum_{k=-\infty}^{\infty} h_{ev}[-(n-k)]g_{ev}[-k] = \sum_{k=-\infty}^{\infty} h_{ev}[(n-k)]g_{ev}[k] \\ &= y[n]. \end{aligned}$$

Hence  $g_{ev}[n] \otimes h_{ev}[n]$  is even.

(b)  $y[n] = g_{ev}[n] \otimes h_{od}[n] = \sum_{k=-\infty}^{\infty} h_{od}[(n-k)]g_{ev}[k]$ . As a result,

$$y[-n] = \sum_{k=-\infty}^{\infty} h_{od}[(-n-k)]g_{ev}[k] = \sum_{k=-\infty}^{\infty} h_{od}[-(n-k)]g_{ev}[-k] = - \sum_{k=-\infty}^{\infty} h_{od}[(n-k)]g_{ev}[k]$$

Hence  $g_{ev}[n] \otimes h_{od}[n]$  is odd.

(c)  $y[n] = g_{od}[n] \otimes h_{od}[n] = \sum_{k=-\infty}^{\infty} h_{od}[n-k]g_{od}[k]$ . As a result,

$$y[-n] = \sum_{k=-\infty}^{\infty} h_{od}[-n-k]g_{od}[k] = \sum_{k=-\infty}^{\infty} h_{od}[-(n-k)]g_{od}[-k] = \sum_{k=-\infty}^{\infty} h_{od}[(n-k)]g_{od}[k]$$

Hence  $g_{od}[n] \otimes h_{od}[n]$  is even.

2.44 (a) The length of  $x[n]$  is  $7 - 4 + 1 = 4$ . Using  $x[n] = \frac{1}{h[0]} \left\{ y[n] - \sum_{k=1}^7 h[k]x[n-k] \right\}$ , we arrive at  $x[n] = \{1 \ 3 \ -2 \ 12\}$ ,  $0 \leq n \leq 3$

(b) The length of  $x[n]$  is  $9 - 5 + 1 = 5$ . Using  $x[n] = \frac{1}{h[0]} \left\{ y[n] - \sum_{k=1}^9 h[k]x[n-k] \right\}$ , we arrive at  $x[n] = \{1 \ 1 \ 1 \ 1 \ 1\}$ ,  $0 \leq n \leq 4$ .

(c) The length of  $x[n]$  is  $5 - 3 + 1 = 3$ . Using  $x[n] = \frac{1}{h[0]} \left\{ y[n] - \sum_{k=1}^5 h[k]x[n-k] \right\}$ , we get  $x[n] = \{-4 + j, \ -0.6923 + j0.4615, \ 3.4556 + j1.1065\}$ ,  $0 \leq n \leq 2$ .

2.45  $y[n] = ay[n-1] + bx[n]$ . Hence,  $y[0] = ay[-1] + bx[0]$ . Next,  $y[1] = ay[0] + bx[1] = a^2y[-1] + a^2bx[0] + bx[1]$ . Continuing further in similar way we obtain  $y[n] = a^{n+1}y[-1] + \sum_{k=0}^n a^{n-k}bx[k]$ .

(a) Let  $y_1[n]$  be the output due to an input  $x_1[n]$ . Then  $y_1[n] = a^{n+1}y[-1] + \sum_{k=0}^n a^{n-k}bx_1[k]$ .

If  $x_1[n] = x[n - n_0]$  then

$$y_1[n] = a^{n+1}y[-1] + \sum_{k=n_0}^n a^{n-k}bx[k - n_0] = a^{n+1}y[-1] + \sum_{r=0}^{n-n_0} a^{n-n_0-r}bx[r].$$

$$\text{However, } y[n - n_0] = a^{n-n_0+1}y[-1] + \sum_{r=0}^{n-n_0} a^{n-n_0-r}bx[r].$$

Hence  $y_1[n] \neq y[n - n_0]$  unless  $y[-1] = 0$ . For example, if  $y[-1] = 1$  then the system is time variant. However if  $y[-1] = 0$  then the system is time -invariant.



- (b) Let  $y_1[n]$  and  $y_2[n]$  be the outputs due to the inputs  $x_1[n]$  and  $x_2[n]$ . Let  $y[n]$  be the output for an input  $\alpha x_1[n] + \beta x_2[n]$ . However,

$$\alpha y_1[n] + \beta y_2[n] = \alpha a^{n+1} y[-1] + \beta a^{n+1} y[-1] + \alpha \sum_{k=0}^n a^{n-k} b x_1[k] + \beta \sum_{k=0}^n a^{n-k} b x_2[k]$$

whereas

$$y[n] = a^{n+1} y[-1] + \alpha \sum_{k=0}^n a^{n-k} b x_1[k] + \beta \sum_{k=0}^n a^{n-k} b x_2[k].$$

Hence the system is not linear if  $y[-1] = 1$  but is linear if  $y[-1] = 0$ .

- (c) Generalizing the above result it can be shown that for an  $N$ -th order causal discrete time system to be linear and time invariant we require  $y[-N] = y[-N+1] = \dots = y[-1] = 0$ .

- 2.46**  $y_{\text{step}}[n] = \sum_{k=0}^n h[k] \mu[n-k] = \sum_{k=0}^n h[k]$ ,  $n \geq 0$ , and  $y_{\text{step}}[n] = 0$ ,  $n < 0$ . Since  $h[k]$  is nonnegative,  $y_{\text{step}}[n]$  is a monotonically increasing function of  $n$  for  $n \geq 0$ , and is not oscillatory. Hence, no overshoot.

- 2.47 (a)**  $f[n] = f[n-1] + f[n-2]$ . Let  $f[n] = \alpha r^n$ , then the difference equation reduces to  $\alpha r^n - \alpha r^{n-1} - \alpha r^{n-2} = 0$  which reduces to  $r^2 - r - 1 = 0$  resulting in  $r = \frac{1 \pm \sqrt{5}}{2}$ .

$$\text{Thus, } f[n] = \alpha_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + \alpha_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n.$$

Since  $f[0] = 0$  hence  $\alpha_1 + \alpha_2 = 0$ . Also  $f[1] = 1$  hence  $\frac{\alpha_1 + \alpha_2}{2} + \sqrt{5} \frac{\alpha_1 - \alpha_2}{2} = 1$ .

Solving for  $\alpha_1$  and  $\alpha_2$ , we get  $\alpha_1 = -\alpha_2 = \frac{1}{\sqrt{5}}$ . Hence,  $f[n] = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n$ .

- (b)  $y[n] = y[n-1] + y[n-2] + x[n-1]$ . As system is LTI, the initial conditions are equal to zero.

Let  $x[n] = \delta[n]$ . Then,  $y[n] = y[n-1] + y[n-2] + \delta[n-1]$ . Hence,  $y[0] = y[-1] + y[-2] = 0$  and  $y[1] = 1$ . For  $n > 1$  the corresponding difference equation is  $y[n] = y[n-1] + y[n-2]$  with initial conditions  $y[0] = 0$  and  $y[1] = 1$ , which are the same as those for the solution of Fibonacci's sequence. Hence the solution for  $n > 1$  is given by

$$y[n] = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

Thus  $f[n]$  denotes the impulse response of a causal LTI system described by the difference equation  $y[n] = y[n-1] + y[n-2] + x[n-1]$ .

- 2.48**  $y[n] = \alpha y[n-1] + x[n]$ . Denoting,  $y[n] = y_{\text{re}}[n] + j y_{\text{im}}[n]$ , and  $\alpha = a + jb$ , we get,

$$y_{\text{re}}[n] + j y_{\text{im}}[n] = (a + jb)(y_{\text{re}}[n-1] + j y_{\text{im}}[n-1]) + x[n].$$

Equating the real and the imaginary parts, and noting that  $x[n]$  is real, we get

$$\begin{aligned} y_{re}[n] &= ay_{re}[n-1] - by_{im}[n-1] + x[n], \\ y_{im}[n] &= by_{re}[n-1] + ay_{im}[n-1] \end{aligned} \quad (1)$$

Therefore

$$y_{im}[n-1] = \frac{1}{a}y_{im}[n] - \frac{b}{a}y_{re}[n-1]$$

Hence a single input, two output difference equation is

$$y_{re}[n] = ay_{re}[n-1] - \frac{b}{a}y_{im}[n] + \frac{b^2}{a}y_{re}[n-1] + x[n]$$

thus  $by_{im}[n-1] = -ay_{re}[n-1] + (a^2 + b^2)y_{re}[n-2] + ax[n-1]$ .

Substituting the above in Eq. (1) we get

$$y_{re}[n] = 2ay_{re}[n-1] - (a^2 + b^2)y_{re}[n-2] + x[n] - ax[n-1]$$

which is a second-order difference equation representing  $y_{re}[n]$  in terms of  $x[n]$ .

**2.49** From Eq. (2.59), the output  $y[n]$  of the factor-of-3 interpolator is given by

$y[n] = x_u[n] + \frac{1}{3}(x_u[n-1] + x_u[n+2]) + \frac{2}{3}(x_u[n-2] + x_u[n+1])$  where  $x_u[n]$  is the output of the factor-of-3 up-sampler for an input  $x[n]$ . To compute the impulse response we set  $x[n] = \delta[n]$ , in which case,  $x_u[n] = \delta[3n]$ . As a result, the impulse response is given by

$$\begin{aligned} h[n] &= \delta[3n] + \frac{1}{3}(\delta[3n-3] + \delta[3n+6]) + \frac{2}{3}(\delta[3n-6] + \delta[3n+3]) \text{ or} \\ &= \frac{1}{3}\delta[n+2] + \frac{2}{3}\delta[n+1] + \delta[n] + \frac{1}{3}\delta[n-1] + \frac{2}{3}\delta[n-2]. \end{aligned}$$

**2.50** The output  $y[n]$  of the factor-of-L interpolator is given by

$$\begin{aligned} y[n] &= x_u[n] + \frac{1}{L}(x_u[n-1] + x_u[n+L-1]) + \frac{2}{L}(x_u[n-2] + x_u[n+L-2]) \\ &\quad + \mathbf{K} + \frac{L-1}{L}(x_u[n-L+1] + x_u[n+1]) \end{aligned}$$

where  $x_u[n]$  is the output of the factor-of-L up-sampler for an input  $x[n]$ . To compute the impulse response we set  $x[n] = \delta[n]$ , in which case,  $x_u[n] = \delta[Ln]$ . As a result, the impulse response is given by

$$\begin{aligned} h[n] &= \delta[Ln] + \frac{1}{L}(\delta[Ln-L] + \delta[Ln+L(L-1)]) + \frac{2}{L}(\delta[Ln-2L] + \delta[Ln+L(L-2)]) \\ &\quad + \mathbf{K} + \frac{L-1}{L}(\delta[Ln-L(L-1)] + \delta[Ln+L]) = \frac{1}{L}\delta[n+(L-1)] + \frac{2}{L}\delta[n+(L-2)] + \mathbf{K} \\ &\quad + \frac{L-1}{L}\delta[n+1] + \delta[n] + \frac{1}{L}\delta[n-1] + \frac{2}{L}\delta[n-2] + \frac{L-1}{L}\delta[n-(L-1)] \end{aligned}$$

**2.51** The first-order causal LTI system is characterized by the difference equation

$y[n] = p_0x[n] + p_1x[n-1] - d_1y[n-1]$ . Letting  $x[n] = \delta[n]$  we obtain the expression for its impulse response  $h[n] = p_0\delta[n] + p_1\delta[n-1] - d_1h[n-1]$ . Solving it for  $n = 0, 1, \text{ and } 2$ , we get  $h[0] = p_0$ ,  $h[1] = p_1 - d_1h[0] = p_1 - d_1p_0$ , and  $h[2] = -d_1h[1] = -d_1(p_1 - d_1p_0)$ . Solving these equations we get  $p_0 = h[0]$ ,  $d_1 = -\frac{h[2]}{h[1]}$ , and  $p_1 = h[1] - \frac{h[2]h[0]}{h[1]}$ .

$$2.52 \quad \sum_{k=0}^M p_k x[n-k] = \sum_{k=0}^N d_k y[n-k]$$

Let the input to the system be  $x[n] = \delta[n]$ . Then,  $\sum_{k=0}^M p_k \delta[n-k] = \sum_{k=0}^N d_k h[n-k]$ . Thus,

$$p_r = \sum_{k=0}^N d_k h[r-k]. \quad \text{Since the system is assumed to be causal, } h[r-k] = 0 \quad \forall k > r.$$

$$p_r = \sum_{k=0}^r d_k h[r-k] = \sum_{k=0}^r h[k] d_{r-k}.$$

2.53 The impulse response of the cascade is given by  $h[n] = h_1[n] \otimes h_2[n]$  where

$$h_1[n] = \alpha^n \mu[n] \quad \text{and} \quad h_2[n] = \beta^n \mu[n]. \quad \text{Hence, } h[n] = \left( \sum_{k=0}^n \alpha^k \beta^{n-k} \right) \mu[n].$$

2.54 Now,  $h[n] = \alpha^n \mu[n]$ . Therefore  $y[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k] = \sum_{k=0}^{\infty} \alpha^k x[n-k]$

$$= x[n] + \sum_{k=1}^{\infty} \alpha^k x[n-k] = x[n] + \alpha \sum_{k=0}^{\infty} \alpha^k x[n-1-k] = x[n] + \alpha y[n-1].$$

Hence,  $x[n] = y[n] - \alpha y[n-1]$ . Thus the inverse system is given by  $y[n] = x[n] - \alpha x[n-1]$ .

The impulse response of the inverse system is given by  $g[n] = \left\{ \underset{\uparrow}{1} \quad -\alpha \right\}$ .

2.55  $y[n] = y[n-1] + y[n-2] + x[n-1]$ . Hence,  $x[n-1] = y[n] - y[n-1] - y[n-2]$ , i.e.

$x[n] = y[n+1] - y[n] - y[n-1]$ . Hence the inverse system is characterised by

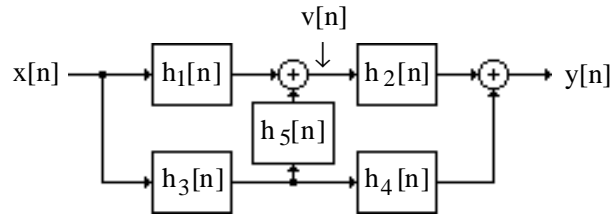
$y[n] = x[n+1] - x[n] - x[n-1]$  with an impulse response given by  $g[n] = \left\{ 1 \quad \underset{\uparrow}{-1} \quad -1 \right\}$ .

2.56  $y[n] = p_0 x[n] + p_1 x[n-1] - d_1 y[n-1]$  which leads to  $x[n] = \frac{1}{p_0} y[n] + \frac{d_1}{p_0} y[n-1] - \frac{p_1}{p_0} x[n-1]$

Hence the inverse system is characterised by the difference equation

$$y_1[n] = \frac{1}{p_0} x_1[n] + \frac{d_1}{p_0} x_1[n-1] - \frac{p_1}{p_0} y_1[n-1].$$

2.57 (a) From the figure shown below we observe



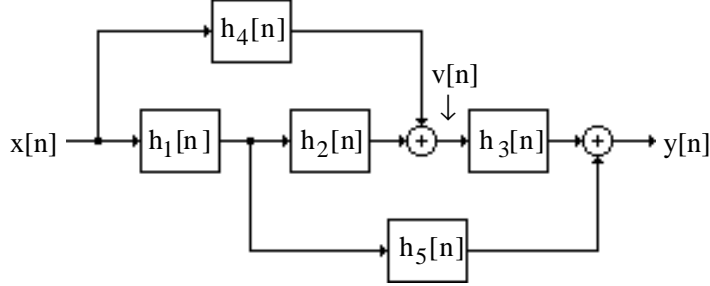
$$v[n] = (h_1[n] + h_3[n] \otimes h_5[n]) \otimes x[n] \text{ and } y[n] = h_2[n] \otimes v[n] + h_3[n] \otimes h_4[n] \otimes x[n].$$

$$\text{Thus, } y[n] = (h_2[n] \otimes h_1[n] + h_2[n] \otimes h_3[n] \otimes h_5[n] + h_3[n] \otimes h_4[n]) \otimes x[n].$$

Hence the impulse response is given by

$$h[n] = h_2[n] \otimes h_1[n] + h_2[n] \otimes h_3[n] \otimes h_5[n] + h_3[n] \otimes h_4[n]$$

(b) From the figure shown below we observe



$$v[n] = h_4[n] \otimes x[n] + h_1[n] \otimes h_2[n] \otimes x[n].$$

$$\text{Thus, } y[n] = h_3[n] \otimes v[n] + h_1[n] \otimes h_5[n] \otimes x[n]$$

$$= h_3[n] \otimes h_4[n] \otimes x[n] + h_3[n] \otimes h_1[n] \otimes h_2[n] \otimes x[n] + h_1[n] \otimes h_5[n] \otimes x[n]$$

Hence the impulse response is given by

$$h[n] = h_3[n] \otimes h_4[n] + h_3[n] \otimes h_1[n] \otimes h_2[n] + h_1[n] \otimes h_5[n]$$

**2.58**  $h[n] = h_1[n] \otimes h_2[n] + h_3[n]$

$$\text{Now } h_1[n] \otimes h_2[n] = (2\delta[n-2] - 3\delta[n+1]) \otimes (\delta[n-1] + 2\delta[n+2])$$

$$= 2\delta[n-2] \otimes \delta[n-1] - 3\delta[n+1] \otimes \delta[n-1] + 2\delta[n-2] \otimes 2\delta[n+2]$$

$$- 3\delta[n+1] \otimes 2\delta[n+2] = 2\delta[n-3] - 3\delta[n] + 4\delta[n] - 6\delta[n+3]$$

$$= 2\delta[n-3] + \delta[n] - 6\delta[n+3]. \text{ Therefore,}$$

$$y[n] = 2\delta[n-3] + \delta[n] - 6\delta[n+3] + 5\delta[n-5] + 7\delta[n-3] + 2\delta[n-1] - \delta[n] + 3\delta[n+1]$$

$$= 5\delta[n-5] + 9\delta[n-3] + 2\delta[n-1] + 3\delta[n+1] - 6\delta[n+3]$$

**2.59** For a filter with complex impulse response, the first part of the proof is same as that for a

filter with real impulse response. Since,  $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$ ,

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]|.$$

Since the input is bounded hence  $0 \leq |x[n]| \leq B_x$ . Therefore,  $|y[n]| \leq B_x \sum_{k=-\infty}^{\infty} |h[k]|$

So if  $\sum_{k=-\infty}^{\infty} |h[k]| = S < \infty$  then  $|y[n]| \leq B_x S$  indicating that  $y[n]$  is also bounded.

To prove the converse we need to show that if a bounded output is produced by a bounded input then  $S < \infty$ . Consider the following bounded input defined by  $x[n] = \frac{h^*[-n]}{|h[-n]|}$ .

$$\text{Then } y[0] = \sum_{k=-\infty}^{\infty} \frac{h^*[k]h[k]}{|h[k]|} = \sum_{k=-\infty}^{\infty} |h[k]| = S. \text{ Now since the output is bounded thus } S < \infty.$$

Thus for a filter with complex response too is BIBO stable if and only if  $\sum_{k=-\infty}^{\infty} |h[k]| = S < \infty$ .

**2.60** The impulse response of the cascade is  $g[n] = h_1[n] \otimes h_2[n]$  or equivalently,

$$g[k] = \sum_{r=-\infty}^{\infty} h_1[k-r]h_2[r]. \text{ Thus,}$$

$$\sum_{k=-\infty}^{\infty} |g[k]| = \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} |h_1[k-r]h_2[r]| \leq \left( \sum_{k=-\infty}^{\infty} |h_1[k]| \right) \left( \sum_{r=-\infty}^{\infty} |h_2[r]| \right).$$

Since  $h_1[n]$  and  $h_2[n]$  are stable,  $\sum_k |h_1[k]| < \infty$  and  $\sum_k |h_2[k]| < \infty$ . Hence,  $\sum_k |g[k]| < \infty$ .

Hence the cascade of two stable LTI systems is also stable.

**2.61** The impulse response of the parallel structure  $g[n] = h_1[n] + h_2[n]$ . Now,

$$\sum_k |g[k]| = \sum_k |h_1[k] + h_2[k]| \leq \sum_k |h_1[k]| + \sum_k |h_2[k]|. \text{ Since } h_1[n] \text{ and } h_2[n] \text{ are stable,}$$

$$\sum_k |h_1[k]| < \infty \text{ and } \sum_k |h_2[k]| < \infty. \text{ Hence, } \sum_k |g[k]| < \infty. \text{ Hence the parallel connection of}$$

two stable LTI systems is also stable.

**2.62** Consider a cascade of two passive systems. Let  $y_1[n]$  be the output of the first system which is the input to the second system in the cascade. Let  $y[n]$  be the overall output of the cascade.

$$\text{The first system being passive we have } \sum_{n=-\infty}^{\infty} |y_1[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

$$\text{Likewise the second system being also passive we have } \sum_{n=-\infty}^{\infty} |y[n]|^2 \leq \sum_{n=-\infty}^{\infty} |y_1[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2,$$

indicating that cascade of two passive systems is also a passive system. Similarly one can prove that cascade of two lossless systems is also a lossless system.

**2.63** Consider a parallel connection of two passive systems with an input  $x[n]$  and output  $y[n]$ . The outputs of the two systems are  $y_1[n]$  and  $y_2[n]$ , respectively. Now,

$$\sum_{n=-\infty}^{\infty} |y_1[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2, \text{ and } \sum_{n=-\infty}^{\infty} |y_2[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

Let  $y_1[n] = y_2[n] = x[n]$  satisfying the above inequalities. Then  $y[n] = y_1[n] + y_2[n] = 2x[n]$  and as a result,  $\sum_{n=-\infty}^{\infty} |y[n]|^2 = 4 \sum_{n=-\infty}^{\infty} |x[n]|^2 > \sum_{n=-\infty}^{\infty} |x[n]|^2$ . Hence, the parallel connection of two passive systems may not be passive.

**2.64** Let  $\sum_{k=0}^M p_k x[n-k] = y[n] + \sum_{k=1}^N d_k y[n-k]$  be the difference equation representing the causal

IIR digital filter. For an input  $x[n] = \delta[n]$ , the corresponding output is then  $y[n] = h[n]$ , the impulse response of the filter. As there are  $M+1$   $\{p_k\}$  coefficients, and  $N$   $\{d_k\}$  coefficients, there are a total of  $N+M+1$  unknowns. To determine these coefficients from the impulse response samples, we compute only the first  $N+M+1$  samples. To illustrate the method, without any loss of generality, we assume  $N = M = 3$ . Then, from the difference equation representation we arrive at the following  $N+M+1 = 7$  equations:

$$\begin{aligned} h[0] &= p_0, \\ h[1] + h[0]d_1 &= p_1, \\ h[2] + h[1]d_1 + h[0]d_2 &= p_2, \\ h[3] + h[2]d_1 + h[1]d_2 + h[0]d_3 &= p_3, \\ h[4] + h[3]d_1 + h[2]d_2 + h[1]d_3 &= 0, \\ h[5] + h[4]d_1 + h[3]d_2 + h[2]d_3 &= 0, \\ h[6] + h[5]d_1 + h[4]d_2 + h[3]d_3 &= 0. \end{aligned}$$

Writing the last three equations in matrix form we arrive at

$$\begin{bmatrix} h[4] \\ h[5] \\ h[6] \end{bmatrix} + \begin{bmatrix} h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and hence, 
$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = - \begin{bmatrix} h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \end{bmatrix}^{-1} \begin{bmatrix} h[4] \\ h[5] \\ h[6] \end{bmatrix}.$$

Substituting these values of  $\{d_i\}$  in the first four equations written in matrix form we get

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 \\ h[2] & h[1] & h[0] & 0 \\ h[3] & h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} 1 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

**2.65**  $y[n] = y[-1] + \sum_{l=0}^n x[l] = y[-1] + \sum_{l=0}^n 1 \mu[l] = y[-1] + \sum_{l=0}^n 1 = y[-1] + \frac{n(n+1)}{2}.$

(a) For  $y[-1] = 0$ ,  $y[n] = \frac{n(n+1)}{2}$

(b) For  $y[-1] = -2$ ,  $y[n] = -2 + \frac{n(n+1)}{2} = \frac{n^2 + n - 4}{2}.$

**2.66**  $y(nT) = y((n-1)T) + \int_{(n-1)T}^{nT} x(\tau) d\tau = y((n-1)T) + T \cdot x((n-1)T)$ . Therefore, the difference equation representation is given by  $y[n] = y[n-1] + T \cdot x[n-1]$ , where  $y[n] = y(nT)$  and  $x[n] = x(nT)$ .

**2.67**  $y[n] = \frac{1}{n} \sum_{l=1}^n x[l] = \frac{1}{n} \sum_{l=1}^{n-1} x[l] + \frac{1}{n} x[n]$ ,  $n \geq 1$ .  $y[n-1] = \frac{1}{n-1} \sum_{l=1}^{n-1} x[l]$ ,  $n \geq 1$ . Hence,  
 $\sum_{l=1}^{n-1} x[l] = (n-1)y[n-1]$ . Thus, the difference equation representation is given by  
 $y[n] = \left(\frac{n-1}{n}\right)y[n-1] + \frac{1}{n}x[n]$ ,  $n \geq 1$ .

**2.68**  $y[n] + 0.5y[n-1] = 2\mu[n]$ ,  $n \geq 0$  with  $y[-1] = 2$ . The total solution is given by  
 $y[n] = y_c[n] + y_p[n]$  where  $y_c[n]$  is the complementary solution and  $y_p[n]$  is the particular solution.

$y_c[n]$  is obtained by solving  $y_c[n] + 0.5y_c[n-1] = 0$ . To this end we set  $y_c[n] = \lambda^n$ , which yields  $\lambda^n + 0.5\lambda^{n-1} = 0$  whose solution gives  $\lambda = -0.5$ . Thus, the complementary solution is of the form  $y_c[n] = \alpha(-0.5)^n$ .

For the particular solution we choose  $y_p[n] = \beta$ . Substituting this solution in the difference equation representation of the system we get  $\beta + 0.5\beta = 2\mu[n]$ . For  $n = 0$  we get  $\beta(1 + 0.5) = 2$  or  $\beta = 4/3$ .

The total solution is therefore given by  $y[n] = y_c[n] + y_p[n] = \alpha(-0.5)^n + \frac{4}{3}$ ,  $n \geq 0$ .

Therefore  $y[-1] = \alpha(-0.5)^{-1} + \frac{4}{3} = 2$  or  $\alpha = -\frac{1}{3}$ . Hence, the total solution is given by

$$y[n] = -\frac{1}{3}(-0.5)^n + \frac{4}{3}, n \geq 0.$$

**2.69**  $y[n] + 0.1y[n-1] - 0.06y[n-2] = 2^n\mu[n]$  with  $y[-1] = 1$  and  $y[-2] = 0$ . The complementary solution  $y_c[n]$  is obtained by solving  $y_c[n] + 0.1y_c[n-1] - 0.06y_c[n-2] = 0$ . To this end we set  $y_c[n] = \lambda^n$ , which yields  $\lambda^n + 0.1\lambda^{n-1} - 0.06\lambda^{n-2} = \lambda^{n-2}(\lambda^2 + 0.1\lambda - 0.06) = 0$  whose solution gives  $\lambda_1 = -0.3$  and  $\lambda_2 = 0.2$ . Thus, the complementary solution is of the form  $y_c[n] = \alpha_1(-0.3)^n + \alpha_2(0.2)^n$ .

For the particular solution we choose  $y_p[n] = \beta(2)^n$ . Substituting this solution in the difference equation representation of the system we get  $\beta 2^n + \beta(0.1)2^{n-1} - \beta(0.06)2^{n-2} = 2^n\mu[n]$ . For  $n = 0$  we get  $\beta + \beta(0.1)2^{-1} - \beta(0.06)2^{-2} = 1$  or  $\beta = 200/207 = 0.9662$ .

The total solution is therefore given by  $y[n] = y_c[n] + y_p[n] = \alpha_1(-0.3)^n + \alpha_2(0.2)^n + \frac{200}{207}2^n$ .

From the above  $y[-1] = \alpha_1(-0.3)^{-1} + \alpha_2(0.2)^{-1} + \frac{200}{207}2^{-1} = 1$  and

$$y[-2] = \alpha_1(-0.3)^{-2} + \alpha_2(0.2)^{-2} + \frac{200}{207}2^{-2} = 0 \text{ or equivalently, } -\frac{10}{3}\alpha_1 + 5\alpha_2 = \frac{107}{207} \text{ and}$$

$$\frac{100}{9}\alpha_1 + 25\alpha_2 = -\frac{50}{207} \text{ whose solution yields } \alpha_1 = -0.1017 \text{ and } \alpha_2 = 0.0356. \text{ Hence, the total}$$

solution is given by  $y[n] = -0.1017(-0.3)^n + 0.0356(0.2)^n + 0.9662(2)^n$ , for  $n \geq 0$ .

**2.70**  $y[n] + 0.1y[n-1] - 0.06y[n-2] = x[n] - 2x[n-1]$  with  $x[n] = 2^n\mu[n]$ , and  $y[-1] = 1$  and  $y[-2] = 0$ . For the given input, the difference equation reduces to

$$y[n] + 0.1y[n-1] - 0.06y[n-2] = 2^n\mu[n] - 2(2^{n-1})\mu[n-1] = \delta[n]. \text{ The solution of this}$$

equation is thus the complementary solution with the constants determined from the given initial conditions  $y[-1] = 1$  and  $y[-2] = 0$ .

From the solution of the previous problem we observe that the complementary is of the form  $y_c[n] = \alpha_1(-0.3)^n + \alpha_2(0.2)^n$ .

For the given initial conditions we thus have

$y[-1] = \alpha_1(-0.3)^{-1} + \alpha_2(0.2)^{-1} = 1$  and  $y[-2] = \alpha_1(-0.3)^{-2} + \alpha_2(0.2)^{-2} = 0$ . Combining these two equations we get  $\begin{bmatrix} -1/0.3 & 1/0.2 \\ 1/0.09 & 1/0.04 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  which yields  $\alpha_1 = -0.18$  and  $\alpha_2 = 0.08$ .

Therefore,  $y[n] = -0.18(-0.3)^n + 0.08(0.2)^n$ .

**2.71** The impulse response is given by the solution of the difference equation  $y[n] + 0.5y[n-1] = \delta[n]$ . From the solution of Problem 2.68, the complementary solution is given by  $y_c[n] = \alpha(-0.5)^n$ . To determine the constant we note  $y[0] = 1$  as  $y[-1] = 0$ . From the complementary solution  $y[0] = \alpha(-0.5)^0 = \alpha$ , hence  $\alpha = 1$ . Therefore, the impulse response is given by  $h[n] = (-0.5)^n$ .

**2.72** The impulse response is given by the solution of the difference equation  $y[n] + 0.1y[n-1] - 0.06y[n-2] = \delta[n]$ . From the solution of Problem 2.69, the complementary solution is given by  $y_c[n] = \alpha_1(-0.3)^n + \alpha_2(0.2)^n$ . To determine the constants  $\alpha_1$  and  $\alpha_2$ , we observe  $y[0] = 1$  and  $y[1] + 0.1y[0] = 0$  as  $y[-1] = y[-2] = 0$ . From the complementary solution  $y[0] = \alpha_1(-0.3)^0 + \alpha_2(0.2)^0 = \alpha_1 + \alpha_2 = 1$ , and  $y[1] = \alpha_1(-0.3)^1 + \alpha_2(0.2)^1 = -0.3\alpha_1 + 0.2\alpha_2 = -0.1$ . Solution of these equations yield  $\alpha_1 = 0.6$  and  $\alpha_2 = 0.4$ . Therefore, the impulse response is given by  $h[n] = 0.6(-0.3)^n + 0.4(0.2)^n$ .

**2.73** Let  $A_n = \left| n^K (\lambda_i)^n \right|$ . Then  $\frac{A_{n+1}}{A_n} = \left| \frac{n+1}{n} \right|^K \lambda_i$ . Now  $\lim_{n \rightarrow \infty} \left| \frac{n+1}{n} \right|^K = 1$ . Since  $\lambda_i < 1$ , there exists a positive integer  $N_o$  such that for all  $n > N_o$ ,  $0 < \frac{A_{n+1}}{A_n} < \frac{1+\lambda_i}{2} < 1$ . Hence  $\sum_{n=0}^{\infty} A_n$  converges.

**2.74 (a)**  $x[n] = \{3 \ -2 \ 0 \ 1 \ 4 \ 5 \ 2\}$ ,  $-3 \leq n \leq 3$ .  $r_{xx}[1] = \sum_{n=-3}^3 x[n]x[n-1]$ . Note,  
 $r_{xx}[-6] = x[3]x[-3] = 2 \times 3 = 6$ ,  $r_{xx}[-5] = x[3]x[-2] + x[2]x[-3] = 2 \times (-2) + 5 \times 3 = 11$ ,  
 $r_{xx}[-4] = x[3]x[-1] + x[2]x[-2] + x[1]x[-3] = 2 \times 0 + 5 \times (-2) + 4 \times 3 = 2$ ,  
 $r_{xx}[-3] = x[3]x[0] + x[2]x[-1] + x[1]x[-2] + x[0]x[-3] = 2 \times 1 + 5 \times 0 + 4 \times (-2) + 1 \times 3 = -3$ ,  
 $r_{xx}[-2] = x[3]x[1] + x[2]x[0] + x[1]x[-1] + x[0]x[-2] + x[-1]x[-3] = 11$ ,  
 $r_{xx}[-1] = x[3]x[2] + x[2]x[1] + x[1]x[0] + x[0]x[-1] + x[-1]x[-2] + x[-2]x[-3] = 28$ ,  
 $r_{xx}[0] = x[3]x[3] + x[2]x[2] + x[1]x[1] + x[0]x[0] + x[-1]x[-1] + x[-2]x[-2] + x[-3]x[-3] = 59$ .  
The samples of  $r_{xx}[1]$  for  $1 \leq l \leq 6$  are determined using the property  $r_{xx}[l] = r_{xx}[-l]$ . Thus,

$$r_{xx}[1] = \{6 \ 11 \ 2 \ -3 \ 11 \ 28 \ 59 \ 28 \ 11 \ -3 \ 2 \ 11 \ 6\}, \quad -6 \leq l \leq 6.$$

$y[n] = \{0 \ 7 \ 1 \ -3 \ 4 \ 9 \ -2\}$ ,  $-3 \leq n \leq 3$ . Following the procedure outlined above we get



$$r_{yy}[l] = \{0 \quad -14 \quad 61 \quad 43 \quad -52 \quad 10 \quad 160 \quad 10 \quad -52 \quad 43 \quad 61 \quad -14 \quad 0\}, \quad -6 \leq l \leq 6.$$

$$w[n] = \{-5 \quad 4 \quad 3 \quad 6 \quad -5 \quad 0 \quad 1\}, \quad -3 \leq n \leq 3. \text{ Thus,}$$

$$r_{ww}[l] = \{-5 \quad 4 \quad 28 \quad -44 \quad -11 \quad -20 \quad 112 \quad -20 \quad -11 \quad -44 \quad 28 \quad 4 \quad -5\}, \quad -6 \leq l \leq 6.$$

$$\text{(b) } r_{xy}[l] = \sum_{n=-3}^3 x[n]y[n-l]. \text{ Note, } r_{xy}[-6] = x[3]y[-3] = 0,$$

$$r_{xy}[-5] = x[3]y[-2] + x[2]y[-3] = 14, \quad r_{xy}[-4] = x[3]y[-1] + x[2]y[-2] + x[1]y[-3] = 37,$$

$$r_{xy}[-3] = x[3]y[0] + x[2]y[-1] + x[1]y[-2] + x[0]y[-3] = 27,$$

$$r_{xy}[-2] = x[3]y[1] + x[2]y[0] + x[1]y[-1] + x[0]y[-2] + x[-1]y[-3] = 4,$$

$$r_{xy}[-1] = x[3]y[2] + x[2]y[1] + x[1]y[0] + x[0]y[-1] + x[-1]y[-2] + x[-2]y[-3] = 27,$$

$$r_{xy}[0] = x[3]y[3] + x[2]y[2] + x[1]y[1] + x[0]y[0] + x[-1]y[-1] + x[-2]y[-2] + x[-3]y[-3] = 40,$$

$$r_{xy}[1] = x[2]y[3] + x[1]y[2] + x[0]y[1] + x[-1]y[0] + x[-2]y[-1] + x[-3]y[-2] = 49,$$

$$r_{xy}[2] = x[1]y[3] + x[0]y[2] + x[-1]y[1] + x[-2]y[0] + x[-3]y[-1] = 10,$$

$$r_{xy}[3] = x[0]y[3] + x[-1]y[2] + x[-2]y[1] + x[-3]y[0] = -19,$$

$$r_{xy}[4] = x[-1]y[3] + x[-2]y[2] + x[-3]y[1] = -6, \quad r_{xy}[5] = x[-2]y[3] + x[-3]y[2] = 31,$$

$$r_{xy}[6] = x[-3]y[3] = -6. \text{ Hence,}$$

$$r_{xy}[l] = \{0 \quad 14 \quad 37 \quad 27 \quad 4 \quad 27 \quad 40 \quad 49 \quad 10 \quad -19 \quad -6 \quad 31 \quad -6\}, \quad -6 \leq l \leq 6.$$

$$r_{xw}[l] = \sum_{n=-3}^3 x[n]w[n-l] = \{-10 \quad -17 \quad 6 \quad 38 \quad 36 \quad 12 \quad -35 \quad 6 \quad 1 \quad 29 \quad -15 \quad -2 \quad 3\}, \quad -6 \leq l \leq 6.$$

$$\text{2.75 (a) } x_1[n] = \alpha^n \mu[n]. \quad r_{xx}[l] = \sum_{n=-\infty}^{\infty} x_1[n]x_1[n-l] = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] \cdot \alpha^{n-l} \mu[n-l]$$

$$= \sum_{n=0}^{\infty} \alpha^{2n-l} \mu[n-l] = \begin{cases} \sum_{n=0}^{\infty} \alpha^{2n-1} = \frac{\alpha^{-1}}{1-\alpha^2}, & \text{for } l < 0, \\ \sum_{n=l}^{\infty} \alpha^{2n-l} = \frac{\alpha^l}{1-\alpha^2} & \text{for } l \geq 0. \end{cases}$$

$$\text{Note for } l \geq 0, \quad r_{xx}[l] = \frac{\alpha^l}{1-\alpha^2}, \text{ and for } l < 0, \quad r_{xx}[l] = \frac{\alpha^{-l}}{1-\alpha^2}. \text{ Replacing } l \text{ with } -l \text{ in the}$$

second expression we get  $r_{xx}[-l] = \frac{\alpha^{-(-l)}}{1-\alpha^2} = \frac{\alpha^l}{1-\alpha^2} = r_{xx}[l]$ . Hence,  $r_{xx}[l]$  is an even function of  $l$ .

Maximum value of  $r_{xx}[l]$  occurs at  $l = 0$  since  $\alpha^l$  is a decaying function for increasing  $l$  when  $|\alpha| < 1$ .

$$\text{(b) } x_2[n] = \begin{cases} 1, & 0 \leq n \leq N-1, \\ 0, & \text{otherwise.} \end{cases} \quad r_{xx}[l] = \sum_{n=0}^{N-1} x_2[n-l], \text{ where } x_2[n-l] = \begin{cases} 1, & 1 \leq n \leq N-1+l, \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{Therefore, } r_{xx}[l] = \begin{cases} 0, & \text{for } l < -(N-1), \\ N+1, & \text{for } -(N-1) \leq l < 0, \\ N, & \text{for } l = 0, \\ N-1, & \text{for } 0 < l \leq N-1, \\ 0, & \text{for } l > N-1. \end{cases}$$

It follows from the above  $r_{xx}[l]$  is a triangular function of  $l$ , and hence is even with a maximum value of  $N$  at  $l = 0$

**2.76 (a)**  $x_1[n] = \cos\left(\frac{\pi n}{M}\right)$  where  $M$  is a positive integer. Period of  $x_1[n]$  is  $2M$ , hence

$$\begin{aligned} r_{xx}[l] &= \frac{1}{2M} \sum_{n=0}^{2M-1} x_1[n]x_1[n+l] = \frac{1}{2M} \sum_{n=0}^{2M-1} \cos\left(\frac{\pi n}{M}\right) \cos\left(\frac{\pi(n+l)}{M}\right) \\ &= \frac{1}{2M} \sum_{n=0}^{2M-1} \cos\left(\frac{\pi n}{M}\right) \left\{ \cos\left(\frac{\pi n}{M}\right) \cos\left(\frac{\pi l}{M}\right) - \sin\left(\frac{\pi n}{M}\right) \sin\left(\frac{\pi l}{M}\right) \right\} = \frac{1}{2M} \cos\left(\frac{\pi l}{M}\right) \sum_{n=0}^{2M-1} \cos^2\left(\frac{\pi n}{M}\right). \end{aligned}$$

From the solution of Problem 2.16  $\sum_{n=0}^{2M-1} \cos^2\left(\frac{\pi n}{M}\right) = \frac{2M}{2} = M$ . Therefore,

$$r_{xx}[l] = \frac{1}{2} \cos\left(\frac{\pi l}{M}\right).$$

**(b)**  $x_2[n] = n \text{ modulo } 6 = \{0 \ 1 \ 2 \ 3 \ 4 \ 5\}$ ,  $0 \leq n \leq 5$ . It is a periodic sequence with a period 6. Thus.,  $r_{xx}[l] = \frac{1}{6} \sum_{n=0}^5 x_2[n]x_2[n+l]$ ,  $0 \leq l \leq 5$ . It is also a periodic sequence with a period 6.

$$\begin{aligned} r_{xx}[0] &= \frac{1}{6} (x_2[0]x_2[0] + x_2[1]x_2[1] + x_2[2]x_2[2] + x_2[3]x_2[3] + x_2[4]x_2[4] + x_2[5]x_2[5]) = \frac{55}{6}, \\ r_{xx}[1] &= \frac{1}{6} (x_2[0]x_2[1] + x_2[1]x_2[2] + x_2[2]x_2[3] + x_2[3]x_2[4] + x_2[4]x_2[5] + x_2[5]x_2[0]) = \frac{40}{6}, \\ r_{xx}[2] &= \frac{1}{6} (x_2[0]x_2[2] + x_2[1]x_2[3] + x_2[2]x_2[4] + x_2[3]x_2[5] + x_2[4]x_2[0] + x_2[5]x_2[1]) = \frac{32}{6}, \\ r_{xx}[3] &= \frac{1}{6} (x_2[0]x_2[3] + x_2[1]x_2[4] + x_2[2]x_2[5] + x_2[3]x_2[0] + x_2[4]x_2[1] + x_2[5]x_2[2]) = \frac{28}{6}, \\ r_{xx}[4] &= \frac{1}{6} (x_2[0]x_2[4] + x_2[1]x_2[5] + x_2[2]x_2[0] + x_2[3]x_2[1] + x_2[4]x_2[2] + x_2[5]x_2[3]) = \frac{31}{6}, \\ r_{xx}[5] &= \frac{1}{6} (x_2[0]x_2[5] + x_2[1]x_2[0] + x_2[2]x_2[1] + x_2[3]x_2[2] + x_2[4]x_2[3] + x_2[5]x_2[4]) = \frac{40}{6}. \end{aligned}$$

**(c)**  $x_3[n] = (-1)^n$ . It is a periodic sequence with a period 2. Hence,

$$\begin{aligned} r_{xx}[l] &= \frac{1}{2} \sum_{n=0}^1 x_3[n]x_3[n+l], \quad 0 \leq l \leq 1. \quad r_{xx}[0] = \frac{1}{2} (x_2[0]x_2[0] + x_2[1]x_2[1]) = 1, \text{ and} \\ r_{xx}[1] &= \frac{1}{2} (x_2[0]x_2[1] + x_2[1]x_2[0]) = -1. \quad \text{It is also a periodic sequence with a period 2.} \end{aligned}$$

$$\begin{aligned} \mathbf{2.77} \quad E\{X+Y\} &= \iint (x+y)p_{XY}(x,y)dx dy = \iint x p_{XY}(x,y)dx dy + \iint y p_{XY}(x,y)dx dy \\ &= \int x \left( \int p_{XY}(x,y)dy \right) dx + \int y \left( \int p_{XY}(x,y)dx \right) dy = \int x p_X(x)dx + \int y p_Y(y)dy \end{aligned}$$

$$= E\{X\} + E\{Y\}.$$

From the above result,  $E\{2X\} = E\{X + X\} = 2 E\{X\}$ . Similarly,  $E\{cX\} = E\{(c-1)X + X\} = E\{(c-1)X\} + E\{X\}$ . Hence, by induction,  $E\{cX\} = cE\{x\}$ .

**2.78**  $C = E\{(X - \kappa)^2\}$ . To find the value of  $\kappa$  that minimize the mean square error  $C$ , we differentiate  $C$  with respect to  $\kappa$  and set it to zero.

Thus  $\frac{dC}{d\kappa} = E\{2(X - \kappa)\} = E\{2X\} - E\{2\kappa\} = 2 E\{X\} - 2\kappa = 0$ . which results in  $\kappa = E\{X\}$ , and the minimum value of  $C$  is  $\sigma_x^2$ .

**2.79** mean =  $m_x = E\{x\} = \int_{-\infty}^{\infty} xp_X(x)dx$

variance =  $\sigma_x^2 = E\{(x - m_x)^2\} = \int_{-\infty}^{\infty} (x - m_x)^2 p_X(x)dx$

(a)  $p_X(x) = \frac{\alpha}{\pi} \left( \frac{1}{x^2 + \alpha^2} \right)$ . Now,  $m_x = \frac{\alpha}{\pi} \int_{-\infty}^{\infty} \frac{x dx}{x^2 + \alpha^2}$ . Since  $\frac{x}{x^2 + \alpha^2}$  is odd hence the integral is zero. Thus  $m_x = 0$ .

$\sigma_x^2 = \frac{\alpha}{\pi} \int_{-\infty}^{\infty} \frac{x^2 dx}{x^2 + \alpha^2}$ . Since this integral does not converge hence variance is not defined for this density function.

(b)  $p_x(x) = \frac{\alpha}{2} e^{-\alpha|x|}$ . Now,  $m_x = \frac{\alpha}{2} \int_{-\infty}^{\infty} x e^{-\alpha|x|} dx = 0$ . Next,

$$\begin{aligned} \sigma_x^2 &= \frac{\alpha}{2} \int_{-\infty}^{\infty} x^2 e^{-\alpha|x|} dx = \alpha \int_0^{\infty} x^2 e^{-\alpha x} dx = \alpha \left\{ \frac{x^2 e^{-\alpha x}}{-\alpha} \Big|_0^{\infty} + \int_0^{\infty} \frac{2x}{\alpha} e^{-\alpha x} dx \right\} \\ &= \alpha \left\{ 0 + \left[ \frac{2x}{\alpha} e^{-\alpha x} \Big|_0^{\infty} + \int_0^{\infty} \frac{2}{\alpha^2} e^{-\alpha x} dx \right] \right\} = \frac{2}{\alpha^2}. \end{aligned}$$

(c)  $p_x(x) = \sum_{l=0}^n \binom{n}{l} p^l (1-p)^{n-l} \delta(x-l)$ . Now,

$$m_x = \int_{-\infty}^{\infty} x \sum_{l=0}^n \binom{n}{l} p^l (1-p)^{n-l} \delta(x-l) dx = \sum_{l=0}^n \binom{n}{l} p^l (1-p)^{n-l} = np$$

$$\begin{aligned} \sigma_x^2 &= E\{x^2\} - m_x^2 = \int_{-\infty}^{\infty} x^2 \sum_{l=0}^n \binom{n}{l} p^l (1-p)^{n-l} \delta(x-l) dx - (np)^2 \\ &= \sum_{l=0}^n l^2 \binom{n}{l} p^l (1-p)^{n-l} - n^2 p^2 = np(1-p). \end{aligned}$$

(d)  $p_x(x) = \sum_{l=0}^{\infty} \frac{e^{-\alpha} \alpha^l}{l!} \delta(x-l)$ . Now,

$$m_x = \int_{-\infty}^{\infty} x \sum_{l=0}^{\infty} \frac{e^{-\alpha} \alpha^l}{l!} \delta(x-l) dx = \sum_{l=0}^{\infty} l \frac{e^{-\alpha} \alpha^l}{l!} = \alpha.$$

$$\sigma_x^2 = E\{x^2\} - m_x^2 = \frac{\alpha}{2} \int_{-\infty}^{\infty} x^2 \sum_{l=0}^{\infty} \frac{e^{-\alpha} \alpha^l}{l!} \delta(x-l) dx - \alpha^2 = \sum_{l=0}^{\infty} l^2 \frac{e^{-\alpha} \alpha^l}{l!} - \alpha^2 = \alpha.$$

(e)  $p_x(x) = \frac{x}{\alpha^2} e^{-x^2/2\alpha^2} \mu(x)$ . Now,

$$m_x = \frac{1}{\alpha^2} \int_{-\infty}^{\infty} x^2 e^{-x^2/2\alpha^2} \mu(x) dx = \frac{1}{\alpha^2} \int_0^{\infty} x^2 e^{-x^2/2\alpha^2} dx = \alpha \sqrt{\pi/2}.$$

$$\sigma_x^2 = E\{x^2\} - m_x^2 = \frac{1}{\alpha^2} \int_{-\infty}^{\infty} x^3 e^{-x^2/2\alpha^2} \mu(x) dx - \frac{\alpha^2 \pi}{2} = \left(2 - \frac{\pi}{2}\right) \alpha^2.$$

**2.80** Recall that random variables  $x$  and  $y$  are linearly independent if and only if

$$E\left\{\left|a_1 x + a_2 y\right|^2\right\} > 0 \quad \forall a_1, a_2 \text{ except when } a_1 = a_2 = 0, \text{ Now,}$$

$$E\left\{\left|a_1\right|^2 |x|^2\right\} + E\left\{\left|a_2\right|^2 |y|^2\right\} + E\{a_1 a_2 x y\} + E\{a_1 a_2 x y\} = |a_1|^2 E\{|x|^2\} + |a_2|^2 E\{|y|^2\} > 0$$

$\forall a_1$  and  $a_2$  except when  $a_1 = a_2 = 0$ .

Hence if  $x, y$  are statistically independent they are also linearly independent.

**2.81**  $\sigma_x^2 = E\{(x - m_x)^2\} = E\{x^2 + m_x^2 - 2xm_x\} = E\{x^2\} + E\{m_x^2\} - 2E\{xm_x\}.$

Since  $m_x$  is a constant, hence  $E\{m_x^2\} = m_x^2$  and  $E\{xm_x\} = m_x E\{x\} = m_x^2$ . Thus,

$$\sigma_x^2 = E\{x^2\} + m_x^2 - 2m_x^2 = E\{x^2\} - m_x^2.$$

**2.82**  $V = aX + bY$ . Therefore,  $E\{V\} = a E\{X\} + b E\{Y\} = a m_x + b m_y$ , and

$$\sigma_v^2 = E\{(V - m_v)^2\} = E\{(a(X - m_x) + b(Y - m_y))^2\}.$$

Since  $X, Y$  are statistically independent hence  $\sigma_v^2 = a^2 \sigma_x^2 + b^2 \sigma_y^2$ .

**2.83**  $v[n] = ax[n] + by[n]$ . Thus  $\phi_{vv}[n] = E\{v[m+n]v[m]\}$

$= E\{a^2 x[m+n]x[m] + b^2 y[m+n]y[m] + ab x[m+n]y[m] + ab x[m]y[m+n]\}$ . Since  $x[n]$  and  $y[n]$  are independent,

$$\phi_{vv}[n] = E\{a^2 x[m+n]x[m]\} + E\{b^2 y[m+n]y[m]\} = a^2 \phi_{xx}[n] + b^2 \phi_{yy}[n].$$

$$\phi_{vx}[n] = E\{v[m+n]x[m]\} = E\{a x[m+n]x[m] + b y[m+n]x[m]\} = a E\{x[m+n]x[m]\} = a \phi_{xx}[n].$$

Likewise,  $\phi_{vy}[n] = b \phi_{yy}[n]$ .

**2.84**  $\phi_{xy}[1] = E\{x[n+1]y^*[n]\}$ ,  $\phi_{xy}[-1] = E\{x[n-1]y^*[n]\}$ ,  $\phi_{yx}[1] = E\{y[n+1]x^*[n]\}$ .

Therefore,  $\phi_{yx}^*[1] = E\{y^*[n+1]x[n]\} = E\{x[n-1]y^*[n]\} = \phi_{xy}[-1]$ .

Hence  $\phi_{xy}[-1] = \phi_{yx}^*[1]$ .

Since  $\gamma_{xy}[1] = \phi_{xy}[1] - m_x(m_y)^*$ . Thus,  $\gamma_{xy}[1] = \phi_{xy}[1] - m_x(m_y)^*$ . Hence,  $\gamma_{xy}[1] = \phi_{xy}[1] - m_x(m_y)^*$ . As a result,  $\gamma_{xy}^*[1] = \phi_{xy}^*[1] - (m_x)^*m_y$ . Hence,  $\gamma_{xy}[-1] = \gamma_{yx}^*[1]$ .

The remaining two properties can be proved in a similar way by letting  $x = y$ .

**2.85**  $E\{[x[n] - x[n-1]]^2\} \geq 0.$

$$E\{x[n]^2\} + E\{x[n-1]^2\} - E\{x[n]x^*[n-1]\} - E\{x^*[n]x[n-1]\} \geq 0$$

$$2\phi_{xx}[0] - 2\phi_{xx}[1] \geq 0$$

$$\phi_{xx}[0] \geq \phi_{xx}[1]$$

Using Cauchy's inequality  $E\{x^2\}E\{y^2\} \leq E^2\{xy\}$ . Hence,  $\phi_{xx}[0]\phi_{yy}[0] \leq |\phi_{xy}[1]|^2$ .

One can show similarly  $\gamma_{xx}[0]\gamma_{yy}[0] \leq |\gamma_{xy}[1]|^2$ .

**2.86** Since there are no periodicities in  $\{x[n]\}$  hence  $x[n], x[n+1]$  become uncorrelated as  $l \rightarrow \infty$ .

Thus  $\lim_{l \rightarrow \infty} \gamma_{xx}[l] = \lim_{l \rightarrow \infty} \phi_{xx}[l] - |m_x|^2 \rightarrow 0$ . Hence  $\lim_{l \rightarrow \infty} \phi_{xx}[l] = |m_x|^2$ .

**2.87**  $\phi_{XX}[1] = \frac{9 + 11l^2 + 14l^4}{1 + 3l^2 + 2l^4}$ . Now,  $|m_{X[n]}|^2 = \lim_{l \rightarrow \infty} \phi_{XX}[l] = \lim_{l \rightarrow \infty} \frac{9 + 11l^2 + 14l^4}{1 + 3l^2 + 2l^4} = 7$ .

Hence,  $m_{X[n]} = m\sqrt{7}$ .  $E\{|X[n]|^2\} = \phi_{XX}[0] = 9$ . Therefore,  $\sigma_X^2 = \phi_{XX}[0] - |m_X|^2 = 9 - 7 = 2$ .

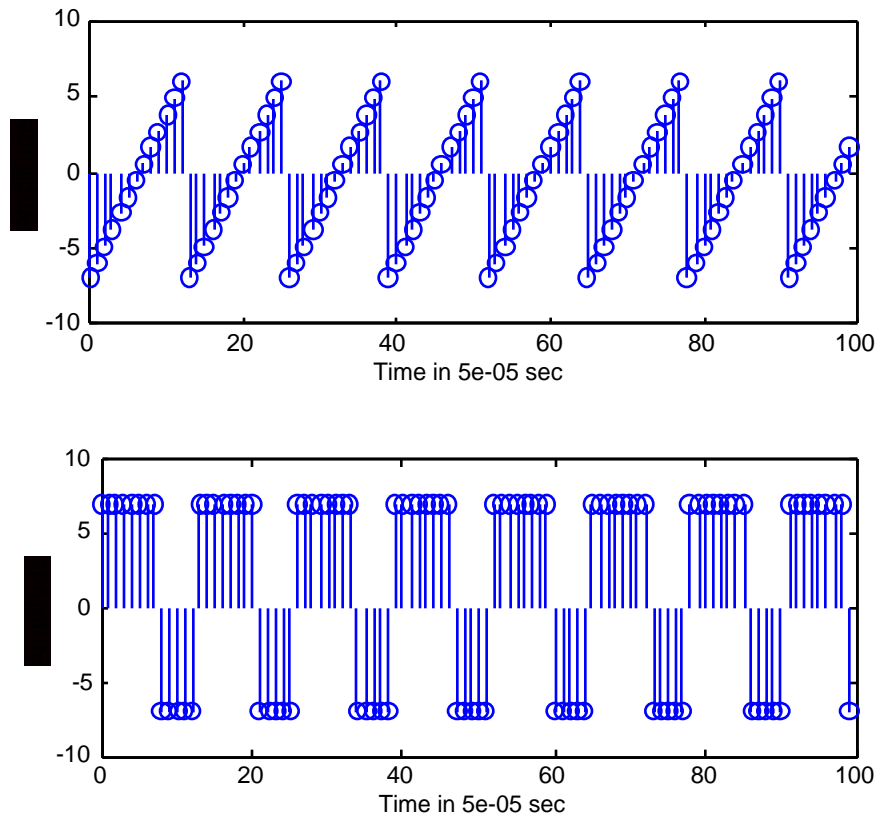
```
M2.1 L = input('Desired length = '); n = 1:L;
FT = input('Sampling frequency = '); T = 1/FT;
imp = [1 zeros(1,L-1)]; step = ones(1,L);
ramp = (n-1).*step;
subplot(3,1,1);
stem(n-1,imp);
xlabel(['Time in ',num2str(T), ' sec']); ylabel('Amplitude');
subplot(3,1,2);
stem(n-1,step);
xlabel(['Time in ',num2str(T), ' sec']); ylabel('Amplitude');
subplot(3,1,3);
stem(n-1,ramp);
xlabel(['Time in ',num2str(T), ' sec']); ylabel('Amplitude');
```

```
M2.2 % Get user inputs
A = input('The peak value = ');
L = input('Length of sequence = ');
N = input('The period of sequence = ');
FT = input('The desired sampling frequency = ');
DC = input('The square wave duty cycle = ');
% Create signals
T = 1/FT;
```

```

t = 0:L-1;
x = A*sawtooth(2*pi*t/N);
y = A*square(2*pi*(t/N),DC);
% Plot
subplot(211)
stem(t,x);
ylabel('Amplitude');
xlabel(['Time in ',num2str(T),'sec']);
subplot(212)
stem(t,y);
ylabel('Amplitude');
xlabel(['Time in ',num2str(T),'sec']);

```

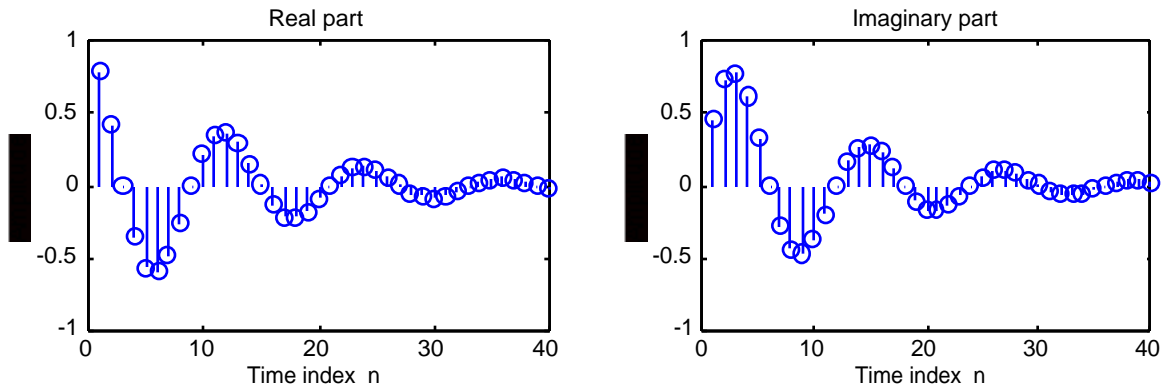


**M2.3 (a)** The input data entered during the execution of Program 2\_1 are

```

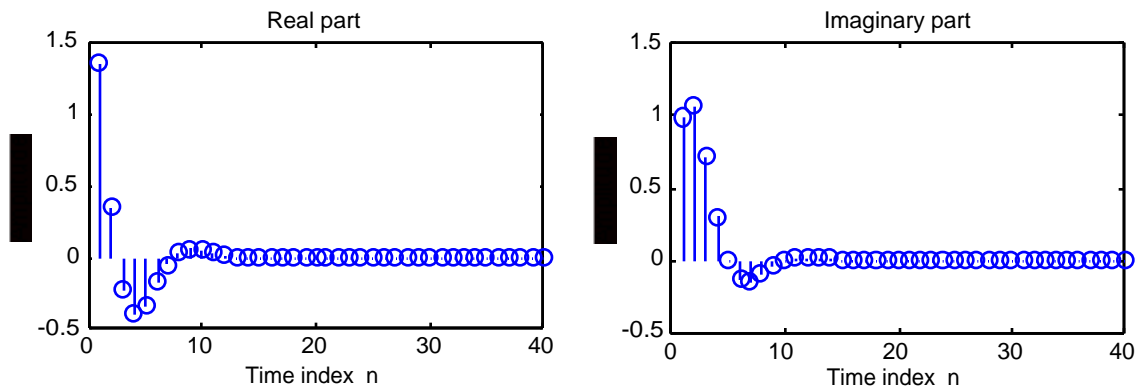
Type in real exponent = -1/12
Type in imaginary exponent = pi/6
Type in the gain constant = 1
Type in length of sequence = 41

```



(b) The input data entered during the execution of Program 2\_1 are

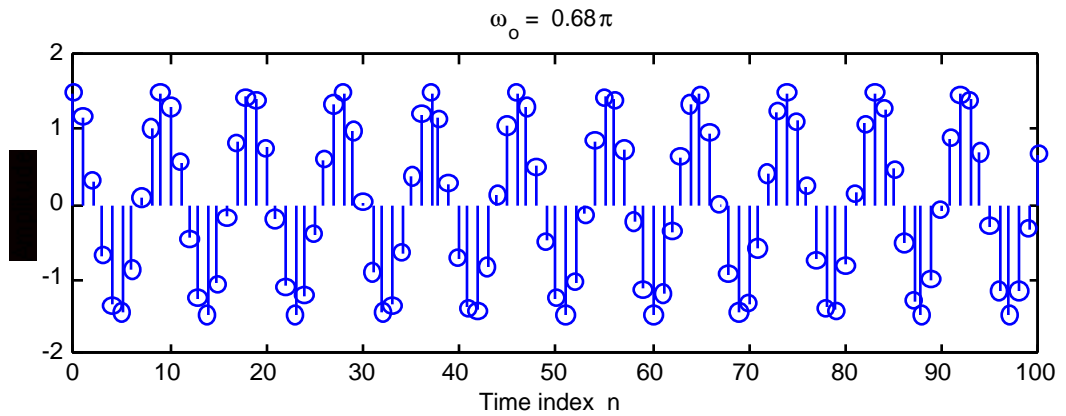
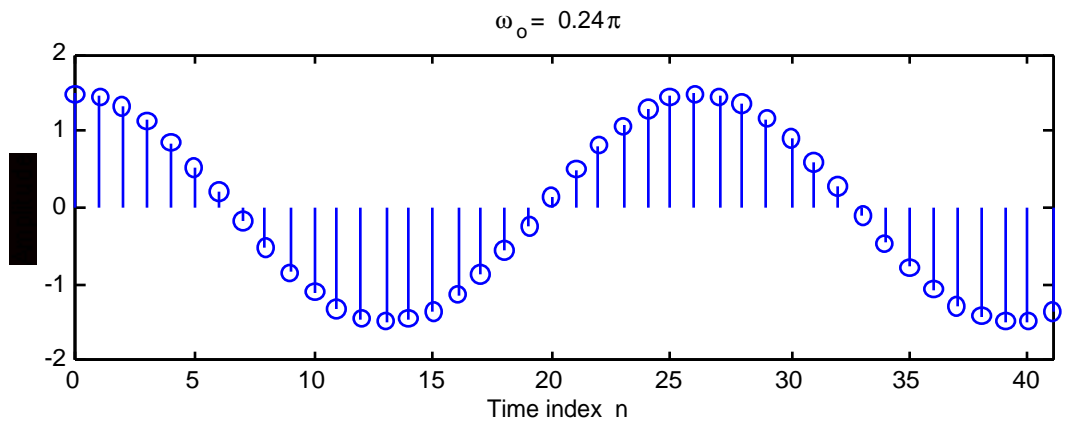
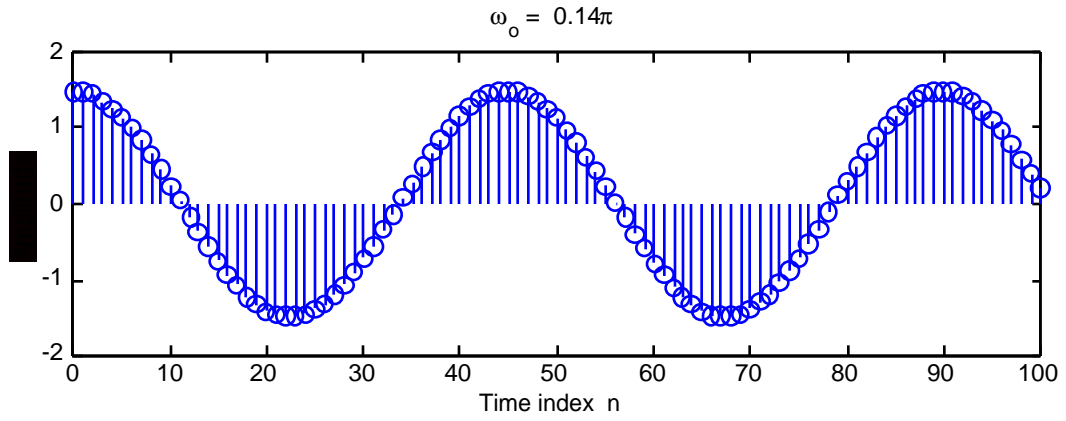
```
Type in real exponent = -0.4
Type in imaginary exponent = pi/5
Type in the gain constant = 2.5
Type in length of sequence = 101
```



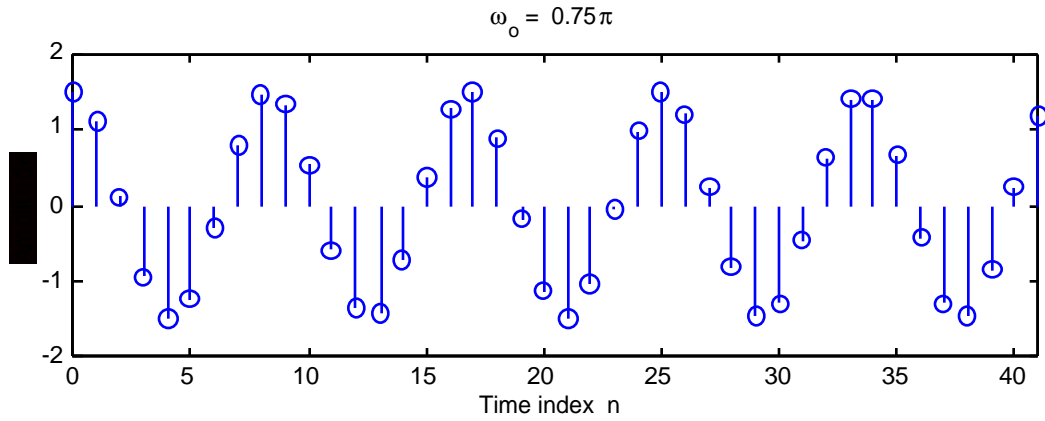
**M2.4 (a)**

```
L = input('Desired length = ');
A = input('Amplitude = ');
omega = input('Angular frequency = ');
phi = input('Phase = ');
n = 0:L-1;
x = A*cos(omega*n + phi);
stem(n,x);
xlabel('Time index');ylabel('Amplitude');
title(['\omega_{o} = ',num2str(omega)]);
```

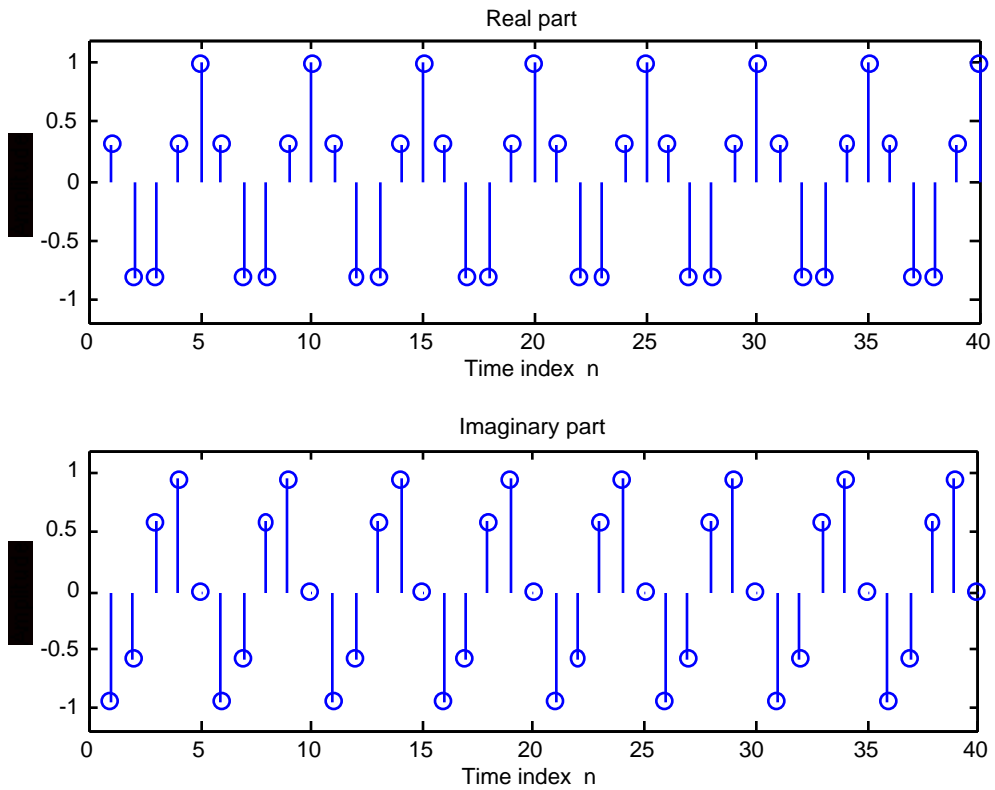
(b)





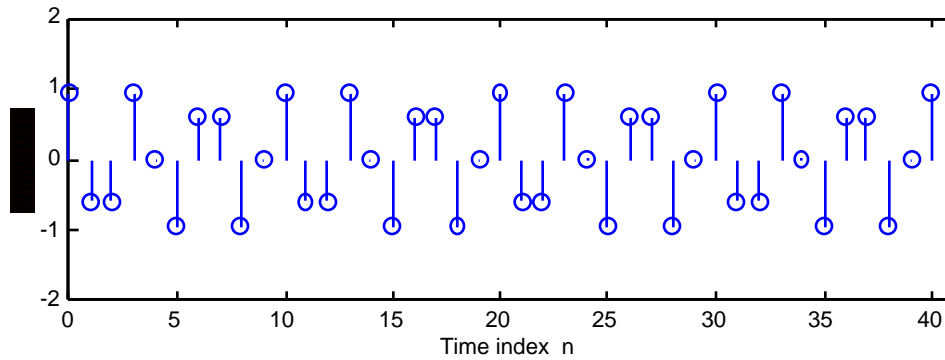


**M2.5 (a)** Using Program 2\_1 we generate the sequence  $\tilde{x}_1[n] = e^{-j0.4\pi n}$  shown below



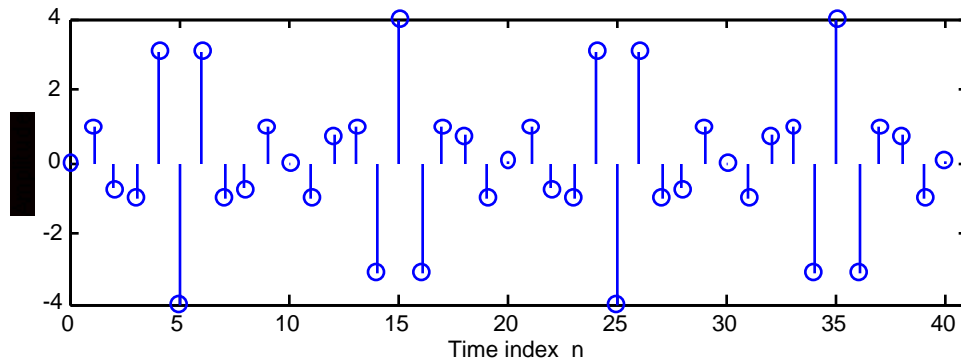
**(b)** Code fragment used to generate  $\tilde{x}_2[n] = \sin(0.6\pi n + 0.6\pi)$  is:

```
x = sin(0.6*pi*n + 0.6*pi);
```



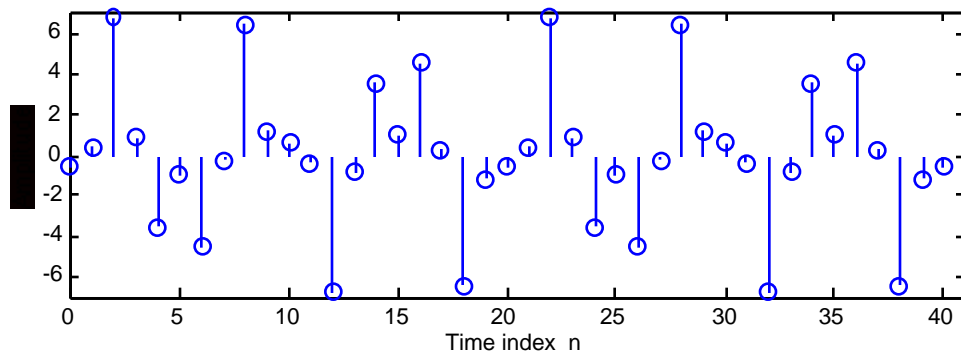
(c) Code fragment used to generate  $\tilde{x}_3[n] = 2 \cos(1.1\pi n - 0.5\pi) + 2 \sin(0.7\pi n)$  is

```
x = 2*cos(1.1*pi*n - 0.5*pi) + 2*sin(0.7*pi*n);
```



(d) Code fragment used to generate  $\tilde{x}_4[n] = 3 \sin(1.3\pi n) - 4 \cos(0.3\pi n + 0.45\pi)$  is:

```
x = 3*sin(1.3*pi*n) - 4*cos(0.3*pi*n+0.45*pi);
```



(e) Code fragment used to generate

$\tilde{x}_5[n] = 5 \sin(1.2\pi n + 0.65\pi) + 4 \sin(0.8\pi n) - \cos(0.8\pi n)$  is:

```
x = 5*sin(1.2*pi*n+0.65*pi)+4*sin(0.8*pi*n)-cos(0.8*pi*n);
```



```

M2.9 alpha = input('Alpha = ');
yo = 1;y1 = 0.5*(yo + (alpha/yo));
while abs(y1 - yo) > 0.00001
    y2 = 0.5*(y1 + (alpha/y1));
    yo = y1; y1 = y2;
end
disp('Square root of alpha is'); disp(y1)
M2.10 alpha = input('Alpha = ');
yo = 0.3; y = zeros(1,61);
L = length(y)-1;
y(1) = alpha - yo*yo + yo;
n = 2;
while abs(y(n-1) - yo) > 0.00001
y2 = alpha - y(n-1)*y(n-1) + y(n-1);
yo = y(n-1); y(n) = y2;
n = n+1;
end
disp('Square root of alpha is'); disp(y(n-1))
m=0:n-2;
err = y(1:n-1) - sqrt(alpha);
stem(m,err);
axis([0 n-2 min(err) max(err)]);
xlabel('Time index n');
ylabel('Error');
title(['\alpha = ',num2str(alpha)])

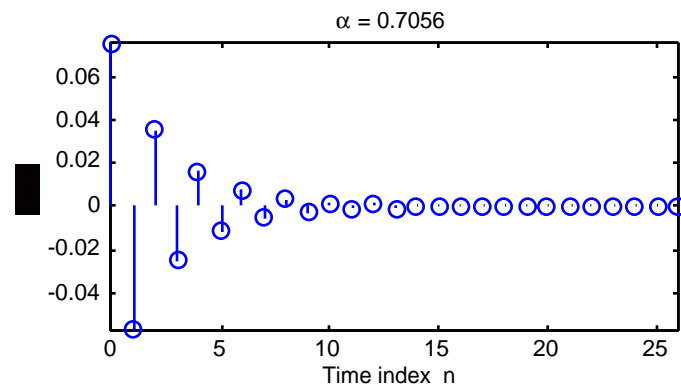
```

The displayed output is

```

Square root of alpha is
0.84000349056114

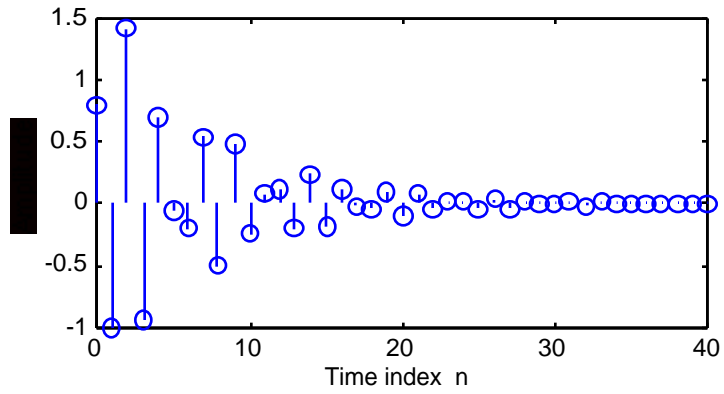
```



```

M2.11 N = input('Desired impulse response length = ');
p = input('Type in the vector p = ');
d = input('Type in the vector d = ');
[h,t] = impz(p,d,N);
n = 0:N-1;
stem(n,h);
xlabel('Time index n');ylabel('Amplitude');

```

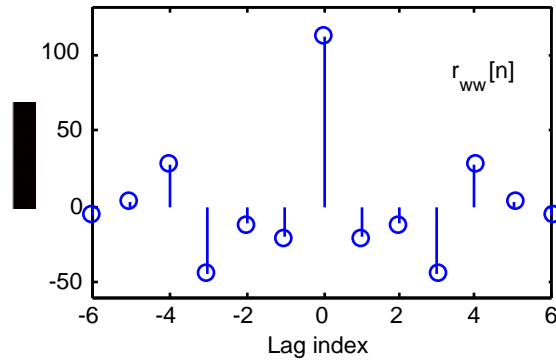
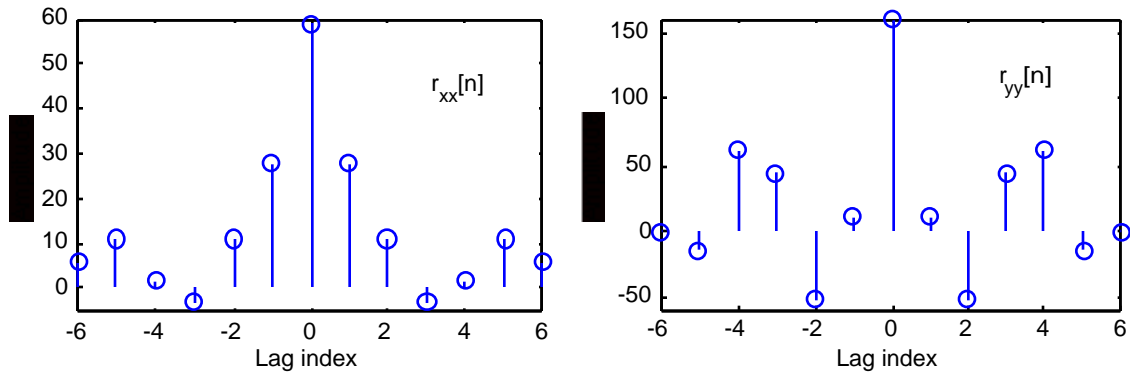


**M2.12**  $x = [3 \ -2 \ 0 \ 1 \ 4 \ 5 \ 2]$   $y = [0 \ 7 \ 1 \ -3 \ 4 \ 9 \ -2]$ ,  $w = [-5 \ 4 \ 3 \ 6 \ -5 \ 0 \ 1]$ .

(a)  $r_{xx}[n] = [6 \ 11 \ 2 \ -3 \ 11 \ 28 \ 59 \ 28 \ 11 \ -3 \ 2 \ 11 \ 6]$ ,

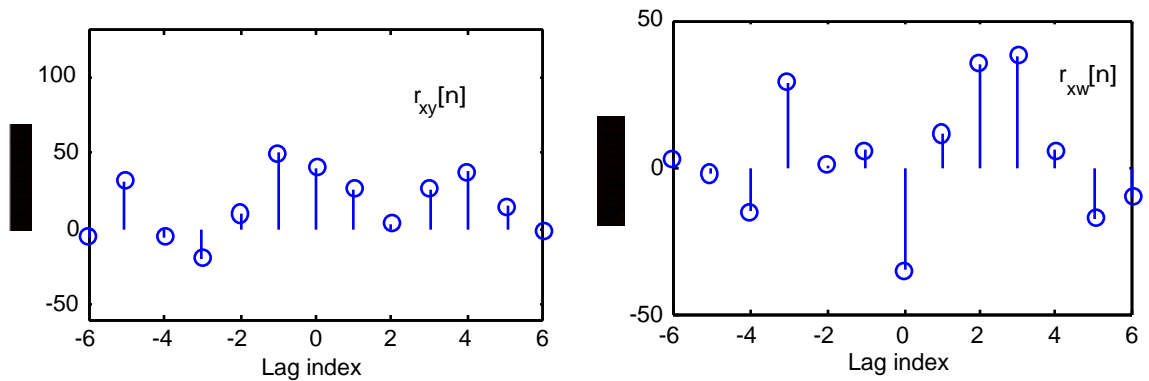
$r_{yy}[n] = [0 \ -14 \ 61 \ 43 \ -52 \ 10 \ 160 \ 10 \ -52 \ 43 \ 61 \ -14 \ 0]$ ,

$r_{ww}[n] = [-5 \ 4 \ 28 \ -44 \ -11 \ -20 \ 112 \ -20 \ -11 \ -44 \ 28 \ 4 \ -5]$ .

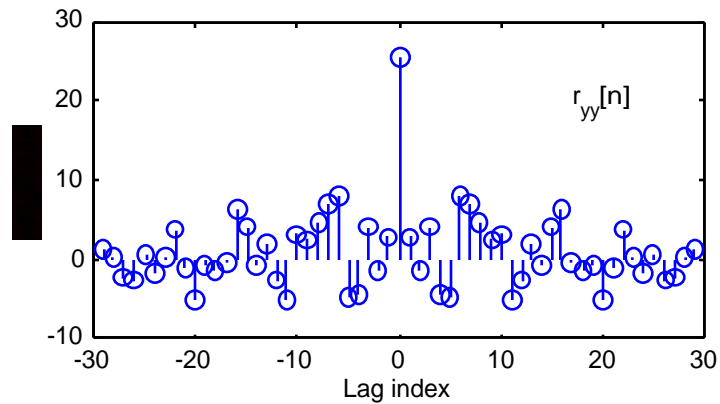


(b)  $r_{xy}[n] = [-6 \ 31 \ -6 \ -19 \ 10 \ 49 \ 40 \ 27 \ 4 \ 27 \ 37 \ 14 \ 0]$ .

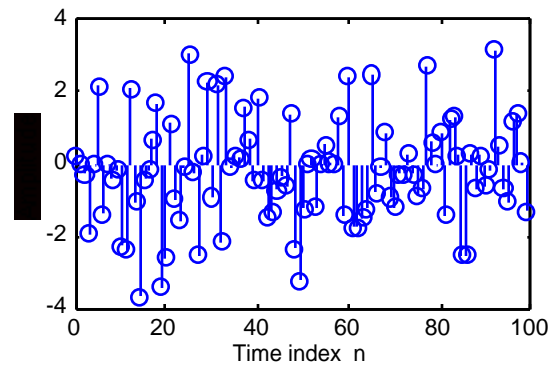
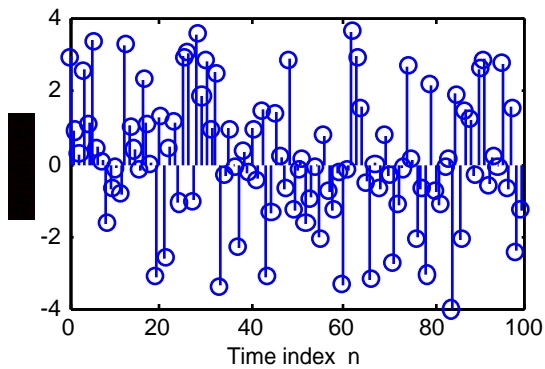
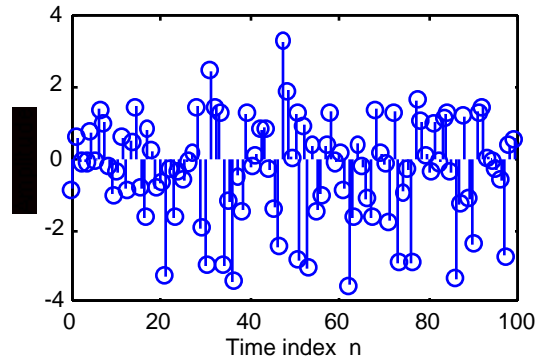
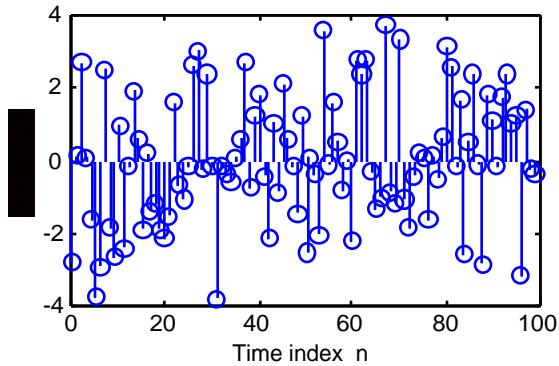
$r_{xw}[n] = [3 \ -2 \ -15 \ 29 \ 1 \ 6 \ -35 \ 12 \ 36 \ 38 \ 6 \ -17 \ -10]$ .



```
M2.13 N = input('Length of sequence = ');
n = 0:N-1;
x = exp(-0.8*n);
y = randn(1,N)+x;
n1 = length(x)-1;
r = conv(y,flipplr(y));
k = (-n1):n1;
stem(k,r);
xlabel('Lag index');ylabel('Amplitude');
gtext('r_{yy}[n]');
```



```
M2.14 (a) n=0:10000;
phi = 2*pi*rand(1,length(n));
A = 4*rand(1,length(n));
x = A.*cos(0.06*pi*n + phi);
stem(n(1:100),x(1:100));%axis([0 50 -4 4]);
xlabel('Time index n');ylabel('Amplitude');
mean = sum(x)/length(x);
var = sum((x - mean).^2)/length(x)
```



```

mean =
0.00491782302432

var =
2.68081196077671

```

From Example 2.44, we note that the mean = 0 and variance =  $8/3 = 2.66667$ .

```

M2.15 n=0:1000;
z = 2*rand(1,length(n));
y = ones(1,length(n));x=z-y;
mean = mean(x)
var = sum((x - mean).^2)/length(x)

```

```

mean =
0.00102235365812

```

```

var =
0.34210530830959

```

Using Eqs. (2.129) and (2.130) we get  $m_X = \frac{1-1}{2} = 0$ , and  $\sigma_X^2 = \frac{(1+1)^2}{22} = \frac{1}{3}$ . It should be noted that the values of the mean and the variance computed using the above MATLAB program get closer to the theoretical values if the length is increased. For example, for a length 100001, the values are

```

mean =
9.122420593670135e-04

```

```
var =  
  0.33486888516819
```



## Chapter 3 (2e)

**3.1**  $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$  where  $x[n]$  is a real sequence. Therefore,

$$X_{\text{re}}(e^{j\omega}) = \text{Re} \left( \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right) = \sum_{n=-\infty}^{\infty} x[n] \text{Re}(e^{-j\omega n}) = \sum_{n=-\infty}^{\infty} x[n] \cos(\omega n), \text{ and}$$

$$X_{\text{im}}(e^{j\omega}) = \text{Im} \left( \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right) = \sum_{n=-\infty}^{\infty} x[n] \text{Im}(e^{-j\omega n}) = - \sum_{n=-\infty}^{\infty} x[n] \sin(\omega n). \text{ Since } \cos(\omega n) \text{ and}$$

$\sin(\omega n)$  are, respectively, even and odd functions of  $\omega$ ,  $X_{\text{re}}(e^{j\omega})$  is an even function of  $\omega$ , and  $X_{\text{im}}(e^{j\omega})$  is an odd function of  $\omega$ .

$|X(e^{j\omega})| = \sqrt{X_{\text{re}}^2(e^{j\omega}) + X_{\text{im}}^2(e^{j\omega})}$ . Now,  $X_{\text{re}}^2(e^{j\omega})$  is the square of an even function and

$X_{\text{im}}^2(e^{j\omega})$  is the square of an odd function, they are both even functions of  $\omega$ . Hence,

$|X(e^{j\omega})|$  is an even function of  $\omega$ .

$\arg\{X(e^{j\omega})\} = \tan^{-1} \left( \frac{X_{\text{im}}(e^{j\omega})}{X_{\text{re}}(e^{j\omega})} \right)$ . The argument is the quotient of an odd function and an even

function, and is therefore an odd function. Hence,  $\arg\{X(e^{j\omega})\}$  is an odd function of  $\omega$ .

$$\mathbf{3.2} \quad X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} = \frac{1}{1 - \alpha e^{-j\omega}} \cdot \frac{1 - \alpha e^{-\omega}}{1 - \alpha e^{-\omega}} = \frac{1 - \alpha e^{-\omega}}{1 - 2\alpha \cos \omega + \alpha^2} = \frac{1 - \alpha \cos \omega - j\alpha \sin \omega}{1 - 2\alpha \cos \omega + \alpha^2}$$

$$\text{Therefore, } X_{\text{re}}(e^{j\omega}) = \frac{1 - \alpha \cos \omega}{1 - 2\alpha \cos \omega + \alpha^2} \text{ and } X_{\text{im}}(e^{j\omega}) = - \frac{\alpha \sin \omega}{1 - 2\alpha \cos \omega + \alpha^2}.$$

$$|X(e^{j\omega})|^2 = X(e^{j\omega}) \cdot X^*(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \cdot \frac{1}{1 - \alpha e^{j\omega}} = \frac{1}{1 - 2\alpha \cos \omega + \alpha^2}.$$

$$\text{Therefore, } |X(e^{j\omega})| = \frac{1}{\sqrt{1 - 2\alpha \cos \omega + \alpha^2}}.$$

$$\tan \theta(\omega) = \frac{X_{\text{im}}(e^{j\omega})}{X_{\text{re}}(e^{j\omega})} = - \frac{\alpha \sin \omega}{1 - \alpha \cos \omega}. \text{ Therefore, } \theta(\omega) = \tan^{-1} \left( - \frac{\alpha \sin \omega}{1 - \alpha \cos \omega} \right).$$

**3.3 (a)**  $y[n] = \mu[n] = y_{\text{ev}}[n] + y_{\text{od}}[n]$ , where  $y_{\text{ev}}[n] = \frac{1}{2}(y[n] + y[-n]) = \frac{1}{2}(\mu[n] + \mu[-n]) = \frac{1}{2} + \frac{1}{2}\delta[n]$ ,

$$\text{and } y_{\text{od}}[n] = \frac{1}{2}(y[n] - y[-n]) = \frac{1}{2}(\mu[n] - \mu[-n]) = \mu[n] - \frac{1}{2} - \frac{1}{2}\delta[n].$$

$$\text{Now, } Y_{\text{ev}}(e^{j\omega}) = \frac{1}{2} \left[ 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k) \right] + \frac{1}{2} = \pi \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k) + \frac{1}{2}.$$

Since  $y_{\text{od}}[n] = \mu[n] - \frac{1}{2} + \frac{1}{2}\delta[n]$ ,  $y_{\text{od}}[n] = \mu[n-1] - \frac{1}{2} + \frac{1}{2}\delta[n-1]$ . As a result,

$y_{od}[n] - y_{od}[n-1] = \mu[n] - \mu[n-1] + \frac{1}{2}\delta[n-1] - \frac{1}{2}\delta[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$ . Taking the DTFT of both sides we then get  $Y_{od}(e^{j\omega}) - e^{-j\omega} Y_{od}(e^{j\omega}) = \frac{1}{2}(1 + e^{-j\omega})$ . or

$$Y_{od}(e^{j\omega}) = \frac{1}{2} \frac{1 + e^{-j\omega}}{1 - e^{-j\omega}} = \frac{1}{1 - e^{-j\omega}} - \frac{1}{2}. \text{ Hence,}$$

$$Y(e^{j\omega}) = Y_{ev}(e^{j\omega}) + Y_{od}(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k).$$

**(b)** Let  $x[n]$  be the sequence with the DTFT  $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$ . Its inverse

$$\text{DTFT is then given by } x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi\delta(\omega - \omega_0) e^{j\omega n} d\omega = e^{j\omega_0 n}.$$

**3.4** Let  $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$ . Its inverse DTFT is then given by

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi\delta(\omega) e^{j\omega n} d\omega = \frac{2\pi}{2\pi} = 1.$$

**3.5 (a)** Let  $y[n] = g[n - n_0]$ , Then  $Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} g[n - n_0] e^{-j\omega n}$

$$= e^{-j\omega n_0} \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n} = e^{-j\omega n_0} G(e^{j\omega}).$$

**(b)** Let  $h[n] = e^{j\omega_0 n} g[n]$ , then  $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} g[n] e^{-j\omega n}$

$$= \sum_{n=-\infty}^{\infty} g[n] e^{-j(\omega - \omega_0)n} = G(e^{j(\omega - \omega_0)}).$$

**(c)**  $G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n}$ . Hence  $\frac{d(G(e^{j\omega}))}{d\omega} = \sum_{n=-\infty}^{\infty} -jng[n] e^{-j\omega n}$ .

Therefore,  $j \frac{d(G(e^{j\omega}))}{d\omega} = \sum_{n=-\infty}^{\infty} ng[n] e^{-j\omega n}$ . Thus the DTFT of  $ng[n]$  is  $j \frac{d(G(e^{j\omega}))}{d\omega}$ .

**(d)**  $y[n] = g[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} g[k] h[n - k]$ . Hence  $Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g[k] h[n - k] e^{-j\omega n}$

$$= \sum_{k=-\infty}^{\infty} g[k] H(e^{j\omega}) e^{-j\omega k} = H(e^{j\omega}) \sum_{k=-\infty}^{\infty} g[k] e^{-j\omega k} = H(e^{j\omega}) G(e^{j\omega}).$$

**(e)**  $y[n] = g[n]h[n]$ . Hence  $Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]h[n] e^{-j\omega n}$

Since  $g[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) e^{j\theta n} d\theta$  we can rewrite the above DTFT as

$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} h[n] e^{-j\omega n} G(e^{j\theta}) e^{j\theta n} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) \sum_{n=-\infty}^{\infty} h[n] e^{-j(\omega-\theta)n} d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) H(e^{j(\omega-\theta)}) d\theta. \end{aligned}$$

$$\begin{aligned} \text{(f)} \quad y[n] &= \sum_{n=-\infty}^{\infty} g[n] h^*[n] = \sum_{n=-\infty}^{\infty} g[n] \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} H^*(e^{j\omega}) e^{-j\omega n} d\omega \right) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H^*(e^{j\omega}) \left( \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n} \right) d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} H^*(e^{j\omega}) G(e^{j\omega}) d\omega. \end{aligned}$$

**3.6** DTFT  $\{x[n]\} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ .

**(a)** DTFT  $\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n] e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m] e^{j\omega m} = X(e^{-j\omega})$ .

**(b)** DTFT  $\{x^*[-n]\} = \sum_{n=-\infty}^{\infty} x^*[-n] e^{-j\omega n} = \left( \sum_{n=-\infty}^{\infty} x[-n] e^{j\omega n} \right)$  using the result of Part (a).

Therefore DTFT  $\{x^*[-n]\} = X^*(e^{j\omega})$ .

**(c)** DTFT  $\{\text{Re}(x[n])\} = \text{DTFT} \left\{ \frac{x[n] + x^*[n]}{2} \right\} = \frac{1}{2} \{X(e^{j\omega}) + X^*(e^{-j\omega})\}$  using the result of Part (b).

**(d)** DTFT  $\{j \text{Im}(x[n])\} = \text{DTFT} \left\{ j \frac{x[n] - x^*[n]}{2j} \right\} = \frac{1}{2} \{X(e^{j\omega}) - X^*(e^{-j\omega})\}$ .

**(e)** DTFT  $\{x_{cs}[n]\} = \text{DTFT} \left\{ \frac{x[n] + x^*[-n]}{2} \right\} = \frac{1}{2} \{X(e^{j\omega}) + X^*(e^{j\omega})\} = \text{Re}\{X(e^{j\omega})\} = X_{\text{re}}(e^{j\omega})$ .

**(f)** DTFT  $\{x_{ca}[n]\} = \text{DTFT} \left\{ \frac{x[n] - x^*[-n]}{2} \right\} = \frac{1}{2} \{X(e^{j\omega}) - X^*(e^{j\omega})\} = jX_{\text{im}}(e^{j\omega})$ .

**3.7**  $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$  where  $x[n]$  is a real sequence. For a real sequence,

$X(e^{j\omega}) = X^*(e^{-j\omega})$ , and IDFT  $\{X^*(e^{-j\omega})\} = x^*[-n]$ .

$$\begin{aligned} \text{(a)} \quad X_{\text{re}}(e^{j\omega}) &= \frac{1}{2} \{X(e^{j\omega}) + X^*(e^{-j\omega})\}. \text{ Therefore, } \text{IDFT}\{X_{\text{re}}(e^{j\omega})\} \\ &= \frac{1}{2} \text{IDFT}\{X(e^{j\omega}) + X^*(e^{-j\omega})\} = \frac{1}{2} \{x[n] + x^*[-n]\} = \frac{1}{2} \{x[n] + x[-n]\} = x_{\text{ev}}[n]. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad jX_{\text{im}}(e^{j\omega}) &= \frac{1}{2} \{X(e^{j\omega}) - X^*(e^{-j\omega})\}. \text{ Therefore, } \text{IDFT}\{jX_{\text{im}}(e^{j\omega})\} \\ &= \frac{1}{2} \text{IDFT}\{X(e^{j\omega}) - X^*(e^{-j\omega})\} = \frac{1}{2} \{x[n] - x^*[-n]\} = \frac{1}{2} \{x[n] - x[-n]\} = x_{\text{od}}[n]. \end{aligned}$$

$$\begin{aligned} \text{3.8} \quad \text{(a)} \quad X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \text{ Therefore, } X^*(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n} = X(e^{-j\omega}), \text{ and hence,} \\ X^*(e^{-j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j\omega}). \end{aligned}$$

$$\text{(b)} \quad \text{From Part (a), } X(e^{j\omega}) = X^*(e^{-j\omega}). \text{ Therefore, } X_{\text{re}}(e^{j\omega}) = X_{\text{re}}(e^{-j\omega}).$$

$$\text{(c)} \quad \text{From Part (a), } X(e^{j\omega}) = X^*(e^{-j\omega}). \text{ Therefore, } X_{\text{im}}(e^{j\omega}) = -X_{\text{im}}(e^{-j\omega}).$$

$$\text{(d)} \quad |X(e^{j\omega})| = \sqrt{X_{\text{re}}^2(e^{j\omega}) + X_{\text{im}}^2(e^{j\omega})} = \sqrt{X_{\text{re}}^2(e^{-j\omega}) + X_{\text{im}}^2(e^{-j\omega})} = |X(e^{-j\omega})|.$$

$$\text{(e)} \quad \arg X(e^{j\omega}) = \tan^{-1} \frac{X_{\text{im}}(e^{j\omega})}{X_{\text{re}}(e^{j\omega})} = -\tan^{-1} \frac{X_{\text{im}}(e^{-j\omega})}{X_{\text{re}}(e^{-j\omega})} = -\arg X(e^{-j\omega})$$

$$\text{3.9} \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega. \text{ Hence, } x^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(e^{j\omega}) e^{-j\omega n} d\omega.$$

**(a)** Since  $x[n]$  is real and even, hence  $X(e^{j\omega}) = X^*(e^{j\omega})$ . Thus,

$$x[-n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega n} d\omega,$$

$$\text{Therefore, } x[n] = \frac{1}{2} (x[n] + x[-n]) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cos(\omega n) d\omega.$$

Now  $x[n]$  being even,  $X(e^{j\omega}) = X(e^{-j\omega})$ . As a result, the term inside the above integral is even,

$$\text{and hence } x[n] = \frac{1}{\pi} \int_0^{\pi} X(e^{j\omega}) \cos(\omega n) d\omega$$

**(b)** Since  $x[n]$  is odd hence  $x[n] = -x[-n]$ .

$$\text{Thus } x[n] = \frac{1}{2} (x[n] - x[-n]) = \frac{j}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \sin(\omega n) d\omega. \text{ Again, since } x[n] = -x[-n],$$

$X(e^{j\omega}) = -X(e^{-j\omega})$ . The term inside the integral is even, hence  $x[n] = \frac{j}{\pi} \int_0^{\pi} X(e^{j\omega}) \sin(\omega n) d\omega$

$$\begin{aligned} \mathbf{3.10} \quad x[n] &= \alpha^n \cos(\omega_0 n + \phi) \mu[n] = A \alpha^n \left( \frac{e^{j\omega_0 n} e^{j\phi} + e^{-j\omega_0 n} e^{-j\phi}}{2} \right) \mu[n] \\ &= \frac{A}{2} e^{j\phi} (\alpha e^{j\omega_0})^n \mu[n] + \frac{A}{2} e^{-j\phi} (\alpha e^{-j\omega_0})^n \mu[n]. \text{ Therefore,} \\ X(e^{j\omega}) &= \frac{A}{2} e^{j\phi} \frac{1}{1 - \alpha e^{-j\omega} e^{j\omega_0}} + \frac{A}{2} e^{-j\phi} \frac{1}{1 - \alpha e^{-j\omega} e^{-j\omega_0}}. \end{aligned}$$

**3.11** Let  $x[n] = \alpha^n \mu[n]$ ,  $|\alpha| < 1$ . From Table 3.1,  $\text{DTFT}\{x[n]\} = X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$ .

$$\begin{aligned} \mathbf{(a)} \quad X_1(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \alpha^n \mu[n+1] e^{-j\omega n} = \sum_{n=-1}^{\infty} \alpha^n e^{-j\omega n} = \alpha^{-1} e^{j\omega} + \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\ &= \alpha^{-1} e^{j\omega} + \frac{1}{1 - \alpha e^{-j\omega}} = \frac{1}{\alpha} \left( \frac{e^{j\omega} - \alpha}{1 - \alpha e^{-j\omega}} \right). \end{aligned}$$

**(b)**  $x_2[n] = n \alpha^n \mu[n]$ . Note that  $x_2[n] = n x[n]$ . Hence, using the differentiation-in-frequency property in Table 3.2, we get  $X_2(e^{j\omega}) = j \frac{dX(e^{j\omega})}{d\omega} = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2}$ .

$$\begin{aligned} \mathbf{(c)} \quad x_3[n] &= \begin{cases} \alpha^{|n|}, & |n| \leq M, \\ 0, & \text{otherwise.} \end{cases} \text{ Then, } X_3(e^{j\omega}) = \sum_{n=0}^M \alpha^n e^{-j\omega n} + \sum_{n=-M}^{-1} \alpha^{-n} e^{-j\omega n} \\ &= \frac{1 - \alpha^{M+1} e^{-j\omega(M+1)}}{1 - \alpha e^{-j\omega}} + \alpha^M e^{j\omega M} \frac{1 - \alpha^{-M} e^{-j\omega M}}{1 - \alpha^{-1} e^{-j\omega}}. \end{aligned}$$

$$\begin{aligned} \mathbf{(d)} \quad X_4(e^{j\omega}) &= \sum_{n=3}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} - 1 - \alpha e^{-j\omega} - \alpha^2 e^{-j2\omega} \\ &= \frac{1}{1 - \alpha e^{-j\omega}} - 1 - \alpha e^{-j\omega} - \alpha^2 e^{-j2\omega}. \end{aligned}$$

$$\begin{aligned} \mathbf{(e)} \quad X_5(e^{j\omega}) &= \sum_{n=-2}^{\infty} n \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} n \alpha^n e^{-j\omega n} - 2\alpha^{-2} e^{j2\omega} - \alpha^{-1} e^{j\omega} \\ &= \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2} - 2\alpha^{-2} e^{j2\omega} - \alpha^{-1} e^{j\omega}. \end{aligned}$$

$$\mathbf{(f)} \quad X_6(e^{j\omega}) = \sum_{n=-\infty}^{-1} \alpha^n e^{-j\omega n} = \sum_{m=1}^{\infty} \alpha^{-m} e^{j\omega m} = \sum_{m=0}^{\infty} \alpha^{-m} e^{j\omega m} - 1 = \frac{1}{1 - \alpha^{-1} e^{j\omega}} - 1 = \frac{e^{j\omega}}{\alpha - e^{j\omega}}.$$

**3.12 (a)**  $y_1[n] = \begin{cases} 1, & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$  Then  $Y_1(e^{j\omega}) = \sum_{n=-N}^N e^{-j\omega n} = e^{j\omega N} \frac{(1 - e^{-j\omega(2N+1)})}{(1 - e^{-j\omega})} = \frac{\sin(\omega[N + \frac{1}{2}])}{\sin(\omega/2)}$

**(b)**  $y_2[n] = \begin{cases} 1 - \frac{|n|}{N}, & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$  Now  $y_2[n] = y_0[n] \circledast y_0[n]$  where

$y_0[n] = \begin{cases} 1, & -N/2 \leq n \leq N/2, \\ 0, & \text{otherwise.} \end{cases}$  Thus  $Y_2(e^{j\omega}) = Y_0^2(e^{j\omega}) = \frac{\sin^2\left(\omega\left[\frac{N+1}{2}\right]\right)}{\sin^2(\omega/2)}$ .

**(c)**  $y_3[n] = \begin{cases} \cos(\pi n/2N), & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$  Then,

$$\begin{aligned} Y_3(e^{j\omega}) &= \frac{1}{2} \sum_{n=-N}^N e^{-j(\pi n/2N)} e^{-j\omega n} + \frac{1}{2} \sum_{n=-N}^N e^{j(\pi n/2N)} e^{-j\omega n} \\ &= \frac{1}{2} \sum_{n=-N}^N e^{-j\left(\omega - \frac{\pi}{2N}\right)n} + \frac{1}{2} \sum_{n=-N}^N e^{-j\left(\omega + \frac{\pi}{2N}\right)n} \\ &= \frac{1}{2} \frac{\sin\left(\left(\omega - \frac{\pi}{2N}\right)(N + \frac{1}{2})\right)}{\sin\left(\left(\omega - \frac{\pi}{2N}\right)/2\right)} + \frac{1}{2} \frac{\sin\left(\left(\omega + \frac{\pi}{2N}\right)(N + \frac{1}{2})\right)}{\sin\left(\left(\omega + \frac{\pi}{2N}\right)/2\right)}. \end{aligned}$$

**3.13** Denote  $x_m[n] = \frac{(n+m-1)!}{n!(m-1)!} \alpha^n \mu[n]$ ,  $|\alpha| < 1$ . We shall prove by induction that

DTFT $\{x_m[n]\} = X_m(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})^m}$ . From Table 3.1, it follows that it holds for  $m = 1$ .

Let  $m = 2$ . Then  $x_2[n] = \frac{(n+1)!}{n!} \alpha^n \mu[n] = (n+1)x_1[n] = n x_1[n] + x_1[n]$ . Therefore,

$X_2(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2} + \frac{1}{1 - \alpha e^{-j\omega}} = \frac{1}{(1 - \alpha e^{-j\omega})^2}$  using the differentiation-in-frequency property of Table 3.2.

Now assume, that it holds for  $m$ . Consider next  $x_{m+1}[n] = \frac{(n+m)!}{n!(m)!} \alpha^n \mu[n]$

$= \left(\frac{n+m}{m}\right) \frac{(n+m-1)!}{n!(m-1)!} \alpha^n \mu[n] = \left(\frac{n+m}{m}\right) x_m[n] = \frac{1}{m} \cdot n \cdot x_m[n] + x_m[n]$ . Hence,

$$\begin{aligned} X_{m+1}(e^{j\omega}) &= \frac{1}{m} j \frac{d}{d\omega} \left\{ \frac{1}{(1 - \alpha e^{-j\omega})^m} \right\} + \frac{1}{(1 - \alpha e^{-j\omega})^m} = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^{m+1}} + \frac{1}{(1 - \alpha e^{-j\omega})^m} \\ &= \frac{1}{(1 - \alpha e^{-j\omega})^{m+1}}. \end{aligned}$$

**3.14 (a)**  $X_a(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k)$ . Hence,  $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} d\omega = 1$ .

**(b)**  $X_b(e^{j\omega}) = \frac{1 - e^{j\omega(N+1)}}{1 - e^{-j\omega}} = \sum_{n=0}^N e^{-j\omega n}$ . Hence,  $x[n] = \begin{cases} 1, & 0 \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$

**(c)**  $X_c(e^{j\omega}) = 1 + 2 \sum_{l=0}^N \cos(\omega l) = \sum_{l=-N}^N e^{-j\omega l}$ . Hence  $x[n] = \begin{cases} 1, & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$

**(d)**  $X_d(e^{j\omega}) = \frac{-j\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2}$ ,  $|\alpha| < 1$ . Now we can rewrite  $X_d(e^{j\omega})$  as

$$X_d(e^{j\omega}) = \frac{d}{d\omega} \frac{1}{(1 - \alpha e^{-j\omega})} = \frac{d}{d\omega} (X_o(e^{j\omega})) \text{ where } X_o(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}.$$

Now  $x_o[n] = \alpha^n \mu[n]$ . Hence, from Table 3.2,  $x_d[n] = -jn\alpha^n \mu[n]$ .

**3.15 (a)**  $H_1(e^{j\omega}) = 1 + 2 \left( \frac{e^{j\omega} + e^{-j\omega}}{2} \right) + 3 \left( \frac{e^{j2\omega} + e^{-j2\omega}}{2} \right) = 1 + e^{j\omega} + e^{-j\omega} + \frac{3}{2} e^{j2\omega} + \frac{3}{2} e^{-j2\omega}$ .

Hence, the inverse of  $H_1(e^{j\omega})$  is a length-5 sequence given by

$$h_1[n] = [1.5 \quad 1 \quad 1 \quad 1 \quad 1.5], \quad -2 \leq n \leq 2.$$

**(b)**  $H_2(e^{j\omega}) = \left[ 3 + 2 \left( \frac{e^{j\omega} + e^{-j\omega}}{2} \right) + 4 \left( \frac{e^{j2\omega} + e^{-j2\omega}}{2} \right) \right] \cdot \left( \frac{e^{j\omega/2} + e^{-j\omega/2}}{2} \right) \cdot e^{-j\omega/2}$   
 $= \frac{1}{2} (2e^{j2\omega} + 3e^{j\omega} + 4 + 4e^{-j\omega} + 3e^{-j2\omega} + 2e^{-j3\omega})$ . Hence, the inverse of  $H_2(e^{j\omega})$  is a length-6 sequence given by  $h_2[n] = [1 \quad 1.5 \quad 2 \quad 2 \quad 1.5 \quad 1]$ ,  $-2 \leq n \leq 3$ .

**(c)**  $H_3(e^{j\omega}) = j \left[ 3 + 4 \left( \frac{e^{j\omega} + e^{-j\omega}}{2} \right) + 2 \left( \frac{e^{j2\omega} + e^{-j2\omega}}{2} \right) \right] \cdot \left( \frac{e^{j\omega/2} - e^{-j\omega/2}}{2j} \right)$   
 $= \frac{1}{2} (e^{j3\omega} + 2e^{j2\omega} + 2e^{j\omega} + 0 - 2e^{-j\omega} - 2e^{-j2\omega} - e^{-j3\omega})$ . Hence, the inverse of  $H_3(e^{j\omega})$  is a length-7 sequence given by  $h_3[n] = [0.5 \quad 1 \quad 1 \quad 0 \quad -1 \quad -1 \quad -0.5]$ ,  $-3 \leq n \leq 3$ .

**(d)**  $H_4(e^{j\omega}) = j \left[ 4 + 2 \left( \frac{e^{j\omega} + e^{-j\omega}}{2} \right) + 3 \left( \frac{e^{j2\omega} + e^{-j2\omega}}{2} \right) \right] \cdot \left( \frac{e^{j\omega/2} - e^{-j\omega/2}}{2j} \right) \cdot e^{j\omega/2}$   
 $= \frac{1}{2} \left( \frac{3}{2} e^{j3\omega} - \frac{1}{2} e^{j2\omega} + 3e^{j\omega} - 3 + \frac{1}{2} e^{-j\omega} - \frac{3}{2} e^{-j2\omega} \right)$ . Hence, the inverse of  $H_4(e^{j\omega})$  is a length-6 sequence given by  $h_4[n] = [0.75 \quad -0.25 \quad 1.5 \quad -1.5 \quad 0.25 \quad -0.75]$ ,  $-3 \leq n \leq 2$ .

**3.16 (a)**  $H_2(e^{j\omega}) = 1 + 2 \cos \omega + \frac{3}{2} (1 + \cos 2\omega) = \frac{3}{4} e^{j2\omega} + e^{j\omega} + \frac{5}{2} + e^{-j\omega} + \frac{3}{4} e^{-j2\omega}$ .

Hence, the inverse of  $H_1(e^{j\omega})$  is a length-5 sequence given by  $h_1[n] = [0.75 \ 1 \ 2.5 \ 1 \ 0.75]$ ,  $-2 \leq n \leq 2$ .

(b)  $H_2(e^{j\omega}) = \left[1 + 3\cos\omega + \frac{4}{2}(1 + \cos 2\omega)\right] \cos(\omega/2)e^{j\omega/2}$   
 $= \frac{1}{2}e^{j3\omega} + \frac{5}{4}e^{j2\omega} + \frac{11}{4}e^{j\omega} + \frac{11}{4} + \frac{5}{4}e^{-j\omega} + \frac{1}{2}e^{-j2\omega}$ . Hence, the inverse of  $H_2(e^{j\omega})$  is a length-6 sequence given by  $h_2[n] = [0.5 \ 1.25 \ 2.75 \ 2.75 \ 1.25 \ 0.5]$ ,  $-3 \leq n \leq 2$ .

(c)  $H_3(e^{j\omega}) = j[3 + 4\cos\omega + (1 + \cos 2\omega)]\sin(\omega)$   
 $= \frac{1}{4}e^{j3\omega} + e^{j2\omega} + \frac{7}{4}e^{j\omega} + 0 - \frac{7}{4}e^{-j\omega} - e^{-j2\omega} - \frac{1}{4}e^{-j3\omega}$ . Hence, the inverse of  $H_3(e^{j\omega})$  is a length-7 sequence given by  $h_3[n] = [0.25 \ 1 \ 1.75 \ 0 \ -1.75 \ -1 \ -0.25]$ ,  $-3 \leq n \leq 3$ .

(d)  $H_4(e^{j\omega}) = j\left[4 + 2\cos\omega + \frac{3}{2}(1 + \cos 2\omega)\right] \sin(\omega/2)e^{-j\omega/2}$   
 $= \frac{1}{2}\left(\frac{3}{4}e^{j2\omega} + \frac{1}{4}e^{j\omega} + \frac{9}{2} - \frac{9}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} - \frac{3}{4}e^{-j3\omega}\right)$ . Hence, the inverse of  $H_4(e^{j\omega})$  is a length-6 sequence given by  $h_4[n] = \left[\frac{3}{8} \ \frac{1}{8} \ \frac{9}{4} \ -\frac{9}{4} \ -\frac{1}{8} \ -\frac{3}{8}\right]$ ,  $-2 \leq n \leq 3$ .

**3.17**  $Y(e^{j\omega}) = X(e^{j3\omega}) = X((e^{j\omega})^3)$ . Now,  $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ . Hence,

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = X((e^{j\omega})^3) = \sum_{n=-\infty}^{\infty} x[n](e^{-j\omega n})^3 = \sum_{m=-\infty}^{\infty} x[m/3]e^{-j\omega m}.$$

Therefore,  $y[n] = \begin{cases} x[n], & n = 0, \pm 3, \pm 6, \dots \\ 0, & \text{otherwise.} \end{cases}$

**3.18**  $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ .

$$X(e^{j\omega/2}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega/2)n}, \text{ and } X(-e^{j\omega/2}) = \sum_{n=-\infty}^{\infty} x[n](-1)^n e^{-j(\omega/2)n}. \text{ Thus,}$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \frac{1}{2}\{X(e^{j\omega/2}) + X(-e^{j\omega/2})\} = \frac{1}{2}\sum_{n=-\infty}^{\infty} (x[n] + x[n](-1)^n)e^{-j(\omega/2)n}. \text{ Thus,}$$

$$y[n] = \frac{1}{2}(x[n] + x[n](-1)^n) = \begin{cases} x[n], & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}.$$

**3.19** From Table 3.3, we observe that even sequences have real-valued DTFTs and odd sequences have imaginary-valued DTFTs.

(a) Since  $| -n | = | n |$ ,  $x_1[n]$  is an even sequence with a real-valued DTFT.

(b) Since  $(-n)^3 = -n^3$ ,  $x_2[n]$  is an odd sequence with an imaginary-valued DTFT.



(c) Since  $\sin(-\omega_c n) = -\sin(\omega_c n)$  and  $\omega_c(-n) = -\omega_c n$ ,  $x_3[n]$  is an even sequence with a real-valued DTFT.

(d) Since  $x_4[n]$  is an odd sequence it has an imaginary-valued DTFT.

(e) Since  $x_5[n]$  is an odd sequence it has an imaginary-valued DTFT.

**3.20 (a)** Since  $Y_1(e^{j\omega})$  is a real-valued function of  $\omega$ , its inverse is an even sequence.

(b) Since  $Y_2(e^{j\omega})$  is an imaginary-valued function of  $\omega$ , its inverse is an odd sequence.

(c) Since  $Y_3(e^{j\omega})$  is an imaginary-valued function of  $\omega$ , its inverse is an odd sequence.

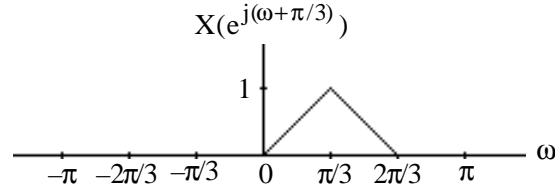
**3.21 (a)**  $H_{LLP}(e^{j\omega})$  is a real-valued function of  $\omega$ . Hence, its inverse is an even sequence.

(b)  $H_{BLDIF}(e^{j\omega})$  is a real-valued function of  $\omega$ . Hence, its inverse is an even sequence.

**3.22** Let  $u[n] = x[-n]$ , and let  $X(e^{j\omega})$  and  $U(e^{j\omega})$  denote the DTFTs of  $x[n]$  and  $u[n]$ , respectively.

From the convolution property of the DTFT given in Table 3.2, the DTFT of  $y[n] = x[n] \circledast u[n]$  is given by  $Y(e^{j\omega}) = X(e^{j\omega}) U(e^{j\omega})$ . From Table 3.3,  $U(e^{j\omega}) = X(e^{-j\omega})$ . But from Table 3.4,  $X(e^{-j\omega}) = X^*(e^{j\omega})$ . Hence,  $Y(e^{j\omega}) = X(e^{j\omega}) X^*(e^{j\omega}) = |X(e^{j\omega})|^2$  which is real-valued function of  $\omega$ .

**3.23** From the frequency-shifting property of the DTFT given in Table 3.2, the DTFT of  $x[n]e^{-j\pi n/3}$  is given by  $X(e^{j(\omega+\pi/3)})$ . A sketch of this DTFT is shown below.



**3.24** The DTFT of  $x[n] = -\alpha^n \mu[-n-1]$  is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{-1} -\alpha^n e^{-j\omega n} = -\sum_{n=1}^{\infty} \alpha^{-n} e^{j\omega n} = -\alpha^{-1} e^{j\omega} \sum_{n=0}^{\infty} \left( \frac{e^{j\omega}}{\alpha} \right)^n.$$

$$\text{For } |\alpha| > 1, X(e^{j\omega}) = -\alpha^{-1} e^{j\omega} \frac{1}{1 - (e^{j\omega}/\alpha)} = \frac{1}{1 - \alpha e^{-j\omega}}. \quad |X(e^{j\omega})|^2 = \frac{1}{1 + \alpha^2 - 2\alpha \cos \omega}.$$

$$\text{From Parseval's relation, } \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

(a)  $|X(e^{j\omega})|^2 = \frac{1}{5 + 4 \cos \omega}$ . Hence,  $\alpha = -2$ . Therefore,  $x[n] = -(-2)^n \mu[-n - 1]$ .

$$\begin{aligned} \text{Now, } 4 \int_0^{\pi} |X(e^{j\omega})|^2 d\omega &= 2 \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 4\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 4\pi \sum_{n=-\infty}^{-1} |(-2)^n|^2 \\ &= 4\pi \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \pi \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{4\pi}{3}. \end{aligned}$$

(b)  $|X(e^{j\omega})|^2 = \frac{1}{3.25 - 3 \cos \omega}$ . Hence,  $\alpha = 1.5$  and therefore,  $x[n] = -(1.5)^n \mu[-n - 1]$ . Now,

$$\begin{aligned} \int_0^{\pi} |X(e^{j\omega})|^2 d\omega &= \frac{1}{2} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = \pi \sum_{n=-\infty}^{-1} |(1.5)^n|^2 = \pi \sum_{n=1}^{\infty} \left(\frac{4}{9}\right)^n \\ &= \frac{4\pi}{9} \sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^n = \frac{4\pi}{9} \cdot \frac{9}{5} = \frac{4\pi}{5}. \end{aligned}$$

(c) Using the differentiation-in-frequency property of the DTFT, the inverse DTFT of

$$X(e^{j\omega}) = j \frac{d}{d\omega} \left( \frac{1}{1 - \alpha e^{-j\omega}} \right) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2} \text{ is } x[n] = -n \alpha^n \mu[-n - 1]. \text{ Hence, the inverse DTFT of}$$

$$\frac{1}{(1 - \alpha e^{-j\omega})^2} \text{ is } -(n+1) \alpha^n \mu[-n - 1].$$

$|Y(e^{j\omega})|^2 = \frac{1}{(5 - 4 \cos \omega)^2}$ . Hence,  $\alpha = 2$  and  $y[n] = -(n+1) 2^n \mu[-n - 1]$ . Now,

$$\begin{aligned} 4 \int_0^{\pi} |X(e^{j\omega})|^2 d\omega &= 2 \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 4\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 4\pi \sum_{n=-\infty}^{-1} (n+1)^2 \cdot 2^{2n} \\ &= \pi \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \cdot n^2 = \pi \frac{9/4}{9/16} = 4\pi. \end{aligned}$$

3.25 (a)  $X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = 3 + 1 - 2 - 3 + 4 + 1 - 1 = 3$ .

(b)  $X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x[n] e^{j\pi n} = -3 - 1 - 2 + 3 - 4 - 1 + 1 = -7$ .

(c)  $\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = 2\pi x[0] = -4\pi$ .

(d)  $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 82\pi$ . (Using Parseval's relation)

(e)  $\int_{-\pi}^{\pi} \left| \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} [n \cdot x[n]]^2 = 378\pi$ . (Using Parseval's relation with differentiation-in-frequency property)

**3.26 (a)**  $X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = -2 + 4 - 1 + 5 - 3 - 2 + 4 + 3 = 8$ .

(b)  $X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} x[n](-1)^n = 2 + 4 + 1 + 5 + 3 - 2 - 4 + 3 = 14$ .

(c)  $\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = 2\pi x[0] = -4\pi$ .

(d)  $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 168\pi$ . (Using Parseval's relation)

(e)  $\int_{-\pi}^{\pi} \left| \frac{dX(e^{j\omega})}{d\omega} \right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} [n \cdot x[n]]^2 = 1238\pi$ .

**3.27** Let  $G_1(e^{j\omega})$  denote the DTFT of  $g_1[n]$ .

(b)  $g_2[n] = g_1[n] + g_1[n-4]$ . Hence, the DTFT of  $g_2[n]$  is given by  $G_2(e^{j\omega}) = G_1(e^{j\omega}) + e^{-j4\omega}G_1(e^{j\omega}) = (1 + e^{-j4\omega})G_1(e^{j\omega})$ .

(c)  $g_3[n] = g_1[-(n-3)] + g_1[n-4]$ . Now, the DTFT of  $g_1[-n]$  is given by  $G_1(e^{-j\omega})$ . Hence, the DTFT of  $g_3[n]$  is given by  $G_3(e^{j\omega}) = e^{-j3\omega}G_1(e^{-j\omega}) + e^{-j4\omega}G_1(e^{j\omega})$ .

(d)  $g_4[n] = g_1[n] + g_1[-(n-7)]$ . Hence, the DTFT of  $g_4[n]$  is given by  $G_4(e^{j\omega}) = G_1(e^{j\omega}) + e^{-j7\omega}G_1(e^{-j\omega})$ .

**3.28**  $Y(e^{j\omega}) = X_1(e^{j\omega}) \cdot X_2(e^{j\omega}) \cdot X_3(e^{j\omega})$ , i.e.,

$$\sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \left( \sum_{n=-\infty}^{\infty} x_1[n]e^{-j\omega n} \right) \left( \sum_{n=-\infty}^{\infty} x_2[n]e^{-j\omega n} \right) \left( \sum_{n=-\infty}^{\infty} x_3[n]e^{-j\omega n} \right)$$

(a) Therefore, setting  $\omega = 0$  we get  $\sum_{n=-\infty}^{\infty} y[n] = \left( \sum_{n=-\infty}^{\infty} x_1[n] \right) \left( \sum_{n=-\infty}^{\infty} x_2[n] \right) \left( \sum_{n=-\infty}^{\infty} x_3[n] \right)$ .

(b) Setting  $\omega = \pi$  we get  $\sum_{n=-\infty}^{\infty} (-1)^n y[n] = \left( \sum_{n=-\infty}^{\infty} (-1)^n x_1[n] \right) \left( \sum_{n=-\infty}^{\infty} (-1)^n x_2[n] \right) \left( \sum_{n=-\infty}^{\infty} (-1)^n x_3[n] \right)$ .

**3.29 (a)**  $x[n] = x_{\text{ev}}[n] + x_{\text{od}}[n]$ . Now, for a causal  $x[n]$ , from the results of Problem 2.4(??), we observe

$$x[n] = 2x_{\text{ev}}[n]\mu[n] - x[0]\delta[n] = h[n] - x[0]\delta[n], \quad (1)$$

$$x[n] = 2x_{\text{od}}[n]\mu[n] + x[0]\delta[n]. \quad (2)$$

Taking the DTFT of both sides of Eq. (2) we get

$$X(e^{j\omega}) = H(e^{j\omega}) - x[0], \quad (3)$$

$$\text{where } H(e^{j\omega}) = \text{DTFT}\{2x_{\text{ev}}[n]\mu[n]\} = \frac{1}{\pi} \int_{-\pi}^{\pi} X_{\text{re}}(e^{j\theta}) \mathbf{m}(e^{j(\omega-\theta)}) d\theta, \quad (4)$$

Note: The DTFT of  $x_{\text{ev}}[n]$  is  $X_{\text{re}}(e^{j\omega})$ , and the DTFT of  $\mu[n]$  is  $\mathbf{m}(e^{j\omega})$ .

$$\text{Now, from Table 3.1, } \mathbf{m}(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k) = \frac{1}{2} - \frac{j}{2} \cot\left(\frac{\omega}{2}\right) + \pi \sum_{k=-\infty}^{\infty} \delta(\omega + 2\pi k).$$

Substituting the above in Eq. (4) we get

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{\pi} \int_{-\pi}^{\pi} X_{\text{re}}(e^{j\theta}) \left\{ \frac{1}{2} - \frac{j}{2} \cot\left(\frac{\theta}{2}\right) + \pi \sum_{k=-\infty}^{\infty} \delta(\theta + 2\pi k) \right\} d\theta \\ &= X_{\text{re}}(e^{j\omega}) + \frac{1}{2\pi} \int_{-\pi}^{\pi} X_{\text{re}}(e^{j\theta}) d\theta - \frac{j}{2\pi} \int_{-\pi}^{\pi} X_{\text{re}}(e^{j\theta}) \cot\left(\frac{\omega - \theta}{2}\right) d\theta. \end{aligned}$$

Substituting the above in Eq. (3) we get

$$\begin{aligned} X(e^{j\omega}) &= X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega}) = H(e^{j\omega}) - x[0] \\ &= X_{\text{re}}(e^{j\omega}) + \frac{1}{2\pi} \int_{-\pi}^{\pi} X_{\text{re}}(e^{j\theta}) d\theta - \frac{j}{2\pi} \int_{-\pi}^{\pi} X_{\text{re}}(e^{j\theta}) \cot\left(\frac{\omega - \theta}{2}\right) d\theta - x[0] \\ &= X_{\text{re}}(e^{j\omega}) - \frac{j}{2\pi} \int_{-\pi}^{\pi} X_{\text{re}}(e^{j\theta}) \cot\left(\frac{\omega - \theta}{2}\right) d\theta, \end{aligned} \quad (5)$$

since  $\frac{1}{2\pi} \int_{-\pi}^{\pi} X_{\text{re}}(e^{j\theta}) d\theta = x[0]$ , as  $x[n]$  is real. Comparing the imaginary part of both sides of

$$\text{Eq. (5) we therefore get } X_{\text{im}}(e^{j\omega}) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} X_{\text{re}}(e^{j\theta}) \cot\left(\frac{\omega - \theta}{2}\right) d\theta.$$

**(b)** Taking the DTFT of both sides of Eq. (2) we get

$$X(e^{j\omega}) = G(e^{j\omega}) + x[0], \quad (6)$$

$$\text{where, } G(e^{j\omega}) = \text{DTFT}\{2x_{\text{od}}[n]\mu[n]\} = \frac{j}{\pi} \int_{-\pi}^{\pi} X_{\text{im}}(e^{j\theta}) \mathbf{m}(e^{j(\omega-\theta)}) d\theta, \quad (7)$$

as  $jX_{\text{im}}(e^{j\omega})$  is the DTFT of  $x_{\text{od}}[n]$ . Substituting the expression for  $\mathbf{m}(e^{j\omega})$  given above in

$$\text{Eq. (7) we get } G(e^{j\omega}) = \frac{j}{\pi} \int_{-\pi}^{\pi} X_{\text{im}}(e^{j\theta}) \left\{ \frac{1}{2} - \frac{j}{2} \cot\left(\frac{\theta}{2}\right) + \pi \sum_{k=-\infty}^{\infty} \delta(\theta + 2\pi k) \right\} d\theta$$

$$= jX_{im}(e^{j\omega}) + \frac{j}{2\pi} \int_{-\pi}^{\pi} X_{im}(e^{j\theta}) d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} X_{im}(e^{j\theta}) \cot\left(\frac{\omega-\theta}{2}\right) d\theta$$

Substituting the above in Eq. (6) we get

$$\begin{aligned} X(e^{j\omega}) &= X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega}) = G(e^{j\omega}) + x[0] \\ &= jX_{im}(e^{j\omega}) + \frac{j}{2\pi} \int_{-\pi}^{\pi} X_{im}(e^{j\theta}) d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} X_{im}(e^{j\theta}) \cot\left(\frac{\omega-\theta}{2}\right) d\theta + x[0] \\ &= jX_{im}(e^{j\omega}) + \frac{1}{2\pi} \int_{-\pi}^{\pi} X_{im}(e^{j\theta}) \cot\left(\frac{\omega-\theta}{2}\right) d\theta + x[0] \end{aligned} \quad (8)$$

as  $\frac{1}{2\pi} \int_{-\pi}^{\pi} X_{im}(e^{j\theta}) d\theta = 0$  since  $X_{im}(e^{j\omega})$  is an odd function of  $\omega$ . Comparing the real parts of

both sides of Eq. (8) we finally arrive at  $X_{re}(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_{im}(e^{j\theta}) \cot\left(\frac{\omega-\theta}{2}\right) d\theta + x[0]$ .

$$\mathbf{3.30} \quad S = \sum_{n=0}^{N-1} W_N^{-(k-1)n} = \sum_{n=0}^{N-1} e^{j2\pi n(k-1)/N}$$

$$\text{If } k-1 \neq rN \text{ then } S = \frac{1 - e^{j2\pi n(k-1)}}{1 - e^{j2\pi n(k-1)/N}} = \frac{1-1}{1 - e^{j2\pi n(k-1)/N}} = 0.$$

$$\text{If } k-1 = rN \text{ then } S = \sum_{n=0}^{N-1} W_N^{-rn} = \sum_{n=0}^{N-1} e^{-j2\pi nr} = \sum_{n=0}^{N-1} 1 = N.$$

$$\text{Hence, } \sum_{n=0}^{N-1} W_N^{-(k-1)n} = \begin{cases} N, & \text{for } k-1 = rN, r \text{ an integer,} \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{3.31} \quad \tilde{y}[n] = \sum_{r=0}^{N-1} \tilde{x}[r] \tilde{h}[n-r]. \text{ Then } \tilde{y}[n+kN] = \sum_{r=0}^{N-1} \tilde{x}[r] \tilde{h}[n+kN-r]. \text{ Since } \tilde{h}[n] \text{ is periodic in } n$$

with a period  $N$ ,  $\tilde{h}[n+kN-r] = \tilde{h}[n-r]$ . Therefore  $\tilde{y}[n+kN] = \sum_{r=0}^{N-1} \tilde{x}[r] \tilde{h}[n-r] = \tilde{y}[n]$ , hence  $\tilde{y}[n]$  is also periodic in  $n$  with a period  $N$ .

$$\mathbf{3.32} \quad \tilde{x}[n] = \{0 \ 1 \ 0 \ -2 \ 3\} \text{ and } \tilde{h}[n] = \{2 \ 0 \ 1 \ 0 \ -2\}.$$

$$\text{Now, } \tilde{y}[0] = \sum_{r=0}^4 \tilde{x}[r] \tilde{h}[0-r] = \tilde{x}[0] \tilde{h}[0] + \tilde{x}[1] \tilde{h}[4] + \tilde{x}[2] \tilde{h}[3] + \tilde{x}[3] \tilde{h}[2] + \tilde{x}[4] \tilde{h}[1] = -4.$$

$$\text{Similarly } \tilde{y}[1] = \sum_{r=0}^3 \tilde{x}[r] \tilde{h}[1-r] = \tilde{x}[0] \tilde{h}[1] + \tilde{x}[1] \tilde{h}[0] + \tilde{x}[2] \tilde{h}[3] + \tilde{x}[3] \tilde{h}[2] = 5.$$

Continuing the process we can show that  $\tilde{y}[2] = 4$ ,  $\tilde{y}[3] = -9$ , and  $\tilde{y}[4] = 6$ .

**3.33**  $\tilde{x}[n] = \{2 \ -1 \ 2 \ 3 \ -2\}$  and  $\tilde{h}[n] = \{1 \ 2 \ -3 \ 0 \ -3\}$ .

$$\text{Now, } \tilde{y}[0] = \sum_{r=0}^4 \tilde{x}[r]\tilde{h}[0-r] = \tilde{x}[0]\tilde{h}[0] + \tilde{x}[1]\tilde{h}[4] + \tilde{x}[2]\tilde{h}[3] + \tilde{x}[3]\tilde{h}[2] + \tilde{x}[4]\tilde{h}[1] = -8.$$

$$\text{Similarly } \tilde{y}[1] = \sum_{r=0}^3 \tilde{x}[r]\tilde{h}[1-r] = \tilde{x}[0]\tilde{h}[1] + \tilde{x}[1]\tilde{h}[0] + \tilde{x}[2]\tilde{h}[3] + \tilde{x}[3]\tilde{h}[2] = 3.$$

Continuing the process we can show that  $\tilde{y}[2] = -15$ ,  $\tilde{y}[3] = 16$ , and  $\tilde{y}[4] = -8$ .

**3.34** Since  $\tilde{\psi}_k[n+rN] = \tilde{\psi}_k[n]$ , hence all the terms which are not in the range  $0, 1, \dots, N-1$  can be accumulated to  $\tilde{\psi}_k[n]$ , where  $0 \leq k \leq N-1$ . Hence in this case the Fourier series representation involves only  $N$  complex exponential sequences. Let

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi kn/N} \text{ then}$$

$$\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi mn/N} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi(k-r)n/N} = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \sum_{n=0}^{N-1} e^{j2\pi(k-r)n/N}.$$

Now from Eq. (3.28), the inner summation is equal to  $N$  if  $k = r$ , otherwise it is equal to  $0$ .

$$\text{Thus } \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi mn/N} = \tilde{X}[r].$$

$$\tilde{X}[k+1N] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi(k+1N)n/N} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi kn/N} e^{-j2\pi n} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j2\pi kn/N} = \tilde{X}[k].$$

**3.35 (a)**  $\tilde{x}_1[n] = \cos\left(\frac{\pi n}{4}\right) = \frac{1}{2} \{e^{j\pi n/4} + e^{-j\pi n/4}\}$ . The period of  $\tilde{x}_1[n]$  is  $N = 8$ .

$$\tilde{X}_1[k] = \frac{1}{2} \left\{ \sum_{n=0}^7 e^{j2\pi n/8} e^{-j2\pi kn/8} + \sum_{n=0}^7 e^{-j2\pi n/8} e^{-j2\pi kn/8} \right\}$$

$$= \frac{1}{2} \left\{ \sum_{n=0}^7 e^{-j2\pi n(k-1)/8} + \sum_{n=0}^7 e^{-j2\pi n(k+1)/8} \right\}. \text{ Now, from Eq. (3.28) we observe}$$

$$\sum_{n=0}^7 e^{-j2\pi n(k-1)/8} = \begin{cases} 8, & \text{for } k=1, \\ 0, & \text{otherwise,} \end{cases} \text{ and } \sum_{n=0}^7 e^{-j2\pi n(k+1)/8} = \begin{cases} 8, & \text{for } k=7, \\ 0, & \text{otherwise.} \end{cases} \text{ Hence,}$$

$$\tilde{X}_1[k] = \begin{cases} 4, & k=1, 7, \\ 0, & \text{otherwise.} \end{cases}$$

**(b)**  $\tilde{x}_2[n] = \sin\left(\frac{\pi n}{3}\right) + 3\cos\left(\frac{\pi n}{4}\right) = \frac{1}{2j} \{e^{j\pi n/3} - e^{-j\pi n/3}\} + \frac{3}{2} \{e^{j\pi n/4} + e^{-j\pi n/4}\}$ . The period of  $\sin\left(\frac{\pi n}{3}\right)$  is  $6$  and the period of  $\cos\left(\frac{\pi n}{4}\right)$  is  $8$ . Hence, the period of  $\tilde{x}_2[n]$  is the gcm of

$$(6, 8) \text{ and is } 24. \tilde{X}_2[k] = \frac{1}{2j} \left\{ \sum_{n=0}^{23} e^{j8\pi n/24} e^{-j2\pi kn/24} - \sum_{n=0}^{23} e^{-j8\pi n/24} e^{-j2\pi kn/24} \right\}$$

$$\begin{aligned}
& + \frac{3}{2} \left\{ \sum_{n=0}^{23} e^{j6\pi n/24} e^{-j2\pi kn/24} + \sum_{n=0}^{23} e^{-j6\pi n/24} e^{-j2\pi kn/24} \right\} \\
& = \frac{1}{2j} \left\{ \sum_{n=0}^{23} e^{-j2\pi n(k-3)/24} - \sum_{n=0}^{23} e^{-j2\pi n(k+3)/24} \right\} + \frac{3}{2} \left\{ \sum_{n=0}^{23} e^{-j2\pi n(k-4)/24} + \sum_{n=0}^{23} e^{-j2\pi n(k+4)/24} \right\}.
\end{aligned}$$

Hence  $\tilde{X}_2[k] = \begin{cases} -j12, & k = 3, \\ j12, & k = 21, \\ 36, & k = 4, 20, \\ 0, & \text{otherwise.} \end{cases}$

**3.36** Since  $\tilde{p}[n]$  is periodic with period  $N$ , then from Eq. (3.168a) Of Problem 3.34,

$$\tilde{p}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{P}[k] e^{-j2\pi kn/N} \quad \text{where using Eq. (3.168b) we get } \tilde{P}[k] = \sum_{n=0}^{N-1} \tilde{p}[n] e^{-j2\pi kn/N} = 1.$$

$$\text{Hence } \tilde{p}[n] = \frac{1}{N} \sum_{k=0}^{N-1} e^{-j2\pi kn/N}.$$

$$\mathbf{3.37} \quad \tilde{X}[k] = X(e^{j\omega}) \Big|_{\omega=2\pi k/N} = X(e^{j2\pi k/N}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi kn/N}, \quad -\infty < k < \infty.$$

$$\mathbf{(a)} \quad \tilde{X}[k+1N] = X(e^{j2\pi(k+1N)/N}) = X(e^{j2\pi k/N} e^{j2\pi 1}) = X(e^{j2\pi k/N}) = \tilde{X}[k].$$

$$\begin{aligned}
\mathbf{(b)} \quad \tilde{x}[n] &= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j2\pi kn/N} = \frac{1}{N} \sum_{k=0}^{N-1} \left( \sum_{l=-\infty}^{\infty} x[l] e^{-j2\pi kl/N} \right) e^{j2\pi kn/N} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=-\infty}^{\infty} x[l] e^{j2\pi k(n-l)/N}. \quad \text{Let } l = n + r \cdot N. \quad \text{Then } \tilde{x}[n] = \frac{1}{N} \sum_{r=-\infty}^{\infty} x[n + r \cdot N] \left( \sum_{k=0}^{N-1} e^{-j2\pi kr} \right).
\end{aligned}$$

$$\text{But } \sum_{k=0}^{N-1} e^{-j2\pi kr} = N. \quad \text{Hence, } \tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n + r \cdot N].$$

$$\mathbf{3.38 (a)} \quad \tilde{G}[k] = \sum_{n=0}^{N-1} \tilde{g}[n] e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} \tilde{x}[n] \tilde{y}[n] e^{-j2\pi kn/N}. \quad \text{Now, } \tilde{x}[n] = \frac{1}{N} \sum_{r=0}^{N-1} \tilde{X}[r] e^{j2\pi rn/N}$$

$$\begin{aligned}
\text{Therefore, } \tilde{G}[k] &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{r=0}^{N-1} \tilde{X}[r] \tilde{y}[n] e^{-j2\pi(k-r)n/N} = \frac{1}{N} \sum_{r=0}^{N-1} \tilde{X}[r] \sum_{n=0}^{N-1} \tilde{y}[n] e^{-j2\pi(k-r)n/N} \\
&= \frac{1}{N} \sum_{r=0}^{N-1} \tilde{X}[r] \tilde{Y}[k-r].
\end{aligned}$$

$$\mathbf{(b)} \quad \tilde{h}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \tilde{Y}[k] e^{j2\pi kn/N} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} \tilde{x}[r] \tilde{Y}[k] e^{j2\pi k(n-r)/N}$$

$$= \sum_{r=0}^{N-1} \tilde{x}[r] \left( \frac{1}{N} \sum_{k=0}^{N-1} \tilde{Y}[k] e^{j2\pi k(n-r)/N} \right) = \sum_{r=0}^{N-1} \tilde{x}[r] \tilde{y}[n-r].$$

**3.39 (a)**  $y[n] = \alpha g[n] + \beta h[n]$ . Therefore

$$Y[k] = \sum_{n=0}^{N-1} y[n] W_N^{nk} = \alpha \sum_{n=0}^{N-1} g[n] W_N^{nk} + \beta \sum_{n=0}^{N-1} h[n] W_N^{nk} = \alpha G[k] + \beta H[k]$$

**(b)**  $x[n] = g[\langle n - n_0 \rangle_N]$ . Therefore  $X[k] = \sum_{n=0}^{N-1} g[\langle n - n_0 \rangle_N] W_N^{nk}$

$$\begin{aligned} &= \sum_{n=0}^{n_0-1} g[N + n - n_0] W_N^{nk} + \sum_{n=n_0}^{N-1} g[n - n_0] W_N^{nk} \\ &= \sum_{n=N-n_0}^{N-1} g[n] W_N^{(n+n_0-N)k} + \sum_{n=0}^{N-n_0-1} g[n] W_N^{(n+n_0)k} = W_N^{n_0 k} \sum_{n=0}^{N-1} g[n] W_N^{nk} = W_N^{n_0 k} G[k]. \end{aligned}$$

**(c)**  $u[n] = W_N^{-k_0 n} g[n]$ . Hence  $U[k] = \sum_{n=0}^{N-1} u[n] W_N^{nk} = \sum_{n=0}^{N-1} g[n] W_N^{(k-k_0)n}$

$$= \begin{cases} \sum_{n=0}^{N-1} W_N^{(k-k_0)n} g[n], & \text{if } k \geq k_0, \\ \sum_{n=0}^{N-1} W_N^{(N+k-k_0)n} g[n], & \text{if } k < k_0. \end{cases}$$

Thus,  $U[k] = \begin{cases} G[k - k_0], & \text{if } k \geq k_0, \\ G[N + k - k_0], & \text{if } k < k_0, \end{cases} = G[\langle k - k_0 \rangle_N]$ .

**(d)**  $h[n] = G[n]$ . Therefore,  $H[k] = \sum_{n=0}^{N-1} h[n] W_N^{nk} = \sum_{n=0}^{N-1} G[n] W_N^{nk} = \sum_{n=0}^{N-1} \sum_{r=0}^{N-1} g[r] W^{nr} W^{kr}$

$$= \sum_{r=0}^{N-1} g[r] \sum_{n=0}^{N-1} W^{(k+r)n}.$$

The second sum is non-zero only if  $k = r = 0$  or else if  $r = N - k$  and  $k \neq 0$ . Hence,

$$H[k] = \begin{cases} Ng[0], & \text{if } k = 0, \\ Ng[N - k], & \text{if } k > 0, \end{cases} = Ng[\langle -k \rangle_N].$$

**(e)**  $u[n] = \sum_{m=0}^{N-1} g[m] h[\langle n - m \rangle_N]$ . Therefore,  $U[k] = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} g[m] h[\langle n - m \rangle_N] W_N^{nk}$

$$= \sum_{m=0}^{N-1} g[m] \sum_{n=0}^{N-1} h[\langle n - m \rangle_N] W_N^{nk} = \sum_{m=0}^{N-1} g[m] H[k] W_N^{mk} = H[k] G[k].$$



(f)  $v[n] = g[n]h[n]$ . Therefore,  $V[k] = \sum_{n=0}^{N-1} g[n]h[n]W_N^{nk} = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{r=0}^{N-1} h[n]G[r]W_N^{nk} W^{-nr} =$   

$$\frac{1}{N} \sum_{r=0}^{N-1} G[r] \sum_{n=0}^{N-1} h[n]W^{(k-r)n} = \frac{1}{N} \sum_{r=0}^{N-1} G[r]H[\langle k-r \rangle_N].$$

(g)  $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]W^{-nk}$ . Thus  $x^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} X^*[k]W^{nk}$ . Therefore,  

$$\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} \frac{1}{N^2} \left( \sum_{r=0}^{N-1} X[r]W^{-nr} \right) \left( \sum_{l=0}^{N-1} X^*[l]W^{nl} \right) = \frac{1}{N^2} \sum_{r=0}^{N-1} \sum_{l=0}^{N-1} X[r]X^*[l] \sum_{n=0}^{N-1} W^{n(l-r)}.$$

Since the inner sum is non-zero only if  $l = r$ , we get  $\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2.$

**3.40**  $X[k] = \sum_{n=0}^{N-1} x[n]W^{nk}.$

(a)  $X^*[k] = \sum_{n=0}^{N-1} x^*[n]W^{-nk}$ . Replacing  $k$  by  $N - k$  on both sides we obtain

$$X^*[N - k] = \sum_{n=0}^{N-1} x^*[n]W^{-n(N-k)} = \sum_{n=0}^{N-1} x^*[n]W^{nk}. \text{ Thus } x^*[n] \Leftrightarrow X^*[N - k] = X^*[\langle -k \rangle_N].$$

(b)  $X^*[k] = \sum_{n=0}^{N-1} x^*[n]W^{-nk}$ . Replacing  $n$  by  $N - n$  in the summation we get

$$X^*[k] = \sum_{n=0}^{N-1} x^*[N - n]W^{-(N-n)k} = \sum_{n=0}^{N-1} x^*[N - n]W^{nk}.$$

Thus  $x^*[N - n] = x^*[\langle -n \rangle_N] \Leftrightarrow X^*[k].$

(c)  $\text{Re}\{x[n]\} = \frac{1}{2} \{x[n] + x^*[n]\}$ . Now taking DFT of both sides and using results of part (a)

we get  $\text{Re}\{x[n]\} \Leftrightarrow \frac{1}{2} \{X[k] + X^*[\langle -k \rangle_N]\}.$

(d)  $j \text{Im}\{x[n]\} = \frac{1}{2} \{x[n] - x^*[n]\}$  this implies  $j \text{Im}\{x[n]\} \Leftrightarrow \frac{1}{2} \{X[k] - X^*[\langle -k \rangle_N]\}.$

(e)  $x_{\text{pcs}}[n] = \frac{1}{2} \{x[n] + x^*[\langle -n \rangle_N]\}$  Using linearity and results of part (b) we get

$$x_{\text{pcs}}[n] \Leftrightarrow \frac{1}{2} \{X[k] + X^*[k]\} = \text{Re}\{X[k]\}.$$

(f)  $x_{\text{pca}}[n] = \frac{1}{2} \{x[n] - x^*[\langle -n \rangle_N]\}$ . Again using results of part (b) and linearity we get

$$x_{\text{pca}}[n] \Leftrightarrow \frac{1}{2} \{X[k] - X^*[k]\} = j \text{Im}\{X[k]\}.$$

$$3.41 \quad X[k] = \text{Re}\{X[k]\} + j \text{Im}\{X[k]\} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}.$$

(a)  $x_{pe}[n] = \frac{1}{2} \{x[n] + x[\langle -n \rangle_N]\}$ . From Table 3.6,  $x^*[\langle -n \rangle_N] \stackrel{\text{DFT}}{\Leftrightarrow} X^*[k]$ . Since  $x[n]$  is real,  $x[\langle -n \rangle_N] = x^*[\langle -n \rangle_N] \stackrel{\text{DFT}}{\Leftrightarrow} X^*[k]$ . Thus,  $X_{pe}[k] = \frac{1}{2} \{X[k] + X^*[k]\} = \text{Re}\{X[k]\}$ .

(b)  $x_{po}[n] = \frac{1}{2} \{x[n] - x[\langle -n \rangle_N]\}$ . As a result,  $X_{po}[k] = \frac{1}{2} \{X[k] - X^*[k]\} = j \text{Im}\{X[k]\}$ .

3.42 Since for a real sequence,  $x[n] = x^*[n]$ , taking DFT of both sides we get  $X[k] =$

$$X^*[\langle -k \rangle_N]. \text{ This implies, } \text{Re}\{X[k]\} + j \text{Im}\{X[k]\} = \text{Re}\{X[\langle -k \rangle_N]\} - j \text{Im}\{X[\langle -k \rangle_N]\}.$$

Comparing real and imaginary parts we get

$$\text{Re}\{X[k]\} = \text{Re}\{X[\langle -k \rangle_N]\} \text{ and } \text{Im}\{X[k]\} = -\text{Im}\{X[\langle -k \rangle_N]\}.$$

$$\begin{aligned} \text{Also } |X[k]| &= \sqrt{(\text{Re}\{X[k]\})^2 + (\text{Im}\{X[k]\})^2} \\ &= \sqrt{(\text{Re}\{X[\langle -k \rangle_N]\})^2 + (-\text{Im}\{X[\langle -k \rangle_N]\})^2} = |X[\langle -k \rangle_N]| \end{aligned}$$

$$\text{and } \arg\{X[k]\} = \tan^{-1} \left( \frac{\text{Im}\{X[k]\}}{\text{Re}\{X[k]\}} \right) = \tan^{-1} \left( \frac{-\text{Im}\{X[\langle -k \rangle_N]\}}{\text{Re}\{X[\langle -k \rangle_N]\}} \right) = -\arg\{X[\langle -k \rangle_N]\}.$$

3.43 (a)  $x_1[\langle -n \rangle_8] = [1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1] = x_1[n]$ . Thus,  $x_1[n]$  is a periodic even sequence, and hence it has a real-valued 8-point DFT.

(b)  $x_2[\langle -n \rangle_8] = [1 \ -1 \ -1 \ 0 \ 0 \ 0 \ 0 \ 1]$ . Thus,  $x_2[n]$  is neither a periodic even or a periodic odd sequence. Hence, its 8-point DFT is a complex sequence.

(c)  $x_3[\langle -n \rangle_8] = [0 \ -1 \ -1 \ 0 \ 0 \ 0 \ 1 \ 1] = -x_3[n]$ . Thus,  $x_3[n]$  is a periodic odd sequence, and hence it has an imaginary-valued 8-point DFT.

(d)  $x_4[\langle -n \rangle_8] = [0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1] = x_4[n]$ . Thus,  $x_4[n]$  is a periodic even sequence, and hence it has a real-valued 8-point DFT.

3.44 (a) Now,  $X[N/2] = \sum_{n=0}^{N-1} x[n] W_N^{nN/2} = \sum_{n=0}^{N-1} (-1)^n x[n]$ . Hence if  $x[n] = x[N-1-n]$  and  $N$  is

even, then  $\sum_{n=0}^{N-1} (-1)^n x[n] = 0$  or  $X[N/2] = 0$ .

(b)  $X[0] = \sum_{n=0}^{N-1} x[n]$  so if  $x[n] = -x[N-1-n]$ , then  $X[0] = 0$ .

$$\begin{aligned}
\text{(c) } X[2l] &= \sum_{n=0}^{N-1} x[n] W^{2nl} = \sum_{n=0}^{\frac{N}{2}-1} x[n] W^{2nl} + \sum_{n=N/2}^{N-1} x[n] W^{2nl} \\
&= \sum_{n=0}^{\frac{N}{2}-1} x[n] W^{2nl} + \sum_{n=0}^{\frac{N}{2}-1} x[n + \frac{N}{2}] W^{2nl} = \sum_{n=0}^{\frac{N}{2}-1} (x[n] + x[n + \frac{N}{2}]) W^{2nl}.
\end{aligned}$$

Hence if  $x[n] = -x[n + \frac{N}{2}] = -x[n + M]$ , then  $X[2l] = 0$ , for  $l = 0, 1, \dots, M - 1$ .

$$\begin{aligned}
\text{3.45 } X[2m] &= \sum_{n=0}^{N-1} x[n] W_N^{2mn} = \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2mn} + \sum_{n=N/2}^{N-1} x[n] W_N^{2mn} \\
&= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2mn} + \sum_{n=0}^{\frac{N}{2}-1} x[n + \frac{N}{2}] W_N^{2m(n + \frac{N}{2})} = \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2mn} + \sum_{n=0}^{\frac{N}{2}-1} x[n + \frac{N}{2}] W_N^{2mn} W_N^{mN} \\
&= \sum_{n=0}^{\frac{N}{2}-1} \left( x[n] + x[n + \frac{N}{2}] \right) W_N^{2mn} = 0, \quad 0 \leq m \leq \frac{N}{2} - 1. \quad \text{This implies } x[n] + x[n + \frac{N}{2}] = 0.
\end{aligned}$$

**3.46 (a)** Using the circular time-shifting property of the DFT given in Table 3.5 we observe  $\text{DFT}\{x[\langle n - m_1 \rangle_N]\} = W_N^{km_1} X[k]$  and  $\text{DFT}\{x[\langle n - m_2 \rangle_N]\} = W_N^{km_2} X[k]$ . Hence,  $W[k] = \text{DFT}\{x[n]\} = \alpha W_N^{km_1} X[k] + \beta W_N^{km_2} X[k] = (\alpha W_N^{km_1} + \beta W_N^{km_2}) X[k]$ . A proof of the circular time-shifting property is given in Problem 3.39.

**(b)**  $g[n] = \frac{1}{2} (x[n] + (-1)^n x[n]) = \frac{1}{2} \left( x[n] + W_N^{-\frac{N}{2}n} x[n] \right)$ , Using the circular frequency-shifting property of the DFT given in Table 3.5, we get  $G[k] = \text{DFT}\{g[n]\} = \frac{1}{2} \left\{ X[k] + X[\langle k - \frac{N}{2} \rangle_N] \right\}$ .

**(c)** Using the circular convolution property of the DFT given in Table 3.5 we get  $Y[k] = \text{DFT}\{y[n]\} = X[k] \cdot X[k] = X^2[k]$ . A proof of the circular convolution property is given in Problem 3.39.

**3.47 (a)**  $\text{DFT}\left\{x\left[n - \frac{N}{2}\right]\right\} = W_N^{k\frac{N}{2}} X[k] = -X[k]$ . Hence,

$$u[k] = \text{DFT}\{u[n]\} = \text{DFT}\left\{x[n] + x\left[n - \frac{N}{2}\right]\right\} = X[k] - X[k] = 0.$$

**(b)**  $V[k] = \text{DFT}\{v[n]\} = \text{DFT}\left\{x[n] - x\left[n - \frac{N}{2}\right]\right\} = X[k] + X[k] = 2X[k]$ .

(c)  $y[n] = (-1)^n x[n] = W_N^{\frac{N}{2}n} x[n]$ . Hence,  $Y[k] = \text{DFT}\{y[n]\} = \text{DFT}\left\{W_N^{\frac{N}{2}n} x[n]\right\} = X\left[\left\langle k - \frac{N}{2} \right\rangle_N\right]$

using the circular frequency-shifting property of the DFT given in Table 3.5.

**3.48 (a)** From the circular frequency-shifting property of the DFT given in Table 3.5,

$$\begin{aligned} \text{IDFT}\{X[\langle k - m_1 \rangle_N]\} &= W_N^{-m_1 n} x[n] \text{ and } \text{IDFT}\{X[\langle k - m_2 \rangle_N]\} = W_N^{-m_2 n} x[n]. \text{ Hence,} \\ w[n] &= \text{IDFT}\{W[k]\} = \text{IDFT}\{\alpha X[\langle k - m_1 \rangle_N] + \beta X[\langle k - m_2 \rangle_N]\} \\ &= \alpha W_N^{-m_1 n} x[n] + \beta W_N^{-m_2 n} x[n] = (\alpha W_N^{-m_1 n} + \beta W_N^{-m_2 n}) x[n]. \end{aligned}$$

(b)  $G[k] = \frac{1}{2} (X[k] + (-1)^k X[k]) = \frac{1}{2} \left( X[k] + W_N^{\frac{-N}{2}k} X[k] \right)$ . Using the circular time-shifting

property of the DFT given in Table 3.5, we get  $g[n] = \text{IDFT}\{G[k]\} = \frac{1}{2} \left( x[n] + x[\langle n - \frac{N}{2} \rangle_N] \right)$ .

(c) Using the modulation property of the DFT given in Table 3.5 we get

$$y[n] = \text{IDFT}\{Y[k]\} = N \cdot x[n] \cdot x[n] = N \cdot x^2[n].$$

**3.49 (a)**  $X[2m] = \sum_{n=0}^{N-1} x[n] W_N^{2mn} = \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2mn} + \sum_{n=\frac{N}{2}}^{N-1} x[n] W_N^{2mn}$

$$\begin{aligned} &= \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2mn} + \sum_{n=0}^{\frac{N}{2}-1} x\left[n + \frac{N}{2}\right] W_N^{2m\left(n + \frac{N}{2}\right)} = \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2mn} + \sum_{n=0}^{\frac{N}{2}-1} x\left[n + \frac{N}{2}\right] W_N^{2mn} W_N^{mN} \\ &= \sum_{n=0}^{\frac{N}{2}-1} \left( x[n] + x\left[n + \frac{N}{2}\right] \right) W_N^{2mn} = \sum_{n=0}^{\frac{N}{2}-1} (x[n] - x[n]) W_N^{2mn} = 0, \quad 0 \leq m \leq \frac{N}{2} - 1. \end{aligned}$$

(b)  $X[4l] = \sum_{n=0}^{N-1} x[n] W_N^{4ln} = \sum_{n=0}^{\frac{N}{4}-1} x[n] W_N^{4ln} + \sum_{n=\frac{N}{4}}^{\frac{N}{2}-1} x[n] W_N^{4ln} + \sum_{n=\frac{N}{2}}^{\frac{3N}{4}-1} x[n] W_N^{4ln} + \sum_{n=\frac{3N}{4}}^{N-1} x[n] W_N^{4ln}$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left( x[n] W_N^{4ln} + x\left[n + \frac{N}{4}\right] W_N^{4l\left(n + \frac{N}{4}\right)} + x\left[n + \frac{N}{2}\right] W_N^{4l\left(n + \frac{N}{2}\right)} + x\left[n + \frac{3N}{4}\right] W_N^{4l\left(n + \frac{3N}{4}\right)} \right)$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left( x[n] + x\left[n + \frac{N}{4}\right] W_N^{lN} + x\left[n + \frac{N}{2}\right] W_N^{2lN} + x\left[n + \frac{3N}{4}\right] W_N^{3lN} \right) W_N^{4ln}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} (x[n] - x[n] + x[n] - x[n]) W_N^{4ln} = 0 \text{ as } W_N^{lN} = W_N^{2lN} = W_N^{3lN} = 1.$$

$$3.50 \text{ (a)} \quad X[N-k] = \sum_{n=0}^{N-1} x[n] W_N^{(N-k)n} = \sum_{n=0}^{N-1} x[n] W_N^{-kn} = X^*[k].$$

$$\text{(b)} \quad X[0] = \sum_{n=0}^{N-1} x[n] W_N^0 = \sum_{n=0}^{N-1} x[n] \text{ which is real.}$$

$$\text{(c)} \quad X\left[\frac{N}{2}\right] = \sum_{n=0}^{N-1} x[n] W_N^{(N/2)n} = \sum_{n=0}^{N-1} (-1)^n x[n] \text{ which is real.}$$

$$3.51 \text{ (a)} \quad H[k] = \text{DFT}\{h[n]\} = \text{DFT}\{g[\langle n-3 \rangle_7]\} = W_7^{3k} G[k] = e^{-j\frac{6\pi k}{7}} G[k]$$

$$= \left[ 1+j2, \quad e^{-j\frac{6\pi}{7}}(-2+j3), \quad e^{-j\frac{12\pi}{7}}(-1-j2), \quad 0, \quad e^{-j\frac{24\pi}{7}}(8+j4), \quad e^{-j\frac{30\pi}{7}}(-3+j2), \quad e^{-j\frac{36\pi}{7}}(2+j5) \right]$$

$$\text{(b)} \quad h[n] = \text{IDFT}\{H[k]\} = \text{IDFT}\{G[\langle k-4 \rangle_7]\} = W_7^{-4n} g[n] = e^{j\frac{8\pi n}{7}} g[n]$$

$$= [-3.1, \quad 2.4e^{j8\pi/7}, \quad 4.5e^{j16\pi/7}, \quad -6e^{j24\pi/7}, \quad e^{j32\pi/7}, \quad -3e^{j40\pi/7}, \quad 7e^{j42\pi/7}].$$

$$3.52 \quad Y[k] = \sum_{n=0}^{MN-1} y[n] W_{MN}^{nk} = \sum_{n=0}^{N-1} x[n] W_{MN}^{nk}. \text{ Thus, } Y[kM] = \sum_{n=0}^{N-1} x[n] W_N^{nk} = X[k].$$

Hence,  $X[k] = Y[kM]$ .

3.53 Note  $X[k]$  is the  $MN$ -point DFT of the sequence  $x_e[n]$  obtained from  $x[n]$  by appending it with  $M(N-1)$  zeros. Thus, the length- $MN$  sequence  $y[n]$  is given by

$$y[n] = \sum_{l=0}^{M-1} x_e[\langle n - Nl \rangle_{MN}], \quad 0 \leq n \leq MN-1. \text{ Taking the } MN\text{-point DFT of both sides we}$$

$$\text{get } Y[k] = \left( \sum_{l=0}^{M-1} W_{MN}^{Nkl} \right) X[k] = \left( \sum_{l=0}^{M-1} W_M^{kl} \right) X[k].$$

$$3.54 \text{ (a)} \quad X[0] = \sum_{n=0}^{11} x[n] = 13.$$

$$\text{(b)} \quad X[6] = \sum_{n=0}^{11} (-1)^n x[n] = -13.$$

$$\text{(c)} \quad \sum_{k=0}^{11} X[k] = 12 \cdot x[0] = 36.$$

(d) The inverse DFT of  $e^{-j(4\pi k/6)} X[k]$  is  $x[\langle n-4 \rangle_{12}]$ . Thus,

$$\sum_{k=0}^{11} e^{-j(4\pi k/6)} X[k] = 12 \cdot x[\langle 0-4 \rangle_{12}] = 12 \cdot x[8] = -48.$$

(e) From Parseval's relation,  $\sum_{k=0}^{11} |X[k]|^2 = 12 \cdot \sum_{n=0}^{11} |x[n]|^2 = 1500$ .

**3.55**  $X[8] = X * \langle -8 \rangle_{14} = X * [6] = -2 + j3$ ,  $X[9] = X * \langle -9 \rangle_{14} = X * [5] = 6 - j3$ ,

$X[10] = X * \langle -10 \rangle_{14} = X * [4] = -2 - j2$ ,  $X[11] = X * \langle -11 \rangle_{14} = X * [3] = 1 + j5$ ,

$X[12] = X * \langle -12 \rangle_{14} = X * [2] = 3 - j4$ ,  $X[13] = X * \langle -13 \rangle_{14} = X * [1] = -1 - j3$ .

(a)  $x[0] = \frac{1}{14} \sum_{k=0}^{13} X[k] = \frac{32}{14} = 2.2857$ ,

(b)  $x[7] = \frac{1}{14} \sum_{k=0}^{13} (-1)^k X[k] = -\frac{12}{14} = -0.8571$ ,

(c)  $\sum_{n=0}^{13} x[n] = X[0] = 12$ ,

(d) Let  $g[n] = e^{j(4\pi n/7)} x[n] = W_{14}^{-4n} x[n]$ . Then  $\text{DFT}\{g[n]\} = \text{DFT}\{W_{14}^{-4n} x[n]\} = X \langle k - 4 \rangle_{14}$   
 $= [X[10] \ X[11] \ X[12] \ X[13] \ X[0] \ X[1] \ X[2] \ X[3] \ X[4] \ X[5] \ X[6] \ X[7] \ X[8] \ X[9]]$

Thus,  $\sum_{n=0}^{13} g[n] = \sum_{n=0}^{13} e^{j(4\pi n/7)} x[n] = X[10] = -2 - j2$ ,

(e) Using Parseval's relation,  $\sum_{n=0}^{13} |x[n]|^2 = \frac{1}{14} \sum_{k=0}^{13} |X[k]|^2 = \frac{498}{14} = 35.5714$ .

**3.56** Now  $y_c[n] = \sum_{k=0}^6 g[k]h \langle n - k \rangle_6$ . Hence,

$y_c[0] = g[0]h[0] + g[1]h[6] + g[2]h[5] + g[3]h[4] + g[4]h[3] + g[5]h[2] + g[6]h[1]$ ,  
 $y_c[1] = g[0]h[1] + g[1]h[0] + g[2]h[6] + g[3]h[5] + g[4]h[4] + g[5]h[3] + g[6]h[2]$ ,  
 $y_c[2] = g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[6] + g[4]h[5] + g[5]h[4] + g[6]h[3]$ ,  
 $y_c[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0] + g[4]h[6] + g[5]h[5] + g[6]h[4]$ ,  
 $y_c[4] = g[0]h[4] + g[1]h[3] + g[2]h[2] + g[3]h[1] + g[4]h[0] + g[5]h[6] + g[6]h[5]$ ,  
 $y_c[5] = g[0]h[5] + g[1]h[4] + g[2]h[3] + g[3]h[2] + g[4]h[1] + g[5]h[0] + g[6]h[6]$ ,  
 $y_c[6] = g[0]h[6] + g[1]h[5] + g[2]h[4] + g[3]h[3] + g[4]h[2] + g[5]h[1] + g[6]h[0]$ .

Likewise,  $y_L[n] = \sum_{k=0}^6 g[k]h[n - k]$ . Hence,

$y_L[0] = g[0]h[0]$ ,  
 $y_L[1] = g[0]h[1] + g[1]h[0]$ ,  
 $y_L[2] = g[0]h[2] + g[1]h[1] + g[2]h[0]$ ,  
 $y_L[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]$ ,  
 $y_L[4] = g[0]h[4] + g[1]h[3] + g[2]h[2] + g[3]h[1] + g[4]h[0]$ ,  
 $y_L[5] = g[0]h[5] + g[1]h[4] + g[2]h[3] + g[3]h[2] + g[4]h[1] + g[5]h[0]$ ,  
 $y_L[6] = g[0]h[6] + g[1]h[5] + g[2]h[4] + g[3]h[3] + g[4]h[2] + g[5]h[1] + g[6]h[0]$ ,

$$\begin{aligned}
y_L[7] &= g[1]h[6] + g[2]h[5] + g[3]h[4] + g[4]h[3] + g[5]h[2] + g[6]h[1], \\
y_L[8] &= g[2]h[6] + g[3]h[5] + g[4]h[4] + g[5]h[3] + g[6]h[2], \\
y_L[9] &= g[3]h[6] + g[4]h[5] + g[5]h[4] + g[6]h[3], \\
y_L[10] &= g[4]h[6] + g[5]h[5] + g[6]h[4], \\
y_L[11] &= g[5]h[6] + g[6]h[5], \\
y_L[12] &= g[6]h[6].
\end{aligned}$$

Comparing  $y_{10}[n]$  with  $y_L[n]$  we observe that

$$\begin{aligned}
y_c[0] &= y_L[0] + y_L[7], \\
y_c[1] &= y_L[1] + y_L[8], \\
y_c[2] &= y_L[2] + y_L[9], \\
y_c[3] &= y_L[3] + y_L[10], \\
y_c[4] &= y_L[4] + y_L[11], \\
y_c[5] &= y_L[5] + y_L[12], \\
y_c[6] &= y_L[6].
\end{aligned}$$

**3.57** Since  $x[n]$  is a real sequence, its DFT satisfies  $X[k] = X^*[\langle -k \rangle_N]$  where  $N = 11$  in this case.

$$\begin{aligned}
\text{Therefore, } X[1] &= X^*[\langle -1 \rangle_{11}] = X^*[10] = \sqrt{3} + j2, \\
X[3] &= X^*[\langle -3 \rangle_{11}] = X^*[8] = -5 + j8, \\
X[5] &= X^*[\langle -5 \rangle_{11}] = X^*[6] = 9 + j6, \\
X[7] &= X^*[\langle -7 \rangle_{11}] = X^*[4] = 2 - j5, \\
X[9] &= X^*[\langle -9 \rangle_{11}] = X^*[2] = -1 - j3.
\end{aligned}$$

**3.58** The  $N$ -point DFT  $X[k]$  of a length- $N$  real sequence  $x[n]$  satisfy  $X[k] = X^*[\langle -k \rangle_N]$ . Here  $N = 11$ . Hence, the remaining 5 samples are  $X[1] = X^*[\langle -1 \rangle_{11}] = X^*[10] = -3.1 - j5.2$ ,  
 $X[4] = X^*[\langle -4 \rangle_{11}] = X^*[7] = -4.1 - j0.2$ ,  $X[6] = X^*[\langle -6 \rangle_{11}] = X^*[5] = 6.5 - j9$ ,  
 $X[8] = X^*[\langle -8 \rangle_{11}] = X^*[3] = 5.3 + j4.1$ ,  $X[9] = X^*[\langle -9 \rangle_{11}] = X^*[2] = -3.2 + j2$ .

**3.59** A length- $N$  periodic even sequence  $x[n]$  satisfying  $x[n] = x^*[\langle -n \rangle_N]$  has a real-valued  $N$ -point DFT  $X[k]$ . Here  $N = 10$ . Hence, the remaining 4 samples of  $x[n]$  are given by  $x[6] = x^*[\langle -6 \rangle_{10}] = x^*[4] = 2.87 - j2$ ,  $x[7] = x^*[\langle -7 \rangle_{10}] = x^*[3] = -2.1 - j4.6$ ,  
 $x[8] = x^*[\langle -8 \rangle_{10}] = x^*[2] = -3.25 - j1.12$ , and  $x[9] = x^*[\langle -9 \rangle_{10}] = x^*[1] = 0.7 + j0.08$ .

**3.60** As  $x[n]$  is a real-valued sequence of length 498, its 498-point DFT  $X[k]$  satisfy  $X[k] = X^*[\langle -k \rangle_{498}] = X^*[498 - k]$  (See Table 3.7).

(a) From the specified DFT samples we observe that  $X[k_1] = X^*[412]$  implying  $k_1 = 498 - 412 = 86$ ,  $X[k_2] = X^*[309]$  implying  $k_2 = 498 - 309 = 189$ ,  $X[k_3] = X^*[112]$  implying  $k_3 = 498 - 112 = 386$ ,  $X[k_4] = X^*[11]$  implying  $k_4 = 498 - 11 = 487$ .

(b) dc value of  $\{x[n]\} = X[0] = 2$ .

$$\text{(c) } x[n] = \frac{1}{498} \sum_{k=0}^{497} X[k] W_{498}^{-kn} = \frac{1}{498} (X[0] + 2 \operatorname{Re}\{X[11] \cdot W_{498}^{-11n}\} + 2 \operatorname{Re}\{X[86] \cdot W_{498}^{-86n}\})$$

$$\begin{aligned}
& + 2 \operatorname{Re}\{X[112] \cdot W_{498}^{-112n}\} + 2 \operatorname{Re}\{X[189] \cdot W_{498}^{-189n}\} + X[249] \cdot W_{498}^{-249n} \\
& = \frac{1}{249} \left\{ -0.45 + 7 \cos\left(\frac{11\pi n}{249}\right) - 3.1 \sin\left(\frac{11\pi n}{249}\right) - 2.2 \cos\left(\frac{86\pi n}{249}\right) + 1.5 \sin\left(\frac{86\pi n}{249}\right) \right. \\
& \left. + 3 \cos\left(\frac{112\pi n}{249}\right) + 0.7 \sin\left(\frac{112\pi n}{249}\right) - 4.7 \cos\left(\frac{189\pi n}{249}\right) - 1.9 \sin\left(\frac{189\pi n}{249}\right) \right\}.
\end{aligned}$$

$$(d) \sum_{n=0}^{497} |x[n]|^2 = \frac{1}{498} \sum_{k=0}^{497} |X[k]|^2 = 0.2275.$$

**3.61** As  $x[n]$  is a real-valued sequence of length 316, its 316-point DFT  $X[k]$  satisfy  $X[k] = X^*[-k]_{316} = X^*[316 - k]$  (See Table 3.7).

(a) From the specified DFT samples we observe that  $X[17] = X^*[k_4]$  implying  $k_4 = 316 - 17 = 299$ ,  $X[k_1] = X^*[210]$  if  $\varepsilon = 0$ , implying  $k_1 = 316 - 210 = 106$ ,  $X[k_2] = X^*[179]$  if  $\delta = 0$  implying  $k_2 = 316 - 179 = 137$ , and  $X[k_3] = X^*[110]$  if  $\gamma = 0$  implying  $k_3 = 316 - 110 = 206$ .

(b) Now,  $X[0] = \sum_{n=0}^{315} x[n]$  which is a real number as  $x[n]$  are real numbers implying  $\alpha = 0$ . Since

the length  $N = 316$  is an even number,  $X[158] = X[N/2] = \sum_{n=0}^{315} (-1)^n x[n]$  is also a real number implying  $\beta = 0$ . We have already shown in Part (a),  $\delta = \varepsilon = \gamma = 0$ .

(c) The dc value of  $\{x[n]\}$  is  $X[0] = 3$ .

$$\begin{aligned}
(d) \quad x[n] &= \frac{1}{316} \{X[0] + 2 \operatorname{Re}\{X[17]W_{316}^{-17n}\} + 2 \operatorname{Re}\{X[106]W_{316}^{-106n}\} + 2 \operatorname{Re}\{X[137]W_{316}^{-137n}\} \\
& + 2 \operatorname{Re}\{X[206]W_{316}^{-206n}\}\} = \frac{1}{316} \{3 + 2(1.5) \cos\left(\frac{17\pi n}{158}\right) + 2(-2.3) \sin\left(\frac{106\pi n}{158}\right) \\
& + 2(4.2) \cos\left(\frac{137\pi n}{158}\right) + 2(1.72) \sin\left(\frac{110\pi n}{158}\right) - 13\} \\
& = \frac{1}{316} \{-10 + 3 \cos\left(\frac{17\pi n}{158}\right) - 4.6 \sin\left(\frac{106\pi n}{158}\right) + 8.4 \cos\left(\frac{137\pi n}{158}\right) + 3.44 \sin\left(\frac{110\pi n}{158}\right)\}.
\end{aligned}$$

$$(e) \sum_{n=0}^{315} |x[n]|^2 = \frac{1}{316} \sum_{k=0}^{315} |X[k]|^2 = 0.6521.$$

**3.62**  $\{x[n]\} = \{-4, 5, 2, -3, 0, -2, 3, 4\}$ ,  $0 \leq n \leq 7$ . Let  $X[k]$  denote the 8-point DFT of  $x[n]$ . Consider the 8-point DFT  $Y[k] = W_4^{3k} X[k] = W_8^{6k} X[k]$ . Using the circular time-shifting property of the DFT given in Table 3.5 we observe that the IDFT  $y[n]$  of  $Y[k]$  is given by  $y[n] = x[\langle n - 6 \rangle_8]$ . Therefore,  $y[0] = x[\langle -6 \rangle_8] = x[2] = 2$ ,  $y[1] = x[\langle 1 - 6 \rangle_8] = x[3] = -3$ ,  $y[2] = x[\langle 2 - 6 \rangle_8] = x[4] = 0$ ,  $y[3] = x[\langle 3 - 6 \rangle_8] = x[5] = -2$ ,  $y[4] = x[\langle 4 - 6 \rangle_8] = x[6] = 3$ ,



$y[5] = x[\langle 5-6 \rangle_8] = x[7] = 4$ ,  $y[6] = x[\langle 6-6 \rangle_8] = x[0] = -4$ ,  $y[7] = x[\langle 7-6 \rangle_8] = x[1] = 5$ . Thus,  $\{y[n]\} = \{2, -3, 0, -2, 3, 4, -4, 5\}$ ,  $0 \leq n \leq 7$ .

**3.63**  $\{x[n]\} = \{1, -1, 2, 3, 0, 0\}$ ,  $0 \leq n \leq 5$ . Let  $X[k]$  denote the 6-point DFT of  $x[n]$ .

Consider the 6-point DFT  $Y[k] = W_3^{2k}X[k] = W_6^{4k}X[k]$ . Using the circular time-shifting property of the DFT given in Table 3.5 we observe that the IDFT  $y[n]$  of  $Y[k]$  is given by  $y[n] = x[\langle n-4 \rangle_6]$ . Therefore,  $y[0] = x[\langle -4 \rangle_6] = x[2] = 2$ ,  $y[1] = x[\langle -3 \rangle_6] = x[3] = 3$ ,  $y[2] = x[\langle -2 \rangle_6] = x[4] = 0$ ,  $y[3] = x[\langle -1 \rangle_6] = x[5] = 0$ ,  $y[4] = x[\langle 0 \rangle_6] = x[0] = 1$ ,  $y[5] = x[\langle 1 \rangle_6] = x[1] = -1$ . Thus,  $\{y[n]\} = \{2, 3, 0, 0, 1, -1\}$ ,  $0 \leq n \leq 5$ .

**3.64 (a)**  $y_L[0] = g[0]h[0] = -6$ ,

$$y_L[1] = g[0]h[1] + g[1]h[0] = 16,$$

$$y_L[2] = g[0]h[2] + g[1]h[1] + g[2]h[0] = 0,$$

$$y_L[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] = -19,$$

$$y_L[4] = g[1]h[3] + g[2]h[2] = 2,$$

$$y_L[5] = g[2]h[3] = 4.$$

**(b)**  $y_c[0] = g_e[0]h[0] + g_e[1]h[3] + g_e[2]h[2] + g_e[3]h[1] = g[0]h[0] + g[1]h[3] + g[2]h[2] = -4$ ,

$$y_c[1] = g_e[0]h[1] + g_e[1]h[0] + g_e[2]h[3] + g_e[3]h[2] = g[0]h[1] + g[1]h[0] + g[2]h[3] = 20,$$

$$y_c[2] = g_e[0]h[2] + g_e[1]h[1] + g_e[2]h[0] + g_e[3]h[3] = g[0]h[2] + g[1]h[1] + g[2]h[0] = 0,$$

$$y_c[3] = g_e[0]h[3] + g_e[1]h[2] + g_e[2]h[1] + g_e[3]h[0] = g[0]h[3] + g[1]h[2] + g[2]h[1] = -19.$$

$$(c) \begin{bmatrix} G_e[0] \\ G_e[1] \\ G_e[2] \\ G_e[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -7-j2 \\ -1 \\ -7+j2 \end{bmatrix}, \quad \begin{bmatrix} H[0] \\ H[1] \\ H[2] \\ H[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2+j5 \\ 5 \\ 2-j5 \end{bmatrix}.$$

$$\begin{bmatrix} Y_c[0] \\ Y_c[1] \\ Y_c[2] \\ Y_c[3] \end{bmatrix} = \begin{bmatrix} G_e[0] \cdot H[0] \\ G_e[1] \cdot H[1] \\ G_e[2] \cdot H[2] \\ G_e[3] \cdot H[3] \end{bmatrix} = \begin{bmatrix} -3 \\ -4-j39 \\ -5 \\ -4+j39 \end{bmatrix}. \quad \text{Therefore}$$

$$\begin{bmatrix} y_c[0] \\ y_c[1] \\ y_c[2] \\ y_c[3] \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} -3 \\ -4-j39 \\ -5 \\ -4+j39 \end{bmatrix} = \begin{bmatrix} -4 \\ 20 \\ 0 \\ -19 \end{bmatrix}.$$

**3.65** We need to show  $g[n] \circledast h[n] = h[n] \circledast g[n]$ .

$$\text{Let } x[n] = g[n] \circledast h[n] = \sum_{m=0}^{N-1} g[m]h[\langle n-m \rangle_N]$$

$$\text{and } y[n] = h[n] \circledast g[n] = \sum_{m=0}^{N-1} h[m]g[\langle n-m \rangle_N]$$

$$\begin{aligned}
&= \sum_{m=0}^n h[m]g[n-m] + \sum_{m=n+1}^{N-1} h[m]g[N+n-m] \\
&= \sum_{m=0}^n h[n-m]g[m] + \sum_{m=n+1}^{N-1} h[N+n-m]g[m] = \sum_{m=0}^{N-1} h[\langle n-m \rangle_N]g[m] = x[n].
\end{aligned}$$

Hence circular convolution is commutative.

**3.66 (a)** Let  $g[n] = x_1[n] \circledast x_2[n] = \sum_{m=0}^{N-1} x_1[m]x_2[\langle n-m \rangle_N]$ . Thus,

$$\sum_{n=0}^{N-1} g[n] = \sum_{m=0}^{N-1} x_1[m] \sum_{n=0}^{N-1} x_2[\langle n-m \rangle_N] = \left( \sum_{n=0}^{N-1} x_1[n] \right) \left( \sum_{m=0}^{N-1} x_2[m] \right).$$

$$\begin{aligned}
&\text{Similarly we can show that if } y[n] = g[n] \circledast x_3[n], \text{ then } \sum_{n=0}^{N-1} y[n] = \sum_{m=0}^{N-1} g[m] \sum_{n=0}^{N-1} x_3[\langle n-m \rangle_N] \\
&= \left( \sum_{n=0}^{N-1} g[n] \right) \left( \sum_{m=0}^{N-1} x_3[m] \right) \\
&= \left( \sum_{n=0}^{N-1} x_1[n] \sum_{m=0}^{N-1} x_2[m] \right) \left( \sum_{m=0}^{N-1} x_3[m] \right).
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad &\sum_{n=0}^{N-1} (-1)^n g[n] = \sum_{m=0}^{N-1} x_1[m] \sum_{n=0}^{N-1} x_2[\langle n-m \rangle_N] (-1)^n \\
&= \left( \sum_{m=0}^{N-1} x_1[m] \right) \left( \sum_{n=0}^{m-1} x_2[N+n-m] (-1)^n + \sum_{n=m}^{N-1} x_2[n-m] (-1)^n \right).
\end{aligned}$$

Replacing  $n$  by  $N+n-m$  in the first sum and by  $n-m$  in the second we obtain

$$\begin{aligned}
\sum_{n=0}^{N-1} (-1)^n g[n] &= \left( \sum_{m=0}^{N-1} x_1[m] \right) \left( \sum_{n=0}^{m-1} x_2[n] (-1)^{n-N+m} + \sum_{n=0}^{N-1} x_2[n] (-1)^{n+m} \right) \\
&= \left( \sum_{n=0}^{N-1} (-1)^n x_1[n] \right) \left( \sum_{n=0}^{N-1} (-1)^n x_2[n] \right).
\end{aligned}$$

$$\sum_{n=0}^{N-1} (-1)^n y[n] = \left( \sum_{n=0}^{N-1} (-1)^n g[n] \right) \left( \sum_{n=0}^{N-1} (-1)^n x_3[n] \right) = \left( \sum_{n=0}^{N-1} (-1)^n x_1[n] \right) \left( \sum_{n=0}^{N-1} (-1)^n x_2[n] \right) \left( \sum_{n=0}^{N-1} (-1)^n x_3[n] \right).$$

$$\text{3.67 } y[n] = \cos\left(\frac{2\pi n}{N}\right) x[n] = \frac{x[n]}{2} \left( e^{-j2\pi n/N} + e^{j2\pi n/N} \right) = \frac{1}{2} x[n] W_N^{nl} + \frac{1}{2} x[n] W_N^{-nl}.$$

$$\text{Hence } Y[k] = \frac{1}{2} X[\langle k+1 \rangle_N] + \frac{1}{2} X[\langle k-1 \rangle_N].$$

$$\text{3.68 } y[n] = x[4n], \quad 0 \leq n \leq \frac{N}{4} - 1. \quad \text{Therefore, } Y[k] = \sum_{n=0}^{\frac{N}{4}-1} y[n] W_{N/4}^{nk} = \sum_{n=0}^{\frac{N}{4}-1} x[4n] W_{N/4}^{nk}.$$

Now,  $x[4n] = \frac{1}{N} \sum_{m=0}^{N-1} X[m] W_N^{-4mn} = \frac{1}{N} \sum_{m=0}^{N-1} X[m] W_{N/4}^{-mn}$ . Hence,

$$Y[k] = \frac{1}{N} \sum_{n=0}^{\frac{N}{4}-1} \sum_{m=0}^{N-1} X[m] W_{N/4}^{-mn} W_{N/4}^{nk} = \frac{1}{N} \sum_{m=0}^{N-1} X[m] \sum_{n=0}^{\frac{N}{4}-1} W_{N/4}^{(k-m)n}. \text{ Since,}$$

$$\sum_{n=0}^{\frac{N}{4}-1} W_{N/4}^{(k-m)n} = \begin{cases} \frac{N}{4}, & m = k, k + \frac{N}{4}, k + \frac{2N}{4}, k + \frac{3N}{4}, k + \frac{4N}{4}, \\ 0, & \text{elsewhere,} \end{cases} \text{ Thus,}$$

$$Y[k] = \frac{1}{4} \left( X[k] + X[k + \frac{N}{4}] + X[k + \frac{2N}{4}] + X[k + \frac{3N}{4}] \right).$$

$V[k] = [-2 + j3, 1 + j5, -4 + j7, 2 + j6, -1 - j3, 4 - j, 3 + j8, j6]$ . Hence,

$V^*[\langle -k \rangle_8] = [-2 - j3, -j6, 3 - j8, 4 + j, -1 + j3, 2 - j6, -4 - j7, 1 - j5]$ . Therefore,,

$X[k] = [-2, 0.5 - j0.5, -0.5 - j0.5, 3 + j3.5, -1, 3 - j3.5, -0.5 + j0.5, 0.5 + j0.5]$  and

$Y[k] = [3, 5.5 - j0.5, 7.5 + j3.5, 2.5 + j, -3, 2.5 - j, 7.5 - j3.5, 5.5 + j0.5]$

**3.69**  $v[n] = x[n] + jy[n]$ . Hence,  $X[k] = \frac{1}{2} \{V[k] + V^*[\langle -k \rangle_8]\}$  is the 8-point DFT of  $x[n]$ , and

$Y[k] = \frac{1}{2j} \{V[k] - V^*[\langle -k \rangle_8]\}$  is the 8-point DFT of  $y[n]$ . Now,

$V[k] = [-2 + j3, 1 + j5, -4 + j7, 2 + j6, -1 - j3, 4 - j, 3 + j8, j6]$

$V^*[\langle -k \rangle_8] = [-2 - j3, -j6, 3 - j8, 4 + j, -1 + j3, 2 - j6, -4 - j7, 1 - j5]$ . Therefore,

$X[k] = [-0.2, 0.5 - j0.5, -0.5 - j0.5, 3 + j3.5, -1, 3 - j3.5, -0.5 + j0.5, 0.5 + j0.5]$

$Y[k] = [3, 5.5 - j0.5, 7.5 + j3.5, 2.5 + j, -3, 2.5 - j, 7.5 - j3.5, 5.5 + j0.5]$

**3.70**  $v[n] = g_e[n] + jh[n] = [-3 + j2, 2 - j4, 4, j]$ . Therefore,

$$\begin{bmatrix} V[0] \\ V[1] \\ V[2] \\ V[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} -3 + j2 \\ 2 - j4 \\ 4 \\ j \end{bmatrix} = \begin{bmatrix} 3 - j \\ -12 \\ -1 + j5 \\ -2 + j4 \end{bmatrix}, \text{ i.e., } \{V[k]\} = [3 - j, -12, -1 + j5, -2 + j4].$$

Thus,  $\{V^*[\langle -k \rangle_4]\} = [3 + j, -2 - j4, -1 - j5, -12]$

Therefore,  $G_e[k] = \frac{1}{2} \{V[k] + V^*[\langle -k \rangle_4]\} = [3, -7 - j2, -1, -7 + j2]$  and

$H[k] = \frac{1}{2j} \{V[k] - V^*[\langle -k \rangle_4]\} = [-1, 2 + j5, 5, 2 - j5]$ .

**3.71**  $v[n] = g[n] + jh[n] = [-2 + j, 1 + j2, -3 - j3, 4 + j2]$ . Therefore,

$$\begin{bmatrix} V[0] \\ V[1] \\ V[2] \\ V[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} -2 + j \\ 1 + j2 \\ -3 - j3 \\ 4 + j2 \end{bmatrix} = \begin{bmatrix} j2 \\ 1 + j7 \\ -10 - j6 \\ 1 + j \end{bmatrix}, \text{ i.e., } \{V[k]\} = [j2, 1 + j7, -10 - j6, 1 + j]$$

Thus,  $\{V^*[\langle -k \rangle_4]\} = [-j2, 1-j, -10+j6, 1-j7]$ .

Therefore,  $G_e[k] = \frac{1}{2}\{V[k] + V^*[\langle -k \rangle_4]\} = [0, 1+j3, -10, 1-j3]$  and

$$H[k] = \frac{1}{2j}\{V[k] - V^*[\langle -k \rangle_4]\} = [2, 4, -6, 4].$$

**3.72 (a)** Let  $p[n] = \text{IDFT}\{P[k]\}$ . Thus,

$$\begin{bmatrix} p[0] \\ p[1] \\ p[2] \\ p[3] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 4 \\ 1+j7 \\ 2 \\ 1-j7 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 8 \\ -12 \\ 4 \\ 16 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 4 \end{bmatrix}. \text{ Similarly, let } d[n] = \text{IDFT}\{D[k]\}.$$

$$\begin{bmatrix} d[0] \\ d[1] \\ d[2] \\ d[3] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 4.5 \\ 1.5+j \\ -5.5 \\ 1.5-j \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 8 \\ -4 \\ 12 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 2 \\ -1 \\ 2 \end{bmatrix}. \text{ Therefore,}$$

$$X(e^{j\omega}) = \frac{2 - 3e^{-j\omega} + e^{-j2\omega} + 4e^{-j3\omega}}{0.5 + 2e^{-j\omega} - e^{-j2\omega} + 3e^{-j3\omega}}.$$

**(b)** Let  $p[n] = \text{IDFT}\{P[k]\}$ . Thus,  $\begin{bmatrix} p[0] \\ p[1] \\ p[2] \\ p[3] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 7 \\ 7+j2 \\ -9 \\ 7-j2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \\ 5 \end{bmatrix}$ . Similarly, let

$d[n] = \text{IDFT}\{D[k]\}$ . Thus,  $\begin{bmatrix} d[0] \\ d[1] \\ d[2] \\ d[3] \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 0 \\ 4+j6 \\ -4 \\ 4-j6 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -3 \\ 4 \end{bmatrix}$ . Therefore,

$$X(e^{j\omega}) = \frac{3 + 3e^{-j\omega} - 4e^{-j2\omega} + 5e^{-j3\omega}}{1 - 2e^{-j\omega} - 3e^{-j2\omega} + 4e^{-j3\omega}}.$$

**3.73**  $X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$  and  $\hat{X}[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/M}$ .

$$\begin{aligned} \text{Now } \hat{x}[n] &= \frac{1}{M} \sum_{k=0}^{M-1} \hat{X}[k] W_M^{-nk} = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{m=0}^{M-1} x[m] e^{-j2\pi km/M} W_M^{-nk} \\ &= \frac{1}{M} \sum_{m=0}^{N-1} x[m] \sum_{k=0}^{M-1} e^{-j2\pi k(m-n)/M} = \sum_{r=-\infty}^{\infty} x[n+rM]. \end{aligned}$$

Thus  $\hat{x}[n]$  is obtained by shifting  $x[n]$  by multiples of  $M$  and adding the shifted copies. Since the new sequence is obtained by shifting in multiples of  $M$ , hence to recover the original sequence take any  $M$  consecutive samples. This would be true only if the shifted copies of  $x[n]$  did not overlap with each other, that is, if only if  $M \geq N$ .

**3.74 (a)**  $X(e^{j\omega}) = \sum_{n=0}^7 x[n]e^{-j\omega n}$ . Therefore,  $X_1[k] = \sum_{n=0}^7 x[n]e^{-j2\pi kn/10}$ . Hence,

$$\begin{aligned}
x_1[n] &= \frac{1}{10} \sum_{k=0}^9 X_1[k] e^{j2\pi kn/10} = \frac{1}{10} \sum_{k=0}^9 \left( \sum_{m=0}^7 x[m] e^{j2\pi km/10} \right) e^{j2\pi kn/10} \\
&= \frac{1}{10} \sum_{m=0}^7 x[m] \sum_{k=0}^9 e^{j2\pi k(n-m)/10} = \sum_{r=-\infty}^{\infty} x[n+10r] \text{ using the result of Problem 3.74.}
\end{aligned}$$

Since  $M = 10$  and  $N = 8$ ,  $M > N$ , and hence  $x[n]$  is recoverable from  $x_1[n]$ . In fact  $\{x_1[n]\} = \{1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 0\}$  and  $x[n]$  is given by the first 8 samples of  $x_1[n]$ .

(b) Here,  $X_2[k] = \sum_{n=0}^7 x[n] e^{-j2\pi kn/6}$ . Hence,

$$\begin{aligned}
x_2[n] &= \frac{1}{6} \sum_{k=0}^5 X_2[k] e^{j2\pi kn/6} = \frac{1}{10} \sum_{k=0}^5 \left( \sum_{m=0}^7 x[m] e^{j2\pi km/6} \right) e^{j2\pi kn/6} \\
&= \frac{1}{10} \sum_{m=0}^7 x[m] \sum_{k=0}^5 e^{j2\pi k(n-m)/6} = \sum_{r=-\infty}^{\infty} x[n+6r]. \text{ Since } M = 6 \text{ and } N = 8, M < N, \text{ the sequence } x[n]
\end{aligned}$$

is not recoverable from  $x_2[n]$ . In fact,  $x_2[n] = \{2 \ 2 \ 1 \ 1 \ 1 \ 1\}$ . As there is time-domain aliasing,  $x[n]$  is not recoverable from  $x_2[n]$ .

**3.75** Since  $F^{-1} = \frac{1}{N}F$  thus  $F = NF^{-1}$ . Thus

$$\begin{aligned}
y[n] &= F \{ F \{ F \{ F \{ F \{ F \{ x[n] \} \} \} \} \} \} \\
&= NF^{-1} \{ F \{ NF^{-1} \{ F \{ NF^{-1} \{ F \{ x[n] \} \} \} \} \} \} = N^3 x[n].
\end{aligned}$$

**3.76**  $y[n] = x[n] \circledast h[n] = \sum_{k=0}^{39} x[k] h[n-k] = \sum_{k=0}^{39} h[k] x[n-k] = \sum_{k=12}^{27} h[k] x[n-k]$ .

$$u[n] = x[n] \circledcirc h[n] = \sum_{k=0}^{39} h[k] x[\langle n-k \rangle_{40}] = \sum_{k=12}^{27} h[k] x[\langle n-k \rangle_{40}].$$

Now for  $n \geq 27$ ,  $x[\langle n-k \rangle_{40}] = x[n-k]$ . Thus  $u[n] = y[n]$  for  $27 \leq n \leq 39$ .

**3.77 (a) Overlap and add method:** Since the impulse response is of length 55 and the DFT size to be used is 64, hence the number of data samples required for each convolution will be  $64 - 54 = 10$ . Thus the total number of DFT's required for length-1100 data sequence is

$$\left\lceil \frac{1100}{10} \right\rceil = 110. \text{ Also the DFT of the impulse response needs to be computed once. Hence, the total number of DFT's used are } = 110 + 1 = 111. \text{ The total number of IDFT's used are } = 110.$$

**(b) Overlap and save method:** In this since the first  $55 - 1 = 54$  points are lost, we need to pad the data sequence with 54 zeros for a total length of 1154. Again each convolution will result in  $64 - 54 = 10$  correct values. Thus the total number of DFT's required for the data are

$$\text{thus } \left\lceil \frac{1154}{10} \right\rceil = 116. \text{ Again 1 DFT is required for the impulse response. Thus}$$

The total number of DFT's used are  $= 116 + 1 = 117$ .

The total number of IDFT's used are  $= 116$ .

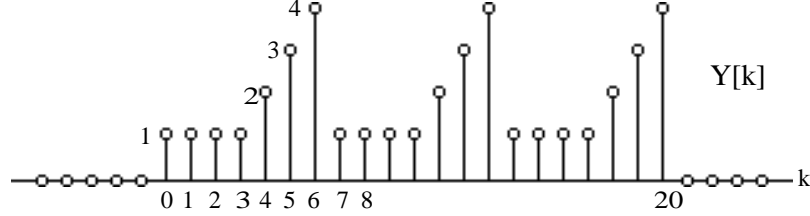
$$3.78 \text{ (a)} \quad y[n] = \begin{cases} x[n/L], & n = 0, L, 2L, \dots, (N-1)L, \\ 0, & \text{elsewhere.} \end{cases}$$

$$Y[k] = \sum_{n=0}^{NL-1} y[n] W_{NL}^{nk} = \sum_{n=0}^{N-1} x[n] W_{NL}^{nLk} = \sum_{n=0}^{N-1} x[n] W_N^{nk}.$$

For  $k \geq N$ , let  $k = k_0 + rN$  where  $k_0 = \langle k \rangle_N$ . Then,

$$Y[k] = Y[k_0 + rN] = \sum_{n=0}^{N-1} x[n] W_N^{n(k_0+rN)} = \sum_{n=0}^{N-1} x[n] W_N^{nk_0} = X[k_0] = X[\langle k \rangle_N].$$

(b) Since  $Y[k] = X[\langle k \rangle_7]$  for  $k = 0, 1, 2, \dots, 20$ , a sketch of  $Y[k]$  is thus as shown below.



$$3.79 \quad x_0[n] = x[2n+1] + x[2n], \quad x_1[n] = x[2n+1] - x[2n], \quad y_1[n] = y[2n+1] + y[2n], \text{ and}$$

$y_0[n] = y[2n+1] - y[2n]$ ,  $0 \leq n \leq \frac{N}{2} - 1$ . Since  $x[n]$  and  $y[n]$  are real, symmetric sequences, it follows that  $x_0[n]$  and  $y_0[n]$  are real, symmetric sequences, and  $x_1[n]$  and  $y_1[n]$  are real, anti-symmetric sequences. Now consider, the  $(N/2)$ -length sequence

$$u[n] = x_0[n] + y_1[n] + j(x_1[n] + y_0[n]). \text{ Its conjugate sequence is given by}$$

$$u^*[n] = x_0[n] + y_1[n] - j(x_1[n] + y_0[n]). \text{ Next we observe that}$$

$$u[\langle -n \rangle_{N/2}] = x_0[\langle -n \rangle_{N/2}] + y_1[\langle -n \rangle_{N/2}] + j(x_1[\langle -n \rangle_{N/2}] + y_0[\langle -n \rangle_{N/2}])$$

$$= x_0[n] - y_1[n] + j(-x_1[n] + y_0[n]). \text{ Its conjugate sequence is given by}$$

$$u^*[\langle -n \rangle_{N/2}] = x_0[n] - y_1[n] - j(-x_1[n] + y_0[n]).$$

By adding the last 4 sequences we get

$$4x_0[n] = u[n] + u^*[n] + u[\langle -n \rangle_{N/2}] + u^*[\langle -n \rangle_{N/2}].$$

From Table 3.6, if  $U[k] = \text{DFT}\{u[n]\}$ , then  $U^*[\langle -k \rangle_{N/2}] = \text{DFT}\{u^*[n]\}$ ,

$$U^*[k] = \text{DFT}\{u^*[\langle -n \rangle_{N/2}]\}, \text{ and } U[\langle -k \rangle_{N/2}] = \text{DFT}\{u[\langle -n \rangle_{N/2}]\}. \text{ Thus,}$$

$$X_0[k] = \text{DFT}\{x_0[n]\} = \frac{1}{4}(U[k] + U^*[\langle -k \rangle_{N/2}] + U[\langle -k \rangle_{N/2}] + U^*[k]). \text{ Similarly,}$$

$$j4x_1[n] = u[n] - u^*[n] - u[\langle -n \rangle_{N/2}] + u^*[\langle -n \rangle_{N/2}]. \text{ Hence,}$$

$$X_1[k] = \text{DFT}\{x_1[n]\} = \frac{1}{4j}(U[k] - U^*[\langle -k \rangle_{N/2}] - U[\langle -k \rangle_{N/2}] + U^*[k]). \text{ Likewise,}$$

$$4y_1[n] = u[n] - u[\langle -n \rangle_{N/2}] + u^*[n] - u^*[\langle -n \rangle_{N/2}]. \text{ Thus,}$$

$$Y_1[k] = \text{DFT}\{y_1[n]\} = \frac{1}{4}(U[k] - U[\langle -k \rangle_{N/2}] + U^*[\langle -k \rangle_{N/2}] - U^*[k]). \text{ Finally,}$$

$$j4y_0[n] = u[n] + u[\langle -n \rangle_{N/2}] - u^*[n] - u^*[\langle -n \rangle_{N/2}]. \text{ Hence,}$$

$$Y_0[k] = \text{DFT}\{y_0[n]\} = \frac{1}{4j}(U[k] + U[\langle -k \rangle_{N/2}] - U^*[\langle -k \rangle_{N/2}] - U^*[k]).$$

**3.80**  $g[n] = \frac{1}{2}(x[2n] + x[2n+1])$ ,  $h[n] = \frac{1}{2}(x[2n] - x[2n+1])$ ,  $0 \leq n \leq \frac{N}{2} - 1$ . Solving for  $x[2n]$  and  $x[2n+1]$ , we get  $x[2n] = g[n] + h[n]$  and  $x[2n+1] = g[n] - h[n]$ ,  $0 \leq n \leq \frac{N}{2} - 1$ . Therefore,

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} x[n]z^{-n} = \sum_{n=0}^{\frac{N}{2}-1} x[2n]z^{-n} + z^{-1} \sum_{n=0}^{\frac{N}{2}-1} x[2n+1]z^{-n} \\ &= \sum_{n=0}^{\frac{N}{2}-1} (g[n] + h[n])z^{-n} + z^{-1} \sum_{n=0}^{\frac{N}{2}-1} (g[n] - h[n])z^{-n} = (1+z^{-1}) \sum_{n=0}^{\frac{N}{2}-1} g[n]z^{-n} + (1-z^{-1}) \sum_{n=0}^{\frac{N}{2}-1} h[n]z^{-n}. \end{aligned}$$

Hence,  $X[k] = X(z)|_{z=W_N^k} = (1+W_N^{-k})G[\langle k \rangle_{N/2}] + (1-W_N^{-k})H[\langle k \rangle_{N/2}]$ ,  $0 \leq k \leq N-1$ .

**3.81**  $g[n] = a_1x[2n] + a_2x[2n+1]$  and  $h[n] = a_3x[2n] + a_4x[2n+1]$ , with  $a_1a_4 \neq a_2a_3$ . Solving for  $x[2n]$  and  $x[2n+1]$ , we get  $x[2n] = \frac{a_4g[n] - a_2h[n]}{a_1a_4 - a_2a_3}$ , and  $x[2n+1] = \frac{-a_3g[n] + a_1h[n]}{a_1a_4 - a_2a_3}$ . Therefore

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} x[n]z^{-n} = \sum_{n=0}^{\frac{N}{2}-1} x[2n]z^{-n} + z^{-1} \sum_{n=0}^{\frac{N}{2}-1} x[2n+1]z^{-n} \\ &= \sum_{n=0}^{\frac{N}{2}-1} \left( \frac{a_4g[n] - a_2h[n]}{a_1a_4 - a_2a_3} \right) z^{-n} + z^{-1} \sum_{n=0}^{\frac{N}{2}-1} \left( \frac{-a_3g[n] + a_1h[n]}{a_1a_4 - a_2a_3} \right) z^{-n} \\ &= \frac{1}{a_1a_4 - a_2a_3} (a_4 - a_3z^{-1}) \sum_{n=0}^{\frac{N}{2}-1} g[n]z^{-n} + (-a_2 + a_1z^{-1}) \sum_{n=0}^{\frac{N}{2}-1} h[n]z^{-n}. \text{ Hence,} \\ X[k] &= \frac{1}{a_1a_4 - a_2a_3} (a_4 - a_3W_N^{-nk})G[\langle k \rangle_{N/2}] + (-a_2 + a_1W_N^{-nk})G[\langle k \rangle_{N/2}], \quad 0 \leq k \leq N-1. \end{aligned}$$

**3.82**  $X_{\text{GDFT}}[k, a, b] = \sum_{n=0}^{N-1} x[n] \exp\left(-j \frac{2\pi(n+a)(k+b)}{N}\right)$ .

$$\begin{aligned} x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k, a, b] \exp\left(j \frac{2\pi(n+a)(k+b)}{N}\right) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} x[r] \exp\left(-j \frac{2\pi(r+a)(k+b)}{N}\right) \exp\left(j \frac{2\pi(n+a)(k+b)}{N}\right) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} x[r] \exp\left(j \frac{2\pi(n+a-r-a)(k+b)}{N}\right) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} x[r] \exp\left(j \frac{2\pi(n-r)(k+b)}{N}\right) \\ &= \frac{1}{N} \sum_{r=0}^{N-1} x[r] \sum_{k=0}^{N-1} \exp\left(j \frac{2\pi(n-r)(k+b)}{N}\right) = \frac{1}{N} \cdot x[n] \cdot N = x[n], \end{aligned}$$

as from Eq. (3.28),  $\sum_{k=0}^{N-1} \exp\left(j \frac{2\pi(n-r)(k+b)}{N}\right) = \begin{cases} N, & \text{if } n = r, \\ 0, & \text{otherwise.} \end{cases}$

**3.83**  $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} x[n]z^{-n}$ . Therefore,

$$\lim_{z \rightarrow \infty} X(z) = \lim_{z \rightarrow \infty} \sum_{n=0}^{\infty} x[n]z^{-n} = \lim_{z \rightarrow \infty} x[0] + \lim_{z \rightarrow \infty} \sum_{n=1}^{\infty} x[n]z^{-n} = x[0].$$

**3.84**  $G(z) = \frac{(z+0.4)(z-0.91)(z^2+0.3z+0.4)}{(z^2-0.6z+0.6)(z^2+3z+5)}$ .  $G(z)$  has poles at  $z = 0.3 \pm j \frac{\sqrt{2.04}}{2}$  and  $z = \frac{3}{2} \pm j \frac{\sqrt{11}}{2}$ .

Hence, there are possible ROCs.

(i)  $\neg_1$ :  $|z| \leq \sqrt{0.6}$ . The inverse z-transform  $g[n]$  in this case is a left-sided sequence.

(ii)  $\neg_2$ :  $\sqrt{0.6} \leq |z| \leq \sqrt{5}$ . The inverse z-transform  $g[n]$  in this case is a two-sided sequence.

(iii)  $\neg_3$ :  $|z| \geq \sqrt{5}$ . The inverse z-transform  $g[n]$  in this case is a right-sided sequence.

**3.85 (a)** (i)  $x_1[n] = (0.4)^n \mu[n]$  is a right-sided sequence. Hence, the ROC of its z-transform is

exterior to a circle. Thus,  $X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]z^{-n} = \sum_{n=0}^{\infty} (0.4)^n z^{-n} = \frac{1}{1-0.4z^{-1}}$ ,  $|z| > 0.4$

The ROC of  $X_1(z)$  is given by  $\neg_1$ :  $|z| > 0.4$ .

(ii)  $x_2[n] = (-0.6)^n \mu[n]$  is a right-sided sequence. Hence, the ROC of its z-transform is exterior to

a circle. Thus,  $X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n]z^{-n} = \sum_{n=0}^{\infty} (-0.6)^n z^{-n} = \frac{1}{1+0.6z^{-1}}$ ,  $|z| > 0.6$

The ROC of  $X_2(z)$  is given by  $\neg_2$ :  $|z| > 0.6$ .

(iii)  $x_3[n] = (0.3)^n \mu[n-4]$  is a right-sided sequence. Hence, the ROC of its z-transform is

exterior to a circle. Thus,  $X_3(z) = \sum_{n=-\infty}^{\infty} x_3[n]z^{-n} = \sum_{n=4}^{\infty} (0.3)^n z^{-n}$

$X_3(z) = \sum_{n=-\infty}^{\infty} x_3[n]z^{-n} = \sum_{n=4}^{\infty} (0.3)^n z^{-n} = \frac{(0.3)^4 z^{-4}}{1-0.3z^{-1}}$ ,  $|z| > 0.3$ . The ROC of  $X_3(z)$  is given by

$\neg_3$ :  $|z| > 0.2$ .



(iv)  $x_4[n] = (-0.3)^n \mu[-n-2]$  is a left-sided sequence. Hence, the ROC of its z-transform is interior to a circle. Thus,

$$X_4(z) = \sum_{n=-\infty}^{\infty} x_4[n]z^{-n} = \sum_{n=-\infty}^{-2} (-0.3)^n z^{-n} = \sum_{m=2}^{\infty} (-0.3)^{-m} z^m = -\frac{2+a^{-1}z}{1+0.3z^{-1}}, \quad |z| < 0.3.$$

The ROC of  $X_4(z)$  is given by  $\neg_4$ :  $|z| < 0.3$ .

**(b)** (i) Now, the ROC of  $X_1(z)$  is given by  $\neg_1$ :  $|z| > 0.4$  and the ROC of  $X_2(z)$  is given by  $\neg_2$ :  $|z| > 0.6$ . Hence, the ROC of  $Y_1(z)$  is given by  $\neg_1 \cap \neg_2 = \neg_2$ :  $|z| > 0.6$

(ii) The ROC of  $Y_2(z)$  is given by  $\neg_1 \cap \neg_3 = \neg_1$ :  $|z| > 0.4$ .

(iii) The ROC of  $Y_3(z)$  is given by  $\neg_1 \cap \neg_4 = \emptyset$ . Hence, the z-transform of the sequence  $y_3[n]$  does not converge anywhere in the z-plane.

(iv) The ROC of  $Y_4(z)$  is given by  $\neg_2 \cap \neg_3 = \neg_2$ :  $|z| > 0.6$ .

(v) The ROC of  $Y_5(z)$  is given by  $\neg_2 \cap \neg_4 = \emptyset$ . Hence, the z-transform of the sequence  $y_5[n]$  does not converge anywhere in the z-plane.

(vi) The ROC of  $Y_6(z)$  is given by  $\neg_3 \cap \neg_4 = \emptyset$ . Hence, the z-transform of the sequence  $y_6[n]$  does not converge anywhere in the z-plane.

**3.86 (a)**  $Z\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} = \delta[0] = 1$ , which converges everywhere in the z-plane.

$$\text{(b)} \quad Z\{\alpha^n \mu[n]\} = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n]z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1-\alpha z^{-1}}, \quad \forall |z| > |\alpha|$$

**(c)** See Example 3.29.

**(d)**  $x[n] = r^n \sin(\omega_0 n) \mu[n] = \frac{r^n}{2j} (e^{j\omega_0 n} - e^{-j\omega_0 n}) \mu[n]$ . Using the results of (iii) and the linearity property of the z-transform we obtain

$$\begin{aligned} Z\{r^n \sin(\omega_0 n) \mu[n]\} &= \frac{1}{2j} \left( \frac{1}{1 - re^{j\omega_0} z^{-1}} \right) - \frac{1}{2j} \left( \frac{1}{1 - re^{-j\omega_0} z^{-1}} \right) \\ &= \frac{\frac{r}{2j} (e^{j\omega_0} - e^{-j\omega_0}) z^{-1}}{1 - rz^{-1} (e^{j\omega_0} + e^{-j\omega_0}) + r^2 z^{-2}} = \frac{r \sin(\omega_0) z^{-1}}{1 - 2r \cos(\omega_0) z^{-1} + r^2 z^{-2}}, \quad \forall |z| > |r| \end{aligned}$$

**3.87 (a)**  $x_1[n] = 6[(0.5)^n - (0.3)^n] \mu[n] = 6(0.5)^n \mu[n] - 6(0.3)^n \mu[n]$ . Therefore,

$$X_1(z) = \frac{6}{1-0.5z^{-1}} - \frac{6}{1-0.3z^{-1}} = \frac{6(1-0.3z^{-1}-1+0.5z^{-1})}{(1-0.5z^{-1})(1-0.3z^{-1})} = \frac{1.2z^{-1}}{(1-0.5z^{-1})(1-0.3z^{-1})}, \quad |z| > 0.5$$

(b)  $x_2[n] = -6(0.3)^n \mu[n] - 6(0.5)^n \mu[-n-1]$ . Now,  $Z\{-6(0.3)^n \mu[n]\} = \frac{-6}{1-0.3z^{-1}}$ ,  $|z| > 0.3$  and

$Z\{-6(0.5)^n \mu[-n-1]\} = \frac{6}{1-0.5z^{-1}}$ ,  $|z| < 0.5$ . Therefore,

$$X_2(z) = \frac{6}{1-0.5z^{-1}} - \frac{6}{1-0.3z^{-1}} = \frac{1.2z^{-1}}{(1-0.5z^{-1})(1-0.3z^{-1})}, \quad 0.3 < |z| < 0.5.$$

(c)  $x_3[n] = 6(0.3)^n \mu[-n-1] - 6(0.5)^n \mu[-n-1]$ . Now,  $Z\{6(0.3)^n \mu[-n-1]\} = \frac{-6}{1-0.3z^{-1}}$ ,  $|z| < 0.3$

and  $Z\{-6(0.5)^n \mu[-n-1]\} = \frac{6}{1-0.5z^{-1}}$ ,  $|z| < 0.5$ . Therefore,

$$X_3(z) = \frac{6}{1-0.5z^{-1}} - \frac{6}{1-0.3z^{-1}} = \frac{1.2z^{-1}}{(1-0.5z^{-1})(1-0.3z^{-1})}, \quad |z| < 0.3.$$

Note:  $X_1(z) = X_2(z) = X_3(z)$ , but their ROCs are all different.

**3.88 (a)**  $x_1[n] = \alpha^n \mu[n] + \beta^n \mu[n]$  with  $|\beta| > |\alpha|$ . Note that  $x_1[n]$  is a right-sided sequence. Hence, the

ROC of its z-transform is exterior to a circle. Now,  $Z\{\alpha^n \mu[n]\} = \frac{1}{1-\alpha z^{-1}}$ , with ROC given

by  $|z| > |\alpha|$  and  $Z\{\beta^n \mu[n]\} = \frac{1}{1-\beta z^{-1}}$ , with ROC given by  $|z| > |\beta|$ . Hence,

$$X_1(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta z^{-1}} = \frac{1-(\alpha+\beta)z^{-1}+z^{-2}}{(1-\alpha z^{-1})(1-\beta z^{-1})}, \quad |z| > |\beta|$$

(b)  $x_2[n] = \alpha^n \mu[-n-1] + \beta^n \mu[n]$ . Note that  $x_2[n]$  is a two-sided sequence. Now,

$Z\{\alpha^n \mu[-n-1]\} = \frac{1}{1-\alpha z^{-1}}$ , with ROC given by  $|z| < |\alpha|$  and  $Z\{\beta^n \mu[n]\} = \frac{1}{1-\beta z^{-1}}$ , with ROC

given by  $|z| > |\beta|$ . Since the regions  $|z| < |\alpha|$  and  $|z| > |\beta|$  do not intersect, the z-transform of  $x_2[n]$  does not converge.

(c)  $x_3[n] = \alpha^n \mu[n] + \beta^n \mu[-n-1]$ . Note, is a two-sided sequence. Now,  $Z\{\alpha^n \mu[n]\} = \frac{1}{1-\alpha z^{-1}}$ ,

with ROC given by  $|z| > |\alpha|$  and  $Z\{\beta^n \mu[-n-1]\} = \frac{1}{1-\beta z^{-1}}$ , with ROC given by  $|z| < |\beta|$ . Since the

regions  $|z| > |\alpha|$  and  $|z| < |\beta|$  intersect, the z-transform  $X_3(z)$  of  $x_3[n]$  converges and is given by

$$X_3(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta z^{-1}} = \frac{1-(\alpha+\beta)z^{-1}+z^{-2}}{(1-\alpha z^{-1})(1-\beta z^{-1})}, \quad \text{Its ROC is an annular region given by}$$

$|\alpha| < |z| < |\beta|$

**3.89 (a)** Let  $y[n] = \alpha g[n] + \beta h[n]$ . Then,

$Y(z) = \sum_{n=-\infty}^{\infty} (\alpha g[n] + \beta h[n]) z^{-n} = \sum_{n=-\infty}^{\infty} \alpha g[n] z^{-n} + \sum_{n=-\infty}^{\infty} \beta h[n] z^{-n} = \alpha G(z) + \beta H(z)$ . In this case,  $Y(z)$  will converge wherever both  $G(z)$  and  $H(z)$  converge. Thus, the ROC of  $Y(z)$  is given by  $R_g \cap R_h$  where  $R_g$  is the ROC of  $G(z)$  and  $R_h$  is the ROC of  $H(z)$ .

**(b)**  $y[n] = g[-n]$ . Then,  $Y(z) = \sum_{n=-\infty}^{\infty} g[-n] z^{-n} = \sum_{n=-\infty}^{\infty} g[n] z^n = \sum_{n=-\infty}^{\infty} g[n] (1/z)^{-n} = G(1/z)$ .  $Y(z)$  will converge wherever  $G(1/z)$  converges. Hence, if  $R_g$  is the ROC of  $G(z)$ , then the ROC of  $Y(z)$  is given by  $1/R_g$  is the ROC of  $G(z)$ .

**(c)**  $y[n] = g[n - n_0]$ . Hence  $Y(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{\infty} g[n - n_0] z^{-n} = \sum_{m=-\infty}^{\infty} g[m] z^{-(m+n_0)}$   
 $= z^{-n_0} \sum_{m=-\infty}^{\infty} g[m] z^{-m} = z^{-n_0} G(z)$ .

In this case the ROC of  $Y(z)$  is the same as that of  $G(z)$  except for the possible addition or elimination of the point  $z = 0$  or  $z = \infty$  (due to the factor  $z^{-n_0}$ ).

**(d)**  $y[n] = \alpha^n g[n]$ . Hence,  $Y(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{\infty} g[n] (z\alpha^{-1})^{-n} = G(z/\alpha)$ .

The ROC of  $Y(z)$  is  $\alpha R_g$ .

**(e)**  $y[n] = ng[n]$ . Hence  $Y(z) = \sum_{n=-\infty}^{\infty} ng[n] z^{-n}$ .

Now  $G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$ . Thus,  $\frac{dG(z)}{dz} = - \sum_{n=-\infty}^{\infty} ng[n] z^{-n-1} \Rightarrow z \frac{dG(z)}{dz} = - \sum_{n=-\infty}^{\infty} ng[n] z^{-n}$ .

Thus  $Y(z) = -z \frac{dG(z)}{dz}$ .

**(f)**  $y[n] = g[n] \otimes h[n] = \sum_{k=-\infty}^{\infty} g[k] h[n - k]$ . Hence,

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} g[k] h[n - k] \right) z^{-n} = \sum_{k=-\infty}^{\infty} g[k] \sum_{n=-\infty}^{\infty} h[n - k] z^{-n} \\ &= \sum_{k=-\infty}^{\infty} g[k] H(z) z^{-k} = H(z) G(z). \end{aligned}$$

In this case also  $Y(z)$  will converge wherever both  $H(z)$  and  $G(z)$  converge. Thus ROC of  $Y(z)$  is  $R_g \cap R_h$ .

**(g)**  $y[n] = g[n]h[n]$ . Hence,  $Y(z) = \sum_{n=-\infty}^{\infty} g[n]h[n]z^{-n}$ . From Eq. (3.??),

$$g[n] = \frac{1}{2\pi j} \oint_C G(v) v^{n-1} dv. \text{ Thus, } Y(z) = \sum_{n=-\infty}^{\infty} h[n] \left( \frac{1}{2\pi j} \oint_C G(v) v^{n-1} dv \right) z^{-n}$$

$$= \frac{1}{2\pi j} \oint_C G(v) \left( \sum_{n=-\infty}^{\infty} h[n] z^{-n} v^{n-1} \right) dv = \frac{1}{2\pi j} \oint_C G(v) H(z/v) v^{-1} dv.$$

$$(h) \sum_{n=-\infty}^{\infty} g[n] h^*[n] = \frac{1}{2\pi j} \oint_C G(v) \sum_{n=-\infty}^{\infty} h^*[n] v^n v^{-1} dv = \frac{1}{2\pi j} \oint_C G(v) H^*(1/v^*) v^{-1} dv.$$

**3.90**  $X(z) = Z\{x[n]\}$  with an ROC given by  $R_x$ . Using the conjugation property of the z-transform given in Table 3.9 we observe that  $Z\{x^*[n]\} = X^*(z^*)$  whose ROC is given by  $R_x$ . Now,

$$\text{Re}(x[n]) = \frac{1}{2}(x[n] + x^*[n]). \text{ Hence, } Z\{\text{Re}(x[n])\} = \frac{1}{2}(X(z) + X^*(z^*)) \text{ whose ROC is also } R_x.$$

Likewise,  $\text{Im}(x[n]) = \frac{1}{2j}(x[n] - x^*[n])$ . Thus  $Z\{\text{Im}(x[n])\} = \frac{1}{2j}(X(z) - X^*(z^*))$  with an ROC given by  $R_x$ .

**3.91**  $\{x[n]\} = \left\{ \begin{matrix} 3 & 0 & 1 & \underset{\uparrow}{-2} & -3 & 4 & 1 & 0 & -1 \end{matrix} \right\}$ . Then,

$\tilde{X}[k] = X(z)|_{z=e^{j\pi k/3}} = X(z)|_{z=e^{j2\pi k/6}} = X(e^{j\omega})|_{\omega=2\pi k/6}$ . Note that  $\tilde{X}[k]$  is a periodic sequence of period 6. Hence, from the results of Prpblem 3.37, the inverse of the discrete Fourier series  $\tilde{X}[k]$

is given by  $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+6r] = x[n-6] + x[n] + x[n+6]$ , for  $0 \leq n \leq 5$ . Let

$$y[n] = x[n-6] + x[n] + x[n+6], -3 \leq n \leq 5. \text{ It follows } \{x[n-6]\} = \left\{ \begin{matrix} 0 & 0 & 0 & \underset{\uparrow}{0} & 0 & 0 & 3 & 0 & -1 \end{matrix} \right\},$$

$$\text{and } \{x[n+6]\} = \left\{ \begin{matrix} 3 & 0 & 1 & \underset{\uparrow}{0} & 0 & 0 & 0 & 0 & 0 \end{matrix} \right\}. \text{ Therefore,}$$

$$\{y[n]\} = \left\{ \begin{matrix} 4 & 0 & 4 & \underset{\uparrow}{-2} & -3 & 4 & 4 & 0 & 0 \end{matrix} \right\}. \text{ Hence, } \{\tilde{x}[n]\} = \left\{ \begin{matrix} -2 & -3 & 4 & 4 & 0 & 0 \end{matrix} \right\}.$$

**3.92**  $X(z) = \sum_{n=0}^{11} x[n] z^{-n}$ .

$$X_o[k] = X(z)|_{z=e^{j2\pi k/9}} = \sum_{n=0}^{11} x[n] e^{-j2\pi kn/9}. \text{ Therefore,}$$

$$x_o[n] = \frac{1}{9} \sum_{k=0}^8 X_o[k] e^{j2\pi kn/9} = \frac{1}{9} \sum_{k=0}^8 \left( \sum_{r=0}^{11} x[r] e^{-j2\pi kr/9} \right) e^{j2\pi kn/9} = \frac{1}{9} \sum_{k=0}^8 \sum_{r=0}^{11} x[r] e^{j2\pi k(n-r)/9}$$

$$= \frac{1}{9} \sum_{r=0}^{11} x[r] \sum_{k=0}^8 e^{j2\pi k(n-r)/9} = \frac{1}{9} \sum_{r=0}^{11} x[r] \sum_{k=0}^8 W_9^{-(r-n)k}$$

But, from Eq. (3.28),  $\sum_{k=0}^8 W_9^{-(r-n)k} = \begin{cases} 9, & \text{for } r-n=9i \\ 0, & \text{otherwise.} \end{cases}$  Hence,

$$x_o[n] = \begin{cases} x[0]+x[9], & \text{for } n=0, \\ x[1]+x[10], & \text{for } n=1, \\ x[2]+x[11], & \text{for } n=2, \\ x[n], & \text{for } 3 \leq n \leq 8, \end{cases} \text{ i.e., } x_o[n] = \{9 \ 1 \ 7 \ 4 \ -3 \ -2 \ 0 \ 1 \ -4\}, 0 \leq n \leq 8.$$

**3.93 (a)**  $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ . Hence,  $X(z^2) = \sum_{n=-\infty}^{\infty} x[n]z^{-2n} = \sum_{\substack{m=-\infty \\ m \text{ even}}}^{\infty} x[m/2]z^{-m}$ . If we define a new

sequence  $g[m] = \begin{cases} x[m/2], & m=0, \pm 2, \pm 4, \mathbf{K} \\ 0, & \text{otherwise,} \end{cases}$ , we can then express  $X(z^2) = \sum_{n=-\infty}^{\infty} g[n]z^{-n}$ . Thus, the

inverse z-transform of  $X(z^2)$  is given by  $g[n]$ . For  $x[n] = (0.4)^n \mu[n]$ ,

$$g[n] = \begin{cases} (0.4)^{n/2}, & n=0, 2, 4, \mathbf{K} \\ 0, & \text{otherwise,} \end{cases}$$

**(b)**  $Y(z) = (1+z^{-1})X(z^2) = X(z^2) + z^{-1}X(z^2)$ . Therefore,

$y[n] = Z^{-1}\{Y(z)\} = Z^{-1}\{X(z^2)\} + Z^{-1}\{z^{-1}X(z^2)\} = g[n] + g[n-1]$ , where  $g[n]$  is the inverse z-

transform of  $X(z^2)$ . Now,  $g[n] = \begin{cases} x[n/2], & n=0, \pm 2, \pm 4, \mathbf{K} \\ 0, & \text{otherwise,} \end{cases}$  and

$g[n-1] = \begin{cases} x[(n-1)/2], & n=\pm 1, \pm 3, \mathbf{K} \\ 0, & \text{otherwise,} \end{cases}$ . Hence,

$y[n] = g[n] + g[n-1] = \begin{cases} x[n/2], & n=0, \pm 2, \pm 4, \mathbf{K} \\ x[(n-1)/2], & n=\pm 1, \pm 3, \pm 5, \mathbf{K} \end{cases}$ . For  $x[n] = (0.4)^n \mu[n]$ , therefore,

$$y[n] = \begin{cases} (0.4)^{n/2}, & n=0, 2, 4, \mathbf{K} \\ (0.4)^{(n-1)/2}, & n=1, 3, 5, \mathbf{K} \\ 0, & n < 0. \end{cases}$$

**3.94 (a)**  $x_1[n] = \alpha^n \mu[n+1]$ ,  $|\alpha| < 1$ . Therefore,  $X_1(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n+1]z^{-n} = \sum_{n=-1}^{\infty} \alpha^n z^{-n}$

$= \frac{z}{\alpha} \sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{\alpha z^{-1}(1-\alpha z^{-1})}$ ,  $|z| > |\alpha|$ . The ROC of  $X_1(z)$  includes the unit circle since  $|\alpha| < 1$ .

On the unit circle  $X_1(e^{j\omega}) = X_1(z)|_{z=e^{j\omega}} = \frac{1}{\alpha e^{-j\omega}(1-\alpha e^{-j\omega})}$ , which is the same as the DTFT of  $x_1[n]$ .

**(b)**  $x_2[n] = n \cdot \alpha^n \mu[n]$ ,  $|\alpha| < 1$ . Therefore,  $X_2(z) = \sum_{n=-\infty}^{\infty} n \cdot \alpha^n \mu[n]z^{-n} = \sum_{n=0}^{\infty} n \cdot \alpha^n z^{-n}$

$= \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$ ,  $|z| > |\alpha|$ . The ROC of  $X_2(z)$  includes the unit circle since  $|\alpha| < 1$ . On the unit circle,

$X_2(e^{j\omega}) = X_2(z)|_{z=e^{j\omega}} = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$ , which is the same as the DTFT of  $x_2[n]$ .

(c)  $x_3[n] = \begin{cases} \alpha^{|n|}, & |n| < M, \\ 0, & \text{otherwise.} \end{cases}$  Therefore,  $X_3(z) = \sum_{n=-M}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^M \alpha^n z^{-n}$   
 $= \alpha^M z^M \frac{1 - \alpha^{-M} z^{-M}}{1 - \alpha^{-1} z^{-1}} + \frac{1 - \alpha^{M+1} z^{-(M+1)}}{1 - \alpha^{-1} z^{-1}}$ . Since  $x_3[n]$  is a finite-length sequence, the ROC is the whole  $z$ -plane except possibly the origin. On the unit circle

$$X(e^{j\omega}) = X(z)|_{z=e^{j\omega}} = \alpha^M e^{j\omega M} \frac{1 - \alpha^{-M} e^{-j\omega M}}{1 - \alpha^{-1} e^{-j\omega}} + \frac{1 - \alpha^{M+1} e^{-j\omega(M+1)}}{1 - \alpha^{-1} e^{-j\omega}}.$$

(d)  $x_4[n] = \alpha^n \mu[n-3]$ ,  $|\alpha| < 1$ . Note that we can express  $x_4[n] = \alpha^4 x_1[n-4]$ , where  $x_1[n] = \alpha^n \mu[n+1]$ , is the sequence considered in Part (a). Therefore,

$$X_4(z) = \alpha^4 z^{-4} X_1(z) = \frac{\alpha^3 z^{-3}}{1 - \alpha z^{-1}}, \quad |z| > |\alpha|$$
 The ROC of  $X_4(z)$  includes the unit circle since  $|\alpha| < 1$ .

On the unit circle,  $X_3(e^{j\omega}) = \frac{\alpha^3 e^{-j3\omega}}{1 - \alpha e^{-j\omega}}$ , which is the same as the DTFT of  $x_4[n]$ .

(e)  $x_5[n] = n \cdot \alpha^n \mu[n+2]$ ,  $|\alpha| < 1$ . Therefore,

$$X_5(z) = \sum_{n=-2}^{\infty} n \cdot \alpha^n z^{-n} = -2\alpha^{-2} z^2 - \alpha^{-1} z + \sum_{n=0}^{\infty} n \cdot \alpha^n z^{-n} = -z(2\alpha^{-2} z + \alpha^{-1}) + \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}, \quad |z| > |\alpha|.$$

The ROC of  $X_5(z)$  includes the unit circle since  $|\alpha| < 1$ . On the unit circle,

$$X_5(e^{j\omega}) = -e^{j\omega}(2\alpha^{-2} e^{j\omega} + \alpha^{-1}) + \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2},$$
 which is the same as the DTFT of  $x_5[n]$ .

(f)  $x_6[n] = \alpha^n \mu[-n-1]$ ,  $|\alpha| > 1$ . Therefore,

$$X_6(z) = \sum_{n=-\infty}^{-1} \alpha^n z^{-n} = \sum_{n=-\infty}^0 \alpha^n z^{-n} - \alpha = \frac{1}{1 - \alpha^{-1} z} - \alpha, \quad |z| < |\alpha|$$
 The ROC of  $X_6(z)$  includes the unit

circle since  $|\alpha| > 1$ . On the unit circle,  $X_6(e^{j\omega}) = \frac{1}{1 - \alpha^{-1} e^{j\omega}} - \alpha$ , which is the same as the DTFT of  $x_6[n]$ .

**3.95 (a)**  $y_1[n] = \begin{cases} 1, & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$  Therefore,  $Y_1(z) = \sum_{n=-N}^N z^{-n} = z^N \frac{(1 - z^{-(2N+1)})}{(1 - z^{-1})}$ . Since  $y_1[n]$  is a finite-length sequence, the ROC of its  $z$ -transform is the whole  $z$ -plane except possibly the origin, and therefore includes the unit circle. On the unit circle,

$$Y_1(e^{j\omega}) = \sum_{n=-N}^N e^{-j\omega n} = e^{j\omega N} \frac{(1 - e^{-j\omega(2N+1)})}{(1 - e^{-j\omega})} = \frac{\sin(\omega[N + \frac{1}{2}])}{\sin(\omega/2)},$$
 which is the same DTFT of  $y_1[n]$ .

(b)  $y_2[n] = \begin{cases} 1 - \frac{|n|}{N}, & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$  Now  $y_2[n] = y_0[n] \circledast y_0[n]$  where

$y_0[n] = \begin{cases} 1, & -N/2 \leq n \leq N/2, \\ 0, & \text{otherwise.} \end{cases}$  Therefore,  $Y_2(z) = Y_0^2(z) = z^N \frac{(1 - z^{-(N+1)})^2}{(1 - z^{-1})^2}$ . Since  $y_2[n]$  is a

finite-length sequence, the ROC of its z-transform is the whole z-plane except possibly the origin, and therefore includes the unit circle. On the unit circle,

$$Y_2(e^{j\omega}) = Y_0^2(e^{j\omega}) = \frac{\sin^2\left(\omega \left[\frac{N+1}{2}\right]\right)}{\sin^2(\omega/2)}$$
 which is the same DTFT of  $y_2[n]$ .

(c)  $y_3[n] = \begin{cases} \cos(\pi n / 2N), & -N \leq n \leq N, \\ 0, & \text{otherwise.} \end{cases}$  Therefore,

$$Y_3(z) = \frac{1}{2} \sum_{n=-N}^N e^{-j(\pi n / 2N)} z^{-n} + \frac{1}{2} \sum_{n=-N}^N e^{j(\pi n / 2N)} z^{-n}$$

$$= \frac{e^{j(\pi/2)} z^N}{2} \left( \frac{1 - e^{-j(2N+1)(\pi/2N)} z^{-(2N+1)}}{1 - e^{-j(\pi/2N)} z^{-1}} \right) + \frac{e^{-j(\pi/2)} z^N}{2} \left( \frac{1 - e^{j(2N+1)(\pi/2N)} z^{-(2N+1)}}{1 - e^{j(\pi/2N)} z^{-1}} \right).$$
 Since

$y_3[n]$  is a finite-length sequence, the ROC of its z-transform is the whole z-plane except possibly the origin, and therefore includes the unit circle. On the unit circle,

$$Y(e^{j\omega}) = \frac{1}{2} \frac{\sin\left((\omega - \frac{\pi}{2N})(N + \frac{1}{2})\right)}{\sin\left((\omega - \frac{\pi}{2N})/2\right)} + \frac{1}{2} \frac{\sin\left((\omega + \frac{\pi}{2N})(N + \frac{1}{2})\right)}{\sin\left((\omega + \frac{\pi}{2N})/2\right)}$$
 which is the same DTFT of  $y_3[n]$ .

**3.96 (a)**  $x_1[n] = -\alpha^n \mu[-n-1]$ . Note, is a left-sided sequence. Hence, the ROC of its z-transform is

interior to a circle. Therefore,  $X_1(z) = - \sum_{n=-\infty}^{\infty} \alpha^n \mu[-n-1] z^{-n} = - \sum_{n=-\infty}^{-1} \alpha^n z^{-n} = - \sum_{m=1}^{\infty} \alpha^{-m} z^m$

$$= - \sum_{m=1}^{\infty} (z/\alpha)^m = - \left( \frac{z/\alpha}{1 - (z/\alpha)} \right) = \frac{z}{z - \alpha}, \quad |z/\alpha| < 1.$$
 The ROC of  $X_1(z)$  is thus given by  $|z| < |\alpha|$

(b)  $x_2[n] = \alpha^n \mu[n+1]$ . Note, is a right-sided sequence. Hence, the ROC of its z-transform is

exterior to a circle. Therefore,  $X_2(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n+1] z^{-n} = \sum_{n=-1}^{\infty} \alpha^n z^{-n}$

$$= \alpha^{-1} z + \sum_{n=0}^{\infty} \alpha^n z^{-n} = \alpha^{-1} z + \frac{1}{1 - \alpha z^{-1}}, \quad |\alpha/z| < 1.$$
 Simplifying we get  $X_2(z) = \frac{z/\alpha}{1 - (z/\alpha)}$  whose ROC is given by  $|z| > |\alpha|$

(c)  $x_3[n] = \alpha^n \mu[-n]$ . Note, is a left-sided sequence. Hence, the ROC of its z-transform is interior to a circle. Therefore,  $X_3(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[-n] z^{-n} = \sum_{n=-\infty}^0 \alpha^n z^{-n} = \sum_{m=0}^{\infty} \alpha^{-m} z^m = \frac{1}{1 - \alpha^{-1}z}$ ,  $|\alpha^{-1}z| < 1$ . Therefore, the ROC of  $X_3(z)$  is given by  $|z| < |\alpha|$

**3.97**  $v[n] = \alpha^{|n|} = \alpha^n \mu[n] + \alpha^{-n} \mu[-n - 1]$ . Now,  $Z\{\alpha^n \mu[n]\} = \frac{1}{1 - \alpha z^{-1}}$ ,  $|z| > |\alpha|$  (see Table 3.8) and

$$Z\{\alpha^{-n} \mu[-n - 1]\} = \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} = \sum_{m=1}^{\infty} \alpha^m z^m = \sum_{m=0}^{\infty} \alpha^m z^m - 1 = \frac{1}{1 - \alpha z} - 1 = \frac{-\alpha z}{\alpha - z^{-1}}, \quad |\alpha z| < 1.$$

Therefore,  $V(z) = \frac{1}{1 - \alpha z^{-1}} + \frac{-\alpha z}{\alpha - z^{-1}} = \frac{z^{-1}(-1)}{(1 - \alpha z^{-1})(\alpha - z^{-1})}$ , with the ROC of  $V(z)$  given by  $|\alpha| < |z| < 1/|\alpha|$

**3.98 (a)**  $Y_1(z) = \frac{z(z-1)}{(z+1)(z+1/3)} = \frac{(1-z^{-1})}{(1+z^{-1})(1+(1/3)z^{-1})} = K + \frac{A}{1+z^{-1}} + \frac{B}{1+(1/3)z^{-1}}$ , where

$$K = Y_1(z)|_{z=0} = 0, \quad A = \left. \frac{(1-z^{-1})}{(1+(1/3)z^{-1})} \right|_{z^{-1}=-1} = 3, \quad \text{and} \quad B = \left. \frac{(1-z^{-1})}{1+z^{-1}} \right|_{z^{-1}=-3} = -2. \quad \text{Thus,}$$

$Y_1(z) = \frac{3}{1+z^{-1}} + \frac{-2}{1+(1/3)z^{-1}}$ ,  $|z| > 1$ . Since, the ROC is exterior to a circle, the inverse z-transform  $y_1[n]$  of  $Y_1(z)$  is a right-sided sequence and is given by

$$y_1[n] = 3(-1)^n \mu[n] - 2(-1/3)^n \mu[n].$$

(b)  $Y_2(z) = \frac{3}{1+z^{-1}} + \frac{-2}{1+(1/3)z^{-1}}$ ,  $|z| < \frac{1}{3}$ . Since, the ROC is interior to a circle, the inverse z-transform  $y_2[n]$  of  $Y_2(z)$  is a left-sided sequence and is given by

$$y_2[n] = -3(-1)^n \mu[-n - 1] + 2(-1/3)^n \mu[-n - 1].$$

(c)  $Y_3(z) = \frac{3}{1+z^{-1}} + \frac{-2}{1+(1/3)z^{-1}}$ ,  $\frac{1}{3} < |z| < 1$ . Since, the ROC is an annular region in the z-plane, the inverse z-transform  $y_3[n]$  of  $Y_3(z)$  is a two-sided sequence and is given by

$$y_3[n] = -3(-1)^n \mu[-n - 1] - 2(-1/3)^n \mu[n].$$

**3.99 (a)**  $X_a(z) = \frac{4 - 3z^{-1} + 3z^{-2}}{(z+2)(z-3)^2} = z^{-3} \left( \frac{4 - 3z^{-1} + 3z^{-2}}{(1+2z^{-1})(1-3z^{-1})^2} \right) = \frac{4z^{-3} - 3z^{-4} + 3z^{-5}}{(1+2z^{-1})(1-3z^{-1})^2}$ .  $\hat{\alpha} \zeta \quad |z| > 3,$



Let  $G(z) = \frac{4 - 3z^{-1} + 3z^{-2}}{(1 + 2z^{-1})(1 - 3z^{-1})^2}$ . A partial-fraction expansion of  $G(z)$  yields

$$G(z) = \frac{A}{1 + 2z^{-1}} + \frac{B}{1 - 3z^{-1}} + \frac{C}{(1 - 3z^{-1})^2}, \quad \text{where } A = \left. \frac{4 - 3z^{-1} + 3z^{-2}}{(1 - 3z^{-1})^2} \right|_{z^{-1} = -1/2} = \frac{25/4}{25/4} = 1,$$

$$C = \left. \frac{4 - 3z^{-1} + 3z^{-2}}{1 + 2z^{-1}} \right|_{z^{-1} = 1/3} = \frac{10/3}{5/3} = 2, \quad \text{and}$$

$$B = \frac{1}{(2-1)!(-3)^{2-1}} \cdot \left. \frac{d}{dz^{-1}} \left[ \frac{4 - 3z^{-1} + 3z^{-2}}{1 + 2z^{-1}} \right] \right|_{z^{-1} = 1/3} = \frac{-25/3}{-25/3} = 1.$$

Therefore  $X_a(z) = z^{-3} \left( \frac{1}{1 + 2z^{-1}} \right) + z^{-3} \left( \frac{1}{1 - 3z^{-1}} \right) + z^{-3} \left( \frac{2}{(1 - 3z^{-1})^2} \right)$ . Since the ROC is  $|z| > 3$ , the inverse z-transform  $x_a[n]$  of  $X_a(z)$  is a right-sided sequence.

$$\text{Thus, } Z^{-1} \left( \frac{1}{1 + 2z^{-1}} \right) = (-2)^n \mu[n], \quad Z^{-1} \left( \frac{1}{1 - 3z^{-1}} \right) = (3)^n \mu[n], \quad \text{and}$$

$$Z^{-1} \left( \frac{3z^{-1}}{(1 - 3z^{-1})^2} \right) = n(3)^n \mu[n]. \quad \text{Thus,}$$

$$x_a[n] = (-2)^{n-3} \mu[n-3] + (3)^{n-3} \mu[n-3] + \frac{2}{3} (n-2)(3)^{n-2} \mu[n-2].$$

**(b)**  $X_b(z) = z^{-3} \left( \frac{1}{1 + 2z^{-1}} \right) + z^{-3} \left( \frac{1}{1 - 3z^{-1}} \right) + z^{-3} \left( \frac{2}{(1 - 3z^{-1})^2} \right)$ ,  $|z| < 2$ . Here the ROC is  $|z| < 2$ .

Hence, the inverse z-transform  $x_b[n]$  of  $X_b(z)$  is a right-sided sequence. Thus,

$$Z^{-1} \left( \frac{1}{1 + 2z^{-1}} \right) = -(-2)^n \mu[-n-1], \quad Z^{-1} \left( \frac{1}{1 - 3z^{-1}} \right) = -(3)^n \mu[-n-1], \quad \text{and}$$

$$Z^{-1} \left( \frac{3z^{-1}}{(1 - 3z^{-1})^2} \right) = -n(3)^n \mu[-n-1]. \quad \text{Therefore,}$$

$$x_b[n] = -(-2)^{n-3} \mu[-n-4] - (3)^{n-3} \mu[-n-4] - \frac{2}{3} n(3)^{n-2} \mu[-n-3].$$

**(c)**  $X_c(z) = z^{-3} \left( \frac{1}{1 + 2z^{-1}} \right) + z^{-3} \left( \frac{1}{1 - 3z^{-1}} \right) + z^{-3} \left( \frac{2}{(1 - 3z^{-1})^2} \right)$ ,  $2 < |z| < 3$ . Here the ROC is an annular region in the z-plane. Hence, the inverse z-transform  $x_c[n]$  of  $X_c(z)$  is a two-sided sequence. Now  $Z^{-1} \left( \frac{1}{1 + 2z^{-1}} \right)$  is right-sided sequence as  $|z| > 2$ , whereas,  $Z^{-1} \left( \frac{1}{1 - 3z^{-1}} \right)$  and

$$Z^{-1} \left( \frac{2z^{-1}}{(1 - 3z^{-1})^2} \right) \text{ are right-sided sequences as } |z| < 3. \quad \text{Thus, } Z^{-1} \left( \frac{1}{1 + 2z^{-1}} \right) = (-2)^n \mu[n],$$

$$Z^{-1}\left(\frac{1}{1-3z^{-1}}\right) = -(3)^n \mu[-n-1], \text{ and } Z^{-1}\left(\frac{3z^{-1}}{(1-3z^{-1})^2}\right) = -n(3)^n \mu[-n-1]. \text{ Therefore,}$$

$$x_c[n] = (-2)^{n-3} \mu[n-3] - (3)^{n-3} \mu[-n-4] - \frac{2}{3} n(3)^{n-2} \mu[-n-3].$$

**3.100**  $G(z) = \frac{P(z)}{D(z)} = \frac{P(z)}{(1-\lambda_1 z^{-1})R(z)}$ . By definition the residue  $\rho_1$  at the pole  $z = \lambda_1$  is given by

$$\rho_1 = \left. \frac{P(z)}{R(z)} \right|_{z=\lambda_1}. \text{ Now, } D'(z) = \frac{dD(z)}{dz^{-1}} = \frac{d[(1-\lambda_1 z^{-1})R(z)]}{dz^{-1}} = -\lambda_1 R(z) + (1-\lambda_1 z^{-1}) \frac{dR(z)}{dz}. \text{ Hence,}$$

$$D'(z)|_{z=\lambda_1} = -\lambda_1 R(z)|_{z=\lambda_1}. \text{ Therefore, } \rho_1 = -\lambda_1 \left. \frac{P(z)}{D'(z)} \right|_{z=\lambda_1}.$$

**3.101**  $G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \mathbf{L} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \mathbf{L} + d_M z^{-M}}$ . Thus,  $G(\infty) = \frac{p_0}{d_0}$ . Now a partial-fraction expansion

of  $G(z)$  in  $z^{-1}$  is given by  $G(z) = \sum_{l=1}^N \frac{\rho_l}{1-\lambda_l z^{-1}}$ , from which we obtain  $G(\infty) = \sum_{l=1}^N \rho_l$ . Therefore,

$$G(\infty) = \sum_{l=1}^N \rho_l = \frac{p_0}{d_0}.$$

**3.102**  $H(z) = \frac{1}{1-2r \cos(\theta)z^{-1} + r^2 z^{-2}}$ ,  $|z| > r > 0$ . By using partial-fraction expansion we write

$$H(z) = \frac{1}{(e^{j\theta} - e^{-j\theta})} \left\{ \frac{e^{j\theta}}{1 - re^{j\theta} z^{-1}} - \frac{e^{-j\theta}}{1 - re^{-j\theta} z^{-1}} \right\} = \frac{1}{2j \sin(\theta)} \left\{ \frac{e^{j\theta}}{1 - re^{j\theta} z^{-1}} - \frac{e^{-j\theta}}{1 - re^{-j\theta} z^{-1}} \right\}. \text{ Thus,}$$

$$h[n] = \frac{1}{2j \sin(\theta)} \left\{ e^{j\theta} r^n e^{jn\theta} \mu[n] - r^n e^{-j\theta} e^{-jn\theta} \mu[n] \right\} = \frac{r^n}{\sin(\theta)} \left\{ \frac{e^{j\theta(n+1)} - e^{-j\theta(n+1)}}{2j} \right\} \mu[n]$$

$$= \frac{r^n \sin((n+1)\theta)}{\sin(\theta)} \mu[n].$$

**3.103**  $G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$  with a ROC given by  $R_g$ .

(a) Therefore  $G^*(z) = \sum_{n=-\infty}^{\infty} g^*[n] (z^*)^{-n}$  and  $G^*(z^*) = \sum_{n=-\infty}^{\infty} g^*[n] z^{-n}$ .

Thus the z-transform of  $g^*[n]$  is  $G^*(z^*)$ .

(b) Replace  $n$  by  $-m$  in the summation. This leads to  $G(z) = \sum_{m=-\infty}^{\infty} g[-m] z^m$ . Therefore

$G(1/z) = \sum_{m=-\infty}^{\infty} g[-m]z^{-m}$ . Thus the  $z$ -transform of  $g[-n]$  is  $G(1/z)$ . Note that since  $z$  has been replaced by  $1/z$ , the ROC of  $G(1/z)$  will be  $1/R_g$ .

(c) Let  $y[n] = \alpha g[n] + \beta h[n]$ . Then,

$$Y(z) = \sum_{n=-\infty}^{\infty} (\alpha g[n] + \beta h[n])z^{-n} = \alpha \sum_{n=-\infty}^{\infty} g[n]z^{-n} + \beta \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \alpha G(z) + \beta H(z)$$

In this case  $Y(z)$  will converge wherever both  $G(z)$  and  $H(z)$  converge. Thus the ROC of  $Y(z)$  is  $R_g \cap R_h$ , where is  $R_g$  the ROC of  $G(z)$  and  $R_h$  is the ROC of  $H(z)$ .

(d)  $y[n] = g[n - n_0]$ . Hence  $Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} g[n - n_0]z^{-n} = \sum_{m=-\infty}^{\infty} g[m]z^{-(m+n_0)}$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} g[m]z^{-m} = z^{-n_0} G(z).$$

In this case the ROC of  $Y(z)$  is the same as that of  $G(z)$  except for the possible addition or elimination of the point  $z = 0$  or  $z = \infty$  (due to the factor  $z^{-n_0}$ ).

(e)  $y[n] = \alpha^n g[n]$ . Hence,  $Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} g[n](z\alpha^{-1})^{-n} = G(z/\alpha)$ .

The ROC of  $Y(z)$  is  $\alpha R_g$ .

(f)  $y[n] = ng[n]$ . Hence  $Y(z) = \sum_{n=-\infty}^{\infty} ng[n]z^{-n}$ .

Now  $G(z) = \sum_{n=-\infty}^{\infty} g[n]z^{-n}$ . Thus,  $\frac{dG(z)}{dz} = - \sum_{n=-\infty}^{\infty} ng[n]z^{-n-1} \Rightarrow z \frac{dG(z)}{dz} = - \sum_{n=-\infty}^{\infty} ng[n]z^{-n}$ .

Thus  $Y(z) = -z \frac{dG(z)}{dz}$ .

(g)  $y[n] = g[n] \otimes h[n] = \sum_{k=-\infty}^{\infty} g[k]h[n-k]$ . Hence,

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} g[k]h[n-k] \right) z^{-n} = \sum_{k=-\infty}^{\infty} g[k] \sum_{n=-\infty}^{\infty} h[n-k]z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} g[k]H(z)z^{-k} = H(z)G(z).$$

In this case also  $Y(z)$  will converge wherever both  $H(z)$  and  $G(z)$  converge. Thus ROC of  $Y(z)$  is  $R_g \cap R_h$ .

(h)  $y[n] = g[n]h[n]$ . Hence,  $Y(z) = \sum_{n=-\infty}^{\infty} g[n]h[n]z^{-n}$ . From Eq. (3.107),

$$g[n] = \frac{1}{2\pi j} \oint_C G(v) v^{n-1} dv. \text{ Thus, } Y(z) = \sum_{n=-\infty}^{\infty} h[n] \left( \frac{1}{2\pi j} \oint_C G(v) v^{n-1} dv \right) z^{-n}$$

$$= \frac{1}{2\pi j} \oint_C G(v) \left( \sum_{n=-\infty}^{\infty} h[n] z^{-n} v^{n-1} \right) dv = \frac{1}{2\pi j} \oint_C G(v) H(z/v) v^{-1} dv.$$

$$(i) \sum_{n=-\infty}^{\infty} g[n] h^*[n] = \frac{1}{2\pi j} \oint_C G(v) \sum_{n=-\infty}^{\infty} h^*[n] v^n v^{-1} dv = \frac{1}{2\pi j} \oint_C G(v) H^*(1/v^*) v^{-1} dv.$$

**3.104**  $x[n] = x_{re}[n] + jx_{im}[n]$ , where  $x_{re}[n] = \frac{1}{2}(x[n] + x^*[n])$ , and  $x_{im}[n] = \frac{1}{2j}(x[n] - x^*[n])$ . From Table 3.9,  $Z\{x^*[n]\} = X^*(z^*)$ , with an ROC  $R_x$ . Therefore,  $Z\{x_{re}[n]\} = \frac{1}{2}\{X(z) + X^*(z^*)\}$ , and  $Z\{x_{im}[n]\} = \frac{1}{2j}\{X(z) - X^*(z^*)\}$ .

**3.105 (a)** Expanding in a power series we get  $X_1(z) = \frac{1}{1-z^{-3}} = \sum_{n=0}^{\infty} z^{-3n}$ ,  $|z| > 1$ .

Thus,  $x_1[n] = \begin{cases} 1, & \text{if } n = 3k \text{ and } n \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$  Using partial fraction, we get

$$X_1(z) = \frac{1}{1-z^{-3}} = \frac{\frac{1}{3}}{1-z^{-1}} + \frac{\frac{1}{3}}{1+(\frac{1}{2}+j\frac{\sqrt{3}}{2})z^{-1}} + \frac{\frac{1}{3}}{1+(\frac{1}{2}-j\frac{\sqrt{3}}{2})z^{-1}}. \text{ Therefore,}$$

$$x_1[n] = \frac{1}{3}\mu[n] + \frac{1}{3}\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)^n \mu[n] + \frac{1}{3}\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)^n \mu[n]$$

$$= \frac{1}{3}\mu[n] + \frac{1}{3}e^{-j2\pi n/3}\mu[n] + \frac{1}{3}e^{j2\pi n/3}\mu[n] = \frac{1}{3}\mu[n] + \frac{2}{3}\cos(2\pi n/3)\mu[n].$$

Thus  $x_1[n] = \begin{cases} 1, & \text{if } n = 3k \text{ and } n \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$

**(b)** Expanding in a power series we get  $X_2(z) = \frac{1}{1-z^{-2}} = \sum_{n=0}^{\infty} z^{-2n}$ ,  $|z| > 1$ .

Thus,  $x_2[n] = \begin{cases} 1, & \text{if } n = 2k \text{ and } n \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$

Using partial fraction, we get  $X_2(z) = \frac{1}{1-z^{-2}} = \frac{\frac{1}{2}}{1+z^{-1}} + \frac{\frac{1}{2}}{1-z^{-1}}$ . Therefore,

$$x_2[n] = \frac{1}{2}\mu[n] + \frac{1}{2}(-1)^n \mu[n]$$

Thus,  $x_2[n] = \begin{cases} 1, & \text{if } n = 2k \text{ and } n \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$

**3.106 (a)**  $X_1(z) = \log(1-\alpha z^{-1})$ ,  $|z| > |\alpha|$ . Expanding  $\log(1-\alpha z^{-1})$  in a power series we get

$$X_1(z) = -\alpha z^{-1} - \frac{\alpha^2 z^{-2}}{2} - \frac{\alpha^3 z^{-3}}{3} - \dots = -\sum_{n=1}^{\infty} \frac{\alpha^n}{n} z^{-n}.$$

Therefore,,  $x_1[n] = -\frac{\alpha^n}{n} \mu[n-1]$ .

(b)  $X_2(z) = \log\left(\frac{\alpha - z^{-1}}{\alpha}\right) = \log(1 - (\alpha z)^{-1})$ ,  $|z| < |\alpha|$ . Expanding  $\log(1 - (\alpha z)^{-1})$  in a power

series we get  $X_2(z) = -(\alpha z)^{-1} - \frac{(\alpha z)^{-2}}{2} - \frac{(\alpha z)^{-3}}{3} - \dots = -\sum_{n=1}^{\infty} \frac{(\alpha z)^{-n}}{n}$ .

Therefore,  $x_2[n] = -\frac{\alpha^{-n}}{n} \mu[n-1]$ .

(c)  $X_3(z) = \log\left(\frac{1}{1 - \alpha z^{-1}}\right) = -\log(1 - \alpha z^{-1})$ ,  $|z| > |\alpha|$ . Expanding  $X_3(z)$  in a power series we get

$$X_1(z) = \alpha z^{-1} + \frac{\alpha^2 z^{-2}}{2} + \frac{\alpha^3 z^{-3}}{3} + \dots = \sum_{n=1}^{\infty} \frac{\alpha^n}{n} z^{-n}.$$

Therefore,  $x_3[n] = \frac{\alpha^n}{n} \mu[n-1]$ .

(d)  $X_4(z) = \log\left(\frac{\alpha}{\alpha - \alpha z^{-1}}\right) = -\log(1 - (\alpha z)^{-1})$ ,  $|z| < |\alpha|$ . Expanding  $X_4(z)$  in a power series we

get  $X_4(z) = (\alpha z)^{-1} + \frac{(\alpha z)^{-2}}{2} + \frac{(\alpha z)^{-3}}{3} + \dots = \sum_{n=1}^{\infty} \frac{(\alpha z)^{-n}}{n}$ .

Therefore,  $x_4[n] = \frac{\alpha^{-n}}{n} \mu[n-1]$ .

**3.107**  $X(z) = \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}$  where  $x[n] = Z^{-1}\{X(z)\}$  is a causal sequence. Now, from Table 3.8,

$$Z\{\alpha^n \mu[n]\} = \frac{1}{1 - \alpha z^{-1}}. \text{ But, } \frac{d}{dz} \left( \frac{1}{1 - \alpha z^{-1}} \right) = \frac{-\alpha z^{-2}}{(1 - \alpha z^{-1})^2}. \text{ Thus, } X(z) = -z \cdot \frac{d}{dz} \left( \frac{1}{1 - \alpha z^{-1}} \right).$$

Therefore,  $x[n] = n \alpha^n \mu[n]$ .

**3.108**  $H(z) = \frac{z^{-1} - 2z^{-2}}{(1 + 0.4z^{-1})(1 - 0.2z^{-1})} = k + \frac{A}{1 + 0.4z^{-1}} + \frac{B}{1 - 0.2z^{-1}}$ , where  $k = H(0) = \frac{-2}{0.4(-0.2)} = 25$ ,

$$A = \frac{z^{-1}(1 - 2z^{-1})}{1 - 0.2z^{-1}} \Bigg|_{z^{-1} = -2.5} = \frac{-2.5(1 - 2(-2.5))}{1 - 0.2(-2.5)} = -10, \quad B = \frac{z^{-1}(1 - 2z^{-1})}{1 + 0.4z^{-1}} \Bigg|_{z^{-1} = 5} = \frac{5(1 - 10)}{1 + 0.4(5)} = -15.$$

Thus,  $H(z) = 25 - \frac{10}{1 + 0.4z^{-1}} - \frac{15}{1 - 0.2z^{-1}}$ . Using the M-file residuz we also arrive at the same partial-fraction expansion.

Therefore,  $h[n] = 25\delta[n] - 10(-0.4)^n \mu[n] - 15(0.2)^n \mu[n]$ .

**3.109** From Eq. (3.177), for  $N = 3$ , we get

$$\mathbf{D}_3 = \begin{bmatrix} 1 & z_0^{-1} & z_0^{-2} \\ 1 & z_1^{-1} & z_1^{-2} \\ 1 & z_2^{-1} & z_2^{-2} \end{bmatrix}. \text{ The determinant of } \mathbf{D}_3 \text{ is given by}$$

$$\begin{aligned} \det(\mathbf{D}_3) &= \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} \\ 1 & z_1^{-1} & z_1^{-2} \\ 1 & z_2^{-1} & z_2^{-2} \end{vmatrix} = \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} \\ 0 & z_1^{-1} - z_0^{-1} & z_1^{-2} - z_0^{-2} \\ 0 & z_2^{-1} - z_0^{-1} & z_2^{-2} - z_0^{-2} \end{vmatrix} = \begin{vmatrix} z_1^{-1} - z_0^{-1} & z_1^{-2} - z_0^{-2} \\ z_2^{-1} - z_0^{-1} & z_2^{-2} - z_0^{-2} \end{vmatrix} \\ &= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1}) \begin{vmatrix} 1 & z_1^{-1} + z_0^{-1} \\ 1 & z_2^{-1} + z_0^{-1} \end{vmatrix} = (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_2^{-1} - z_1^{-1}) = \prod_{2 \geq k > l \geq 0} (z_k^{-1} - z_l^{-1}). \end{aligned}$$

From Eq. (3.177), for  $N = 4$ , we get

$$\mathbf{D}_4 = \begin{bmatrix} 1 & z_0^{-1} & z_0^{-2} & z_0^{-3} \\ 1 & z_1^{-1} & z_1^{-2} & z_1^{-3} \\ 1 & z_2^{-1} & z_2^{-2} & z_2^{-3} \\ 1 & z_3^{-1} & z_3^{-2} & z_3^{-3} \end{bmatrix}. \text{ The determinant of } \mathbf{D}_4 \text{ is given by}$$

$$\begin{aligned} \det(\mathbf{D}_4) &= \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} & z_0^{-3} \\ 1 & z_1^{-1} & z_1^{-2} & z_1^{-3} \\ 1 & z_2^{-1} & z_2^{-2} & z_2^{-3} \\ 1 & z_3^{-1} & z_3^{-2} & z_3^{-3} \end{vmatrix} = \begin{vmatrix} 1 & z_0^{-1} & z_0^{-2} & z_0^{-3} \\ 0 & z_1^{-1} - z_0^{-1} & z_1^{-2} - z_0^{-2} & z_1^{-3} - z_0^{-3} \\ 0 & z_2^{-1} - z_0^{-1} & z_2^{-2} - z_0^{-2} & z_2^{-3} - z_0^{-3} \\ 0 & z_3^{-1} - z_0^{-1} & z_3^{-2} - z_0^{-2} & z_3^{-3} - z_0^{-3} \end{vmatrix} \\ &= \begin{vmatrix} z_1^{-1} - z_0^{-1} & z_1^{-2} - z_0^{-2} & z_1^{-3} - z_0^{-3} \\ z_2^{-1} - z_0^{-1} & z_2^{-2} - z_0^{-2} & z_2^{-3} - z_0^{-3} \\ z_3^{-1} - z_0^{-1} & z_3^{-2} - z_0^{-2} & z_3^{-3} - z_0^{-3} \end{vmatrix} = (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_3^{-1} - z_0^{-1}) \begin{vmatrix} 1 & z_1^{-1} + z_0^{-1} & z_1^{-2} + z_1^{-1}z_0^{-1} + z_0^{-2} \\ 1 & z_2^{-1} + z_0^{-1} & z_2^{-2} + z_2^{-1}z_0^{-1} + z_0^{-2} \\ 1 & z_3^{-1} + z_0^{-1} & z_3^{-2} + z_3^{-1}z_0^{-1} + z_0^{-2} \end{vmatrix} \\ &= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_3^{-1} - z_0^{-1}) \begin{vmatrix} 1 & z_1^{-1} + z_0^{-1} & z_1^{-2} + z_1^{-1}z_0^{-1} + z_0^{-2} \\ 0 & z_2^{-1} - z_1^{-1} & (z_2^{-1} - z_1^{-1})(z_2^{-1} + z_1^{-1} + z_0^{-1}) \\ 0 & z_3^{-1} - z_1^{-1} & (z_3^{-1} - z_1^{-1})(z_3^{-1} + z_1^{-1} + z_0^{-1}) \end{vmatrix} \\ &= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_3^{-1} - z_0^{-1}) \begin{vmatrix} z_2^{-1} - z_1^{-1} & (z_2^{-1} - z_1^{-1})(z_2^{-1} + z_1^{-1} + z_0^{-1}) \\ z_3^{-1} - z_1^{-1} & (z_3^{-1} - z_1^{-1})(z_3^{-1} + z_1^{-1} + z_0^{-1}) \end{vmatrix} \\ &= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_3^{-1} - z_0^{-1}) \begin{vmatrix} 1 & z_2^{-1} + z_1^{-1} + z_0^{-1} \\ 1 & z_3^{-1} + z_1^{-1} + z_0^{-1} \end{vmatrix} \\ &= (z_1^{-1} - z_0^{-1})(z_2^{-1} - z_0^{-1})(z_3^{-1} - z_0^{-1})(z_2^{-1} - z_1^{-1})(z_3^{-1} - z_1^{-1})(z_3^{-1} - z_2^{-1}) = \prod_{3 \geq k > l \geq 0} (z_k^{-1} - z_l^{-1}). \end{aligned}$$

Hence, in the general case,  $\det(\mathbf{D}_N) = \prod_{N-1 \geq k > l \geq 0} (z_k^{-1} - z_l^{-1})$ . It follows from this expression that the determinant is non-zero, i.e.  $\mathbf{D}_N$  is non-singular, if the sampling points  $z_k$  are distinct.

**3.110**  $X_{\text{NDFT}}[0] = X(z_0) = 4 + 4 + 12 - 8 = 12$ ,  $X_{\text{NDFT}}[1] = X(z_1) = 4 - 2 + 3 + 1 = 6$ ,

$X_{\text{NDFT}}[2] = X(z_2) = 4 - 4 + 12 + 8 = 20$ ,  $X_{\text{NDFT}}[3] = X(z_3) = 4 - 4 + 12 + 8 = 52$ .

$I_0(z) = (1 - z^{-1})(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1}) = 1 - \frac{11}{6}z^{-1} + z^{-2} - \frac{1}{6}z^{-3}$ . Thus,  $I_0(-\frac{1}{2}) = 10$ ,

$I_1(z) = (1 + \frac{1}{2}z^{-1})(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1}) = 1 - \frac{1}{3}z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{12}z^{-3}$ . Thus,  $I_1(1) = \frac{1}{2}$ ,

$I_2(z) = (1 + \frac{1}{2}z^{-1})(1 - z^{-1})(1 - \frac{1}{3}z^{-1}) = 1 - \frac{5}{6}z^{-1} - \frac{1}{3}z^{-2} + \frac{1}{6}z^{-3}$ . Thus,  $I_2(2) = -\frac{2}{3}$  and

$I_3(z) = (1 + \frac{1}{2}z^{-1})(1 - z^{-1})(1 - \frac{1}{2}z^{-1}) = 1 - z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{4}z^{-3}$ . Thus,  $I_3(\frac{1}{3}) = \frac{5}{2}$ . Therefore,

$$X(z) = \frac{12}{10}I_0(z) + \frac{6}{1/2}I_1(z) - \frac{20}{2/3}I_2(z) + \frac{52}{5/2}I_3(z) = 4 - 2z^{-1} + 3z^{-2} + z^{-3}.$$

**3.111**  $x_e[n] = \begin{cases} x[n], & 0 \leq n \leq N-1, \\ 0, & N \leq n \leq 2N-1. \end{cases}$   $y[n] = x_e[n] + x_e[2N-1-n]$ . Therefore,

$$\begin{aligned} Y[k] &= \sum_{n=0}^{2N-1} y[n] W_{2N}^{nk} = \sum_{n=0}^{N-1} x[n] W_{2N}^{nk} + \sum_{n=N}^{2N-1} x[2N-1-n] W_{2N}^{nk} \\ &= \sum_{n=0}^{N-1} x[n] W_{2N}^{nk} + \sum_{n=0}^{N-1} x[n] W_{2N}^{(2N-1-n)k} = \sum_{n=0}^{N-1} x[n] (W_{2N}^{nk} + W_{2N}^{-k} W_{2N}^{-nk}). \end{aligned}$$

$$\begin{aligned} \text{Thus, } C_x[k] &= W_{2N}^{k/2} Y[k] = \sum_{n=0}^{N-1} x[n] (W_{2N}^{k(n+1/2)} + W_{2N}^{-k(n+1/2)}) \\ &= \sum_{n=0}^{N-1} 2x[n] \cos\left(\frac{\pi(2n+1)k}{2N}\right), \quad 0 \leq k \leq N-1. \end{aligned}$$

**3.112**  $Y[k] = \begin{cases} W_{2N}^{-k/2} C_x[k], & 0 \leq k \leq N-1, \\ 0, & k = N, \\ -W_{2N}^{-k/2} C_x[2N-k], & N+1 \leq k \leq 2N-1. \end{cases}$  Thus,

$$\begin{aligned} y[n] &= \frac{1}{2N} \sum_{k=0}^{2N-1} Y[k] W_{2N}^{-nk} = \frac{1}{2N} \sum_{k=0}^{N-1} C_x[k] W_{2N}^{-(n+1/2)k} - \frac{1}{2N} \sum_{k=N+1}^{2N-1} C_x[2N-k] W_{2N}^{-(n+1/2)k} \\ &= \frac{1}{2N} \sum_{k=0}^{N-1} C_x[k] W_{2N}^{-(n+1/2)k} - \frac{1}{2N} \sum_{k=1}^{N-1} C_x[k] W_{2N}^{-(n+1/2)(2N-k)} \\ &= \frac{1}{2N} \sum_{k=0}^{N-1} C_x[k] W_{2N}^{-(n+1/2)k} + \frac{1}{2N} \sum_{k=1}^{N-1} C_x[k] W_{2N}^{(n+1/2)k} \\ &= \frac{C_x[0]}{2N} + \frac{1}{N} \sum_{k=1}^{N-1} C_x[k] \cos\left(\frac{\pi k(2n+1)}{2N}\right), \end{aligned}$$

Hence,  $\frac{C_x[0]}{2N} + \frac{1}{N} \sum_{k=1}^{N-1} C_x[k] \cos\left(\frac{\pi k(2n+1)}{2N}\right)$ , where  $w[k] = \begin{cases} 1/2, & k = 0, \\ 1, & 1 \leq k \leq N-1. \end{cases}$

Moreover,  $x[n] = \begin{cases} y[n], & 0 \leq n \leq N-1, \\ 0, & \text{elsewhere,} \end{cases} = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} w[k] C_x[k] \cos\left(\frac{\pi k(2n+1)}{2N}\right), & 0 \leq n \leq N-1, \\ 0, & \text{elsewhere.} \end{cases}$

(b) From Eq. (3.163),  $x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} w[k] C_x[k] \cos\left(\frac{(2n+1)\pi k}{2N}\right), & 0 \leq n \leq N-1, \\ 0, & \text{elsewhere.} \end{cases}$  Hence,

$$\begin{aligned} \sum_{m=0}^{N-1} 2x[n] \cos\left(\frac{(2n+1)\pi m}{2N}\right) &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} w[k] C_x[k] \cos\left(\frac{(2n+1)\pi k}{2N}\right) \cos\left(\frac{(2n+1)\pi m}{2N}\right) \\ &= \frac{1}{N} \sum_{k=0}^{N-1} w[k] C_x[k] \sum_{m=0}^{N-1} \cos\left(\frac{(2n+1)\pi k}{2N}\right) \cos\left(\frac{(2n+1)\pi m}{2N}\right). \end{aligned} \quad (10)$$

Now,  $\sum_{m=0}^{N-1} \cos\left(\frac{(2n+1)\pi k}{2N}\right) \cos\left(\frac{(2n+1)\pi m}{2N}\right) = \begin{cases} N, & \text{if } k = m = 0, \\ N/2, & \text{if } k = m, \\ 0, & \text{elsewhere.} \end{cases}$

Thus, Eq. (10) reduces to

$$\begin{aligned} \sum_{m=0}^{N-1} 2x[n] \cos\left(\frac{(2n+1)\pi m}{2N}\right) &= \begin{cases} \frac{1}{N} w[0] C_x[0] \cdot N, & m = 0, \\ \frac{1}{N} w[m] C_x[m] \cdot N, & 1 \leq m \leq N-1, \end{cases} \\ &= \begin{cases} C_x[0], & m = 0, \\ C_x[m], & 1 \leq m \leq N-1, \end{cases} = C_x[m], \quad 0 \leq m \leq N-1. \end{aligned}$$

**3.113**  $y[n] = \alpha g[n] + \beta h[n]$ . Thus,

$$\begin{aligned} C_y[k] &= \sum_{n=0}^{N-1} y[n] \cos\left(\frac{\pi k(2n+1)}{2N}\right) = \sum_{n=0}^{N-1} (\alpha g[n] + \beta h[n]) \cos\left(\frac{\pi k(2n+1)}{2N}\right) = \\ &= \alpha \sum_{n=0}^{N-1} g[n] \cos\left(\frac{\pi k(2n+1)}{2N}\right) + \beta \sum_{n=0}^{N-1} h[n] \cos\left(\frac{\pi k(2n+1)}{2N}\right) = \alpha C_g[k] + \beta C_h[k]. \end{aligned}$$

**3.114**  $C_x[k] = \sum_{n=0}^{N-1} x[n] \cos\left(\frac{\pi k(2n+1)}{2N}\right) \Rightarrow C_x^*[k] = \sum_{n=0}^{N-1} x^*[n] \cos\left(\frac{\pi k(2n+1)}{2N}\right)$ .

Thus the DCT coefficients of  $x^*[n]$  are given by  $C_x^*[k]$ .

**3.115** Note that  $\sum_{n=0}^{N-1} \cos\left(\frac{\pi k(2n+1)}{2N}\right) \cos\left(\frac{\pi m(2n+1)}{2N}\right) = \begin{cases} N, & \text{if } k = m = 0, \\ N/2, & \text{if } k = m \text{ and } k \neq 0, \\ 0, & \text{otherwise.} \end{cases}$

Now,  $x[n]x^*[n] = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \alpha[k] \alpha^*[m] C_x^*[m] C_x[k] \cos\left(\frac{\pi(2n+1)k}{2N}\right) \cos\left(\frac{\pi(2n+1)m}{2N}\right)$

Thus,  $\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} \alpha[k] \alpha^*[m] C_x^*[m] C_x[k] \sum_{n=0}^{N-1} \cos\left(\frac{\pi(2n+1)k}{2N}\right) \cos\left(\frac{\pi(2n+1)m}{2N}\right)$



Now, using the orthogonality property mentioned above  $\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{2N} \sum_{k=0}^{N-1} \alpha[k] |C_x[k]|^2$ .

**3.116**  $X_{\text{DHT}}[k] = \sum_{n=0}^{N-1} x[n] \left( \cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right)$ . Now,

$$X_{\text{DHT}}[k] \left( \cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right) \right) = \sum_{n=0}^{N-1} x[n] \left( \cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right) \left( \cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right) \right).$$

Therefore,

$$\sum_{k=0}^{N-1} X_{\text{DHT}}[k] \left( \cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right) \right) = \sum_{n=0}^{N-1} x[n] \sum_{k=0}^{N-1} \left( \cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right) \left( \cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right) \right)$$

It can be shown that  $\sum_{k=0}^{N-1} \cos\left(\frac{2\pi nk}{N}\right) \cos\left(\frac{2\pi mk}{N}\right) = \begin{cases} N, & \text{if } m = n = 0, \\ N/2, & \text{if } m = n \neq 0, \\ N/2, & \text{if } m = N - n, \\ 0, & \text{elsewhere,} \end{cases}$

$$\sum_{k=0}^{N-1} \sin\left(\frac{2\pi nk}{N}\right) \sin\left(\frac{2\pi mk}{N}\right) = \begin{cases} N/2, & \text{if } m = n \neq 0, \\ -N/2, & \text{if } m = N - n, \\ 0, & \text{elsewhere,} \end{cases} \quad \text{and}$$

$$\sum_{k=0}^{N-1} \sin\left(\frac{2\pi nk}{N}\right) \cos\left(\frac{2\pi mk}{N}\right) = \sum_{k=0}^{N-1} \cos\left(\frac{2\pi nk}{N}\right) \sin\left(\frac{2\pi mk}{N}\right) = 0.$$

Hence,  $x[m] = \frac{1}{N} \sum_{k=0}^{N-1} X_{\text{DHT}}[k] \left( \cos\left(\frac{2\pi mk}{N}\right) + \sin\left(\frac{2\pi mk}{N}\right) \right)$ .

**3.117 (a)**  $y[n] = x(\langle n - n_0 \rangle_N) = \begin{cases} x[n - n_0 + N], & 0 \leq n \leq n_0 - 1, \\ x[n - n_0], & n_0 \leq n \leq N - 1. \end{cases}$

$$Y_{\text{DHT}}[k] = \sum_{n=0}^{N-1} y[n] \left( \cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right) = \sum_{n=0}^{n_0-1} x[n - n_0 + N] \left( \cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right) + \sum_{n=n_0}^{N-1} x[n - n_0] \left( \cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right).$$

Replacing  $n - n_0 + N$  by  $n$  in the first sum and  $n - n_0$  by  $n$  in the second sum we get

$$Y_{\text{DHT}}[k] = \sum_{n=N-n_0}^{N-1} x[n] \left( \cos\left(\frac{2\pi(n+n_0)k}{N}\right) + \sin\left(\frac{2\pi(n+n_0)k}{N}\right) \right) + \sum_{n=0}^{n_0-1} x[n] \left( \cos\left(\frac{2\pi(n+n_0)k}{N}\right) + \sin\left(\frac{2\pi(n+n_0)k}{N}\right) \right)$$

$$\begin{aligned}
&= \sum_{n=0}^{N-1} x[n] \left( \cos\left(\frac{2\pi(n+n_0)k}{N}\right) + \sin\left(\frac{2\pi(n+n_0)k}{N}\right) \right) \\
&= \cos\left(\frac{2\pi n_0 k}{N}\right) \sum_{n=0}^{N-1} x[n] \left( \cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right) \\
&\quad + \sin\left(\frac{2\pi n_0 k}{N}\right) \sum_{n=0}^{n_0-1} x[n] \left( \cos\left(\frac{2\pi nk}{N}\right) - \sin\left(\frac{2\pi nk}{N}\right) \right) \\
&= \cos\left(\frac{2\pi n_0 k}{N}\right) X_{\text{DHT}}[k] + \sin\left(\frac{2\pi n_0 k}{N}\right) X_{\text{DHT}}[-k].
\end{aligned}$$

(b) The  $N$ -point DHT of  $x[\langle -n \rangle_N]$  is  $X_{\text{DHT}}[-k]$ .

$$\begin{aligned}
\text{(c)} \quad \sum_{n=0}^{N-1} x^2[n] &= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} X_{\text{DHT}}[k] X_{\text{DHT}}[l] \times \\
&\quad \left( \sum_{n=0}^{N-1} \left( \cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right) \left( \cos\left(\frac{2\pi nl}{N}\right) + \sin\left(\frac{2\pi nl}{N}\right) \right) \right).
\end{aligned}$$

Using the orthogonality property, the product is non-zero if  $k = l$  and is equal to  $N$ .

$$\text{Thus } \sum_{n=0}^{N-1} x^2[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_{\text{DHT}}^2[k].$$

$$\text{3.118} \quad \cos\left(\frac{2\pi nk}{N}\right) = \frac{1}{2} (W_N^{nk} + W_N^{-nk}), \text{ and } \sin\left(\frac{2\pi nk}{N}\right) = \frac{1}{2j} (W_N^{nk} - W_N^{-nk}).$$

$$X_{\text{DHT}}[k] = \sum_{n=0}^{N-1} x[n] \left( \frac{e^{j2\pi nk/N} + e^{-j2\pi nk/N}}{2} + \frac{e^{j2\pi nk/N} - e^{-j2\pi nk/N}}{2j} \right)$$

$$\text{Therefore } X_{\text{DHT}}[k] = \frac{1}{2} (X[N-k] + X[k] - jX[N-k] + jX[k]).$$

$$\begin{aligned}
\text{3.119} \quad y[n] &= x[n] \circledast g[n]. \text{ Thus, } Y_{\text{DHT}}[k] = \sum_{n=0}^{N-1} y[n] \left( \cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right) \\
&= \sum_{r=0}^{N-1} x[r] \sum_{n=0}^{N-1} g[\langle n-r \rangle_N] \left( \cos\left(\frac{2\pi nk}{N}\right) + \sin\left(\frac{2\pi nk}{N}\right) \right).
\end{aligned}$$

From results of Problem 3.117

$$\begin{aligned}
Y_{\text{DHT}}[k] &= \sum_{l=0}^{N-1} x[l] \left( G_{\text{DHT}}[k] \cos\left(\frac{2\pi lk}{N}\right) + G_{\text{DHT}}[\langle -k \rangle_N] \sin\left(\frac{2\pi lk}{N}\right) \right) \\
&= G_{\text{DHT}}[k] \sum_{l=0}^{N-1} x[l] \cos\left(\frac{2\pi lk}{N}\right) + G_{\text{DHT}}[\langle -k \rangle_N] \sum_{l=0}^{N-1} x[l] \sin\left(\frac{2\pi lk}{N}\right)
\end{aligned}$$

$$= \frac{1}{2} G_{\text{DHT}}[k](X_{\text{DHT}}[k] + X_{\text{DHT}}[\langle -k \rangle_N]) \\ + \frac{1}{2} G_{\text{DHT}}[\langle -k \rangle_N](X_{\text{DHT}}[k] - X_{\text{DHT}}[\langle -k \rangle_N]).$$

$$\text{or } Y_{\text{DHT}}[k] = \frac{1}{2} X_{\text{DHT}}[k](G_{\text{DHT}}[k] + G_{\text{DHT}}[\langle -k \rangle_N]) \\ + \frac{1}{2} X_{\text{DHT}}[\langle -k \rangle_N](G_{\text{DHT}}[k] - G_{\text{DHT}}[\langle -k \rangle_N]).$$

$$\mathbf{3.120} \quad (\mathbf{a}) \quad \mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{H}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad \text{and } \mathbf{H}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

(b) From the structure of  $\mathbf{H}_2$ ,  $\mathbf{H}_4$  and  $\mathbf{H}_8$  it can be seen that

$$\mathbf{H}_4 = \begin{bmatrix} \mathbf{H}_2 & \mathbf{H}_2 \\ \mathbf{H}_2 & -\mathbf{H}_2 \end{bmatrix}, \quad \text{and } \mathbf{H}_8 = \begin{bmatrix} \mathbf{H}_4 & \mathbf{H}_4 \\ \mathbf{H}_4 & -\mathbf{H}_4 \end{bmatrix}.$$

(c)  $\mathbf{X}_{\text{HT}} = \mathbf{H}_N \mathbf{x}$ . Therefore  $\mathbf{x} = \mathbf{H}_N^{-1} \mathbf{X}_{\text{HT}} = \mathbf{N} \mathbf{H}_N^T \mathbf{X}_{\text{HT}} = \mathbf{N} \mathbf{H}_N^* \mathbf{X}_{\text{HT}}$ . Hence,

$$x[n] = \sum_{k=0}^{N-1} X_{\text{HT}}[k] (-1)^{\sum_{i=0}^{l-1} b_i^{(n)} b_i^{(k)}}.$$

where  $b_i(r)$  is the  $i^{\text{th}}$  bit in the binary representation of  $r$ .

$$\mathbf{3.121} \quad X(z_1) = \sum_{n=0}^{N-1} x[n] z_1^{-n} = \sum_{n=0}^{N-1} x[n] A^{-n} V^{ln} = \sum_{n=0}^{N-1} x[n] A^{-n} V^{1^2/2} V^{n^2/2} V^{-(l-n)^2/2} \\ = V^{1^2/2} \sum_{n=0}^{N-1} g[n] h[l-n], \quad \text{where } g[n] = x[n] A^{-n} V^{n^2/2} \quad \text{and } h[n] = V^{-n^2/2}.$$

A block-diagram representation for the computation of  $X(z_1)$  using the above scheme is thus precisely Figure P3.6.

**3.122**  $z_1 = \alpha^l$ . Hence,  $A_0 V_0^{-1} e^{j\theta_0} e^{-jl\phi_0} = \alpha^l$ . Since  $\alpha$  is real, we have

$$A_0 = 1, \quad V_0 = 1/\alpha, \quad \theta_0 = 0 \quad \text{and } \phi_0 = 0.$$

**3.123** (i)  $N = 3$ .

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2}, \quad \text{and } H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2}$$

$$Y_L(z) = h[0]x[0] + (h[1]x[0] + x[1]h[0])z^{-1} + (h[2]x[0] + h[1]x[1] + h[0]x[2])z^{-2}$$

$$+ (h[1]x[2] + h[2]x[1])z^{-3} + h[2]x[2]z^{-4}.$$

On the other hand,

$$Y_c(z) = (h[0]x[0] + h[1]x[2] + h[2]x[0]) + (h[0]x[1] + h[1]x[0] + h[2]x[2])z^{-1} \\ + (h[0]x[2] + h[1]x[1] + h[2]x[0])z^{-2}.$$

It is easy to see that in this case  $Y_c(z) = Y_L(z) \bmod(1 - z^{-3})$ .

(ii)  $N = 4$ .

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} \text{ and } H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3}.$$

$$Y_L(z) = h[0]x[0] + (h[1]x[0] + h[0]x[1])z^{-1} + (h[0]x[2] + h[1]x[1] + h[2]x[0])z^{-2} \\ + (h[0]x[3] + h[1]x[2] + h[2]x[1] + h[3]x[0])z^{-3} + (h[1]x[3] + h[2]x[2] + h[3]x[1])z^{-4} \\ + (h[2]x[3] + h[3]x[2])z^{-5} + h[3]x[3]z^{-6},$$

$$\text{whereas, } Y_c(z) = (h[0]x[0] + h[1]x[3] + h[2]x[2] + h[3]x[1]) \\ + (h[0]x[1] + h[1]x[0] + h[2]x[3] + h[3]x[2])z^{-1} \\ + (h[0]x[2] + h[1]x[1] + h[2]x[0] + h[3]x[3])z^{-2} \\ + (h[0]x[3] + h[1]x[2] + h[2]x[1] + h[3]x[0])z^{-3}.$$

Again it can be seen that  $Y_c(z) = Y_L(z) \bmod(1 - z^{-4})$ .

(ii)  $N = 5$ .

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + x[4]z^{-4} \text{ and}$$

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4}.$$

$$Y_L(z) = h[0]x[0] + (h[1]x[0] + h[0]x[1])z^{-1} + (h[0]x[2] + h[1]x[1] + h[2]x[0])z^{-2} \\ + (h[0]x[3] + h[1]x[2] + h[2]x[1] + h[3]x[0])z^{-3} \\ + (h[0]x[4] + h[1]x[3] + h[2]x[2] + h[3]x[1] + h[4]x[0])z^{-4} \\ + (h[1]x[4] + h[2]x[3] + h[3]x[2] + h[4]x[1])z^{-5} + (h[2]x[4] + h[3]x[3] + h[4]x[2])z^{-6} \\ + (h[4]x[3] + h[3]x[4])z^{-7} + h[4]x[4]z^{-8},$$

$$\text{whereas, } Y_c(z) = (h[0]x[0] + h[4]x[1] + h[3]x[2] + h[2]x[3] + h[1]x[4]) \\ + (h[0]x[1] + h[1]x[0] + h[2]x[4] + h[3]x[3] + h[4]x[2])z^{-1} \\ + (h[0]x[2] + h[1]x[1] + h[2]x[0] + h[3]x[4] + h[4]x[3])z^{-2} \\ + (h[0]x[3] + h[1]x[2] + h[2]x[1] + h[3]x[0] + h[4]x[4])z^{-3} \\ + (h[0]x[4] + h[1]x[3] + h[2]x[2] + h[3]x[1] + h[4]x[0])z^{-4}.$$

Again it can be seen that  $Y_c(z) = Y_L(z) \bmod(1 - z^{-5})$ .

**3.124 (a)**  $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ . Let  $\hat{X}(z) = \log(X(z)) \Rightarrow X(z) = e^{\hat{X}(z)}$ . Thus,  $X(e^{j\omega}) = e^{\hat{X}(e^{j\omega})}$

(b)  $\hat{x}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(X(e^{j\omega})) e^{j\omega n} d\omega$ . If  $x[n]$  is real, then  $X(e^{j\omega}) = X^*(e^{-j\omega})$ . Therefore,

$$\log(X(e^{j\omega})) = \log(X^*(e^{-j\omega})).$$

$$\begin{aligned} \hat{x}^*[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(X^*(e^{j\omega})) e^{-j\omega n} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(X(e^{-j\omega})) e^{-j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(X(e^{j\omega})) e^{j\omega n} d\omega = \hat{x}[n]. \end{aligned}$$

$$\begin{aligned} \text{(c) } \hat{x}_{\text{ev}}[n] &= \frac{\hat{x}[n] + \hat{x}[-n]}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(X(e^{j\omega})) \left( \frac{e^{j\omega n} + e^{-j\omega n}}{2} \right) d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(X(e^{j\omega})) \cos(\omega n) d\omega, \end{aligned}$$

$$\begin{aligned} \text{and similarly, } \hat{x}_{\text{od}}[n] &= \frac{\hat{x}[n] - \hat{x}[-n]}{2} = \frac{j}{2\pi} \int_{-\pi}^{\pi} \log(X(e^{j\omega})) \left( \frac{e^{j\omega n} - e^{-j\omega n}}{2j} \right) d\omega \\ &= \frac{j}{2\pi} \int_{-\pi}^{\pi} \log(X(e^{j\omega})) \sin(\omega n) d\omega. \end{aligned}$$

**3.125**  $x[n] = a\delta[n] + b\delta[n-1]$  and  $X(z) = a + bz^{-1}$ . Also,

$$\hat{X}(z) = \log(a + bz^{-1}) = \log(a) + \log(1 + b/az^{-1}) = \log(a) + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(b/a)^n}{n} z^{-n}. \text{ Therefore,}$$

$$\hat{x}[n] = \begin{cases} \log(a), & \text{if } n = 0, \\ (-1)^{n-1} \frac{(b/a)^n}{n}, & \text{for } n > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

**3.126 (a)**  $\hat{X}(z) = \log(K) + \sum_{k=1}^{N_{\alpha}} \log(1 - \alpha_k z^{-1}) + \sum_{k=1}^{N_{\gamma}} \log(1 - \gamma_k z) - \sum_{k=1}^{N_{\beta}} \log(1 - \beta_k z^{-1}) - \sum_{k=1}^{N_{\delta}} \log(1 - \delta_k z)$ .

$$\hat{X}(z) = \log(K) - \sum_{k=1}^{N_{\alpha}} \sum_{n=1}^{\infty} \frac{\alpha_k^n}{n} z^{-n} - \sum_{k=1}^{N_{\gamma}} \sum_{n=1}^{\infty} \frac{\gamma_k^n}{n} z^n + \sum_{k=1}^{N_{\beta}} \sum_{n=1}^{\infty} \frac{\beta_k^n}{n} z^{-n} + \sum_{k=1}^{N_{\delta}} \sum_{n=1}^{\infty} \frac{\delta_k^n}{n} z^n.$$

$$\text{Thus, } \hat{x}[n] = \begin{cases} \log(K), & n = 0, \\ \sum_{k=1}^{N_{\beta}} \frac{\beta_k^n}{n} - \sum_{k=1}^{N_{\alpha}} \frac{\alpha_k^n}{n}, & n > 0, \\ \sum_{k=1}^{N_{\gamma}} \frac{\gamma_k^{-n}}{n} - \sum_{k=1}^{N_{\delta}} \frac{\delta_k^{-n}}{n}, & n < 0. \end{cases}$$

(b)  $|\hat{x}[n]| < N \frac{|r|^n}{|n|}$  as  $n \rightarrow \infty$ , where  $r$  is the maximum value of  $\alpha_k, \beta_k, \gamma_k$  and  $\delta_k$  for all values of  $k$ , and  $N$  is a constant. Thus  $\hat{x}[n]$  is a decaying bounded sequence as  $n \rightarrow \infty$ .

(c) From Part (a) if  $\alpha_k = \beta_k = 0$  then  $\hat{x}[n] = 0$  for all  $n > 0$ , and is thus anti-causal.

(d) If  $\gamma_k = \delta_k = 0$  then  $\hat{x}[n] = 0$  for all  $n < 0$  and is thus a causal sequence.

**3.127** If  $X(z)$  has no poles and zeros on the unit circle then from Problem 3.95,  $\gamma_k = \delta_k = 0$  and  $\hat{x}[n] = 0$  for all  $n < 0$ .

$$\hat{X}(z) = \log(X(z)) \text{ therefore } \frac{d\hat{X}(z)}{dz} = \frac{1}{X(z)} \frac{dX(z)}{dz}. \text{ Thus, } z \frac{d\hat{X}(z)}{dz} = z X(z) \frac{d\hat{X}(z)}{dz}.$$

$$\text{Taking the inverse } z\text{-transform we get } n x[n] = \sum_{k=-\infty}^{\infty} k \hat{x}[k] x[n-k], n \neq 0.$$

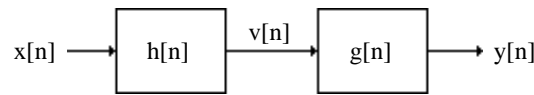
$$\text{Since } x[n] = 0 \text{ and } \hat{x}[n] = 0 \text{ for } n < 0, \text{ thus } x[n] = \sum_{k=0}^n \frac{k}{n} \hat{x}[k] x[n-k], n \neq 0.$$

$$\text{Or, } x[n] = \sum_{k=0}^{n-1} \frac{k}{n} \hat{x}[k] x[n-k] + \hat{x}[n] x[0]. \text{ Hence, } \hat{x}[n] = \frac{x[n]}{x[0]} - \sum_{k=0}^{n-1} \left( \frac{k}{n} \right) \frac{\hat{x}[k] x[n-k]}{x[0]}, n \neq 0.$$

For  $n = 0$ ,  $\hat{x}[0] = \hat{X}(z)|_{z=\infty} = X(z)|_{z=\infty} = \log(x[0])$ . Thus,

$$\hat{x}[n] = \begin{cases} 0, & n < 0, \\ \log(x[0]), & n = 0, \\ \frac{x[n]}{x[0]} - \sum_{k=0}^{n-1} \left( \frac{k}{n} \right) \frac{\hat{x}[k] x[n-k]}{x[0]}, & n > 0. \end{cases}$$

**3.128** This problem is easy to solve using the method discussed in Section 4.13.2.



$$h[n] = 0.6^n \mu[n]. \text{ Thus, } H(z) = \frac{1}{1 - 0.6z^{-1}}. \quad g[n] = 0.8^n \mu[n]. \text{ Thus, } G(z) = \frac{1}{1 - 0.8z^{-1}}.$$

$$\text{From Eq. (4.212) we get } \Phi_{vv}(z) = H(z)H(z^{-1})\Phi_{xx}(z), \quad (\text{A})$$

$$\text{and } \Phi_{yy}(z) = G(z)G(z^{-1})\Phi_{vv}(z) = G(z)G(z^{-1})H(z)H(z^{-1})\Phi_{xx}(z). \quad (\text{B})$$

$$\text{Now, } H(z)H(z^{-1}) = \frac{1}{(1 - 0.6z^{-1})(1 - 0.6z)} = \frac{z^{-1}}{-0.6 + 1.36z^{-1} - 0.6z^{-2}} = \frac{1.5625}{1 - 0.6z^{-1}} + \frac{-1.5625}{1 - 1.6667z^{-1}}.$$

Thus, using Eq. (A) we get

$$\Phi_{vv}(z) = H(z)H(z^{-1})\sigma_x^2 = 1.5625\sigma_x^2 \left( \frac{1}{1-0.6z^{-1}} - \frac{1}{1-1.6667z^{-1}} \right), \quad 0.6 < |z| < 1.6667$$

Taking the inverse transform of the above we get

$$\phi_{vv}[n] = 1.5625\sigma_x^2 (0.6^n \mu[n] - 1.6667^n \mu[-n-1]) = 1.5625\sigma_x^2 0.6^{|n|}.$$

As  $m_x = 0$  and  $m_v = 0$ , we have  $\sigma_v^2 = \phi_{vv}[0] = 1.5625\sigma_x^2$ .

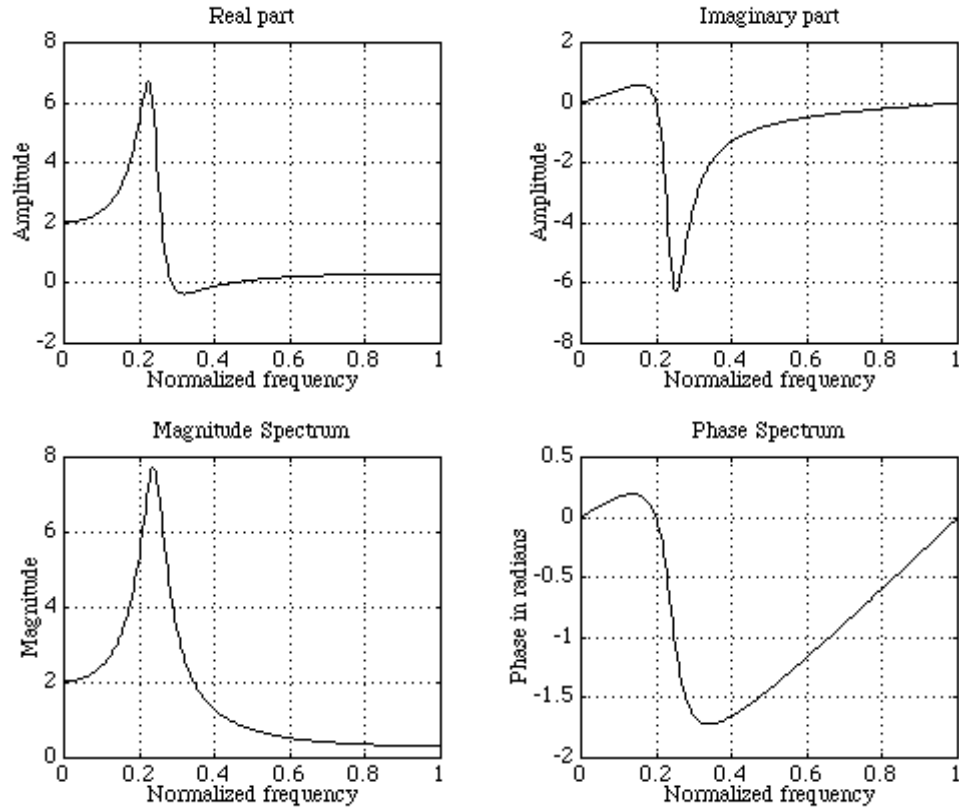
$$\begin{aligned} \text{Next we observe } G(z)G(z^{-1})H(z)H(z^{-1}) &= \frac{1}{(1-0.6z^{-1})(1-0.6z)(1-0.8z^{-1})(1-0.8z)} \\ &= \frac{-15.0240}{1-0.6z^{-1}} + \frac{5.4087}{1-1.6667z^{-1}} + \frac{26.7094}{1-0.8z^{-1}} + \frac{-17.094}{1-1.25z^{-1}}. \end{aligned}$$

Using Eq. (B) and taking the inverse we get

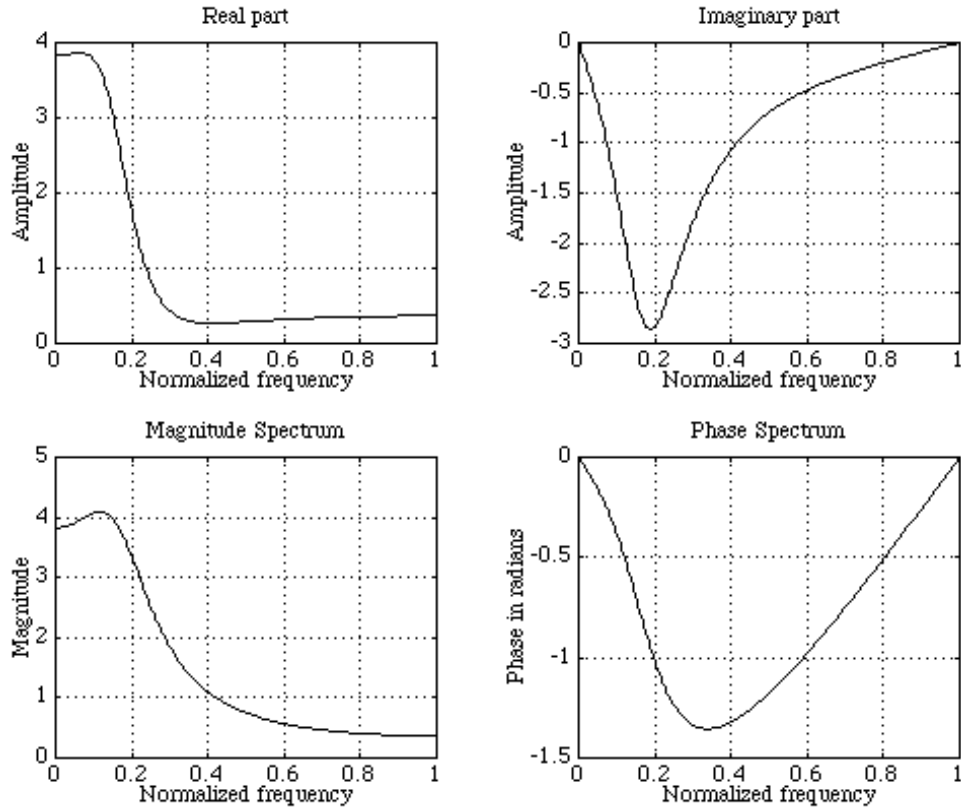
$$\begin{aligned} \phi_{yy}[n] &= \sigma_x^2 (-15.024(0.6)^n \mu[n] - 5.4087(1.6667)^n \mu[-n-1] \\ &\quad + 26.7094(0.8)^n \mu[n] + 17.094(1.25)^n \mu[-n-1]). \end{aligned}$$

As  $m_v = 0$  and  $m_y = 0$ , we have  $\sigma_v^2 = \phi_{yy}[0] = (-15.024 + 26.7094)\sigma_x^2 = 11.6854\sigma_x^2$ .

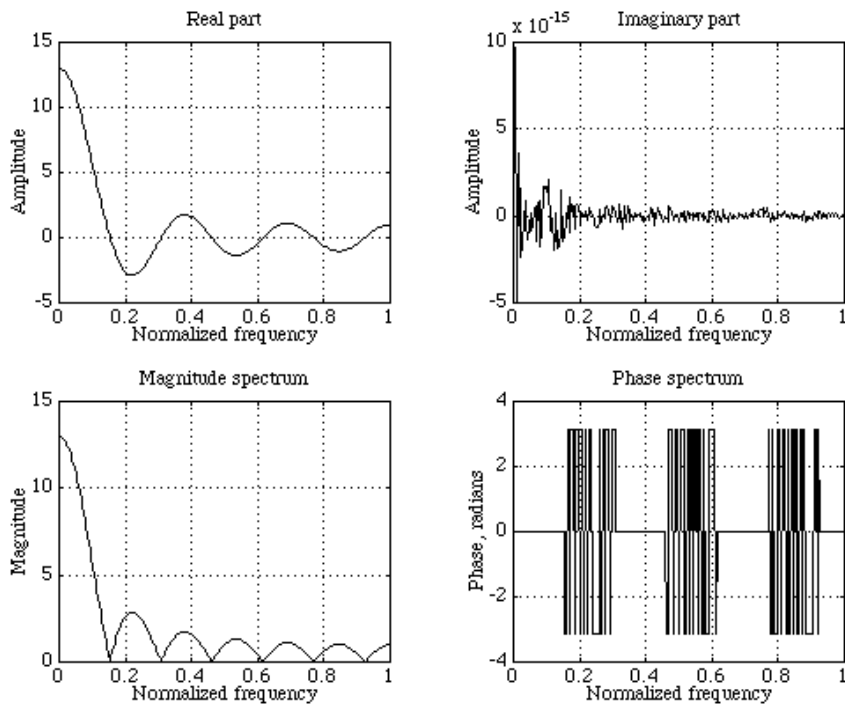
**M3.1 (a)**  $r = 0.9$ ,  $\theta = 0.75$ . The various plots generated by the program are shown below:



**(b)**  $r = 0.7$ ,  $\theta = 0.5$ . The various plots generated by the program are shown below:



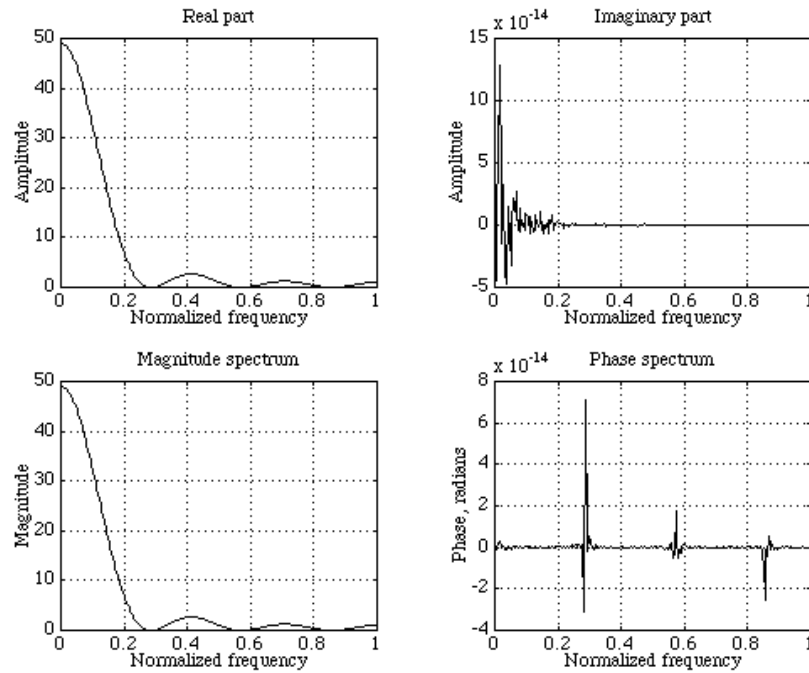
**M3.2 (a)**  $Y_1(e^{j\omega}) = \frac{1 - e^{-j(2N+1)\omega}}{e^{-jN\omega} - e^{-j(N+1)\omega}}$ . For example, for  $N = 6$ ,  $Y_1(e^{j\omega}) = \frac{1 - e^{-j13\omega}}{e^{-j6\omega} - e^{-j7\omega}}$ .



**(b)**  $Y_2(e^{j\omega}) = \frac{1 - 2e^{-j(N+1)\omega} + e^{-j2(N+1)\omega}}{e^{-jN\omega} - 2e^{-j(N+1)\omega} + e^{-j(N+2)\omega}}$ . For example, for  $N = 6$ ,

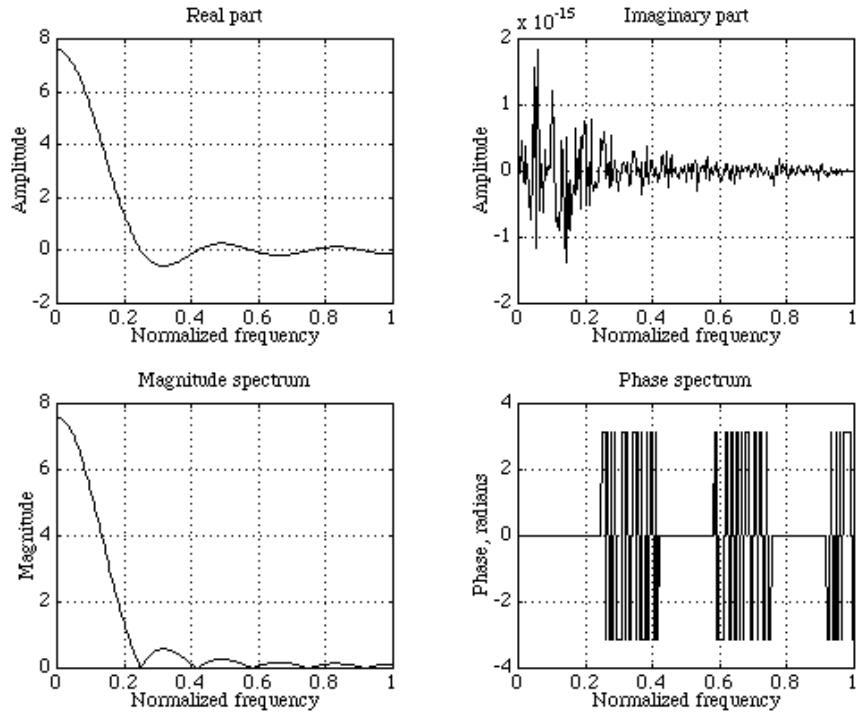


$$Y_2(e^{j\omega}) = \frac{1 - 2e^{-j7\omega} + e^{-j14\omega}}{e^{-j6\omega} - 2e^{-j7\omega} + e^{-j8\omega}}$$

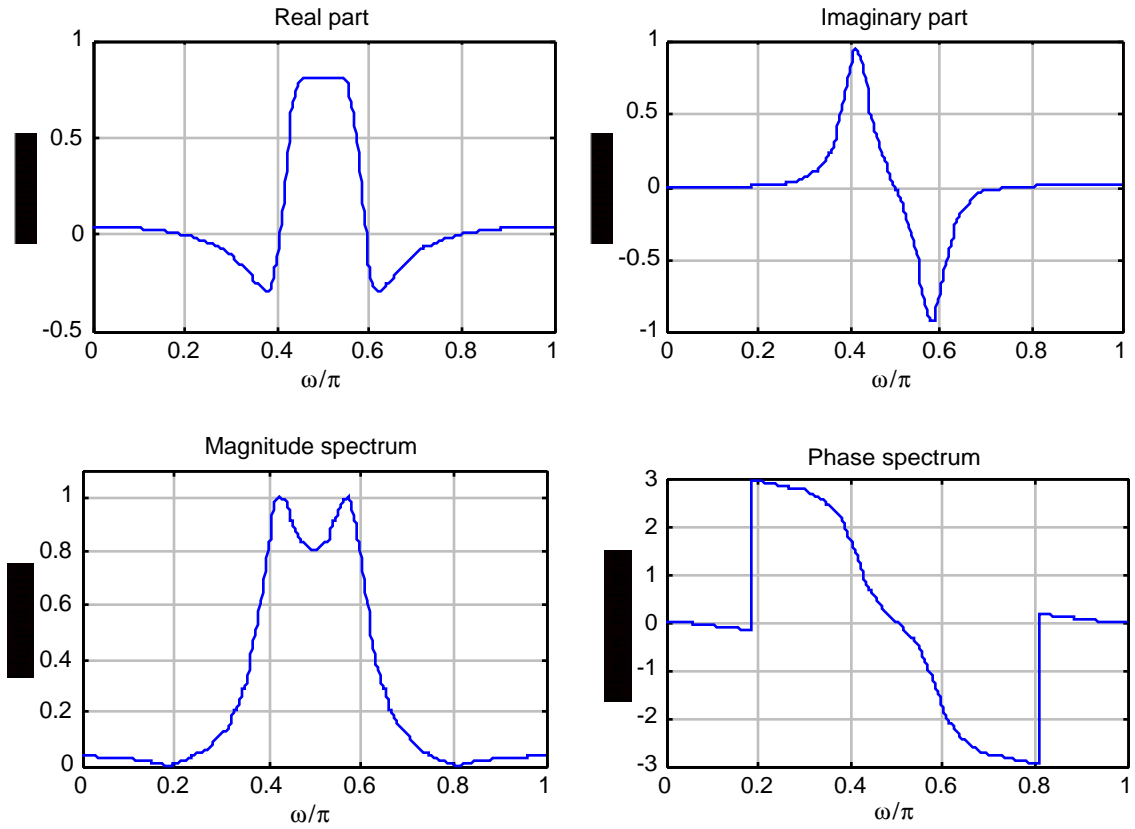


$$\begin{aligned} \text{(c) } Y_3(e^{j\omega}) &= \frac{1}{2} \sum_{n=-N}^N \left( e^{-j\pi n/2N} + e^{j\pi n/2N} \right) e^{-j\omega n} = \sum_{n=-N}^N \cos\left(\frac{\pi n}{2N}\right) e^{-j\omega n} \\ &= e^{jN\omega} \sum_{n=-N}^N \cos\left(\frac{\pi n}{2N}\right) e^{-j(N+n)\omega} = \frac{\sum_{n=-N}^N \cos\left(\frac{\pi n}{2N}\right) e^{-j(N+n)\omega}}{e^{-jN\omega}}. \end{aligned}$$

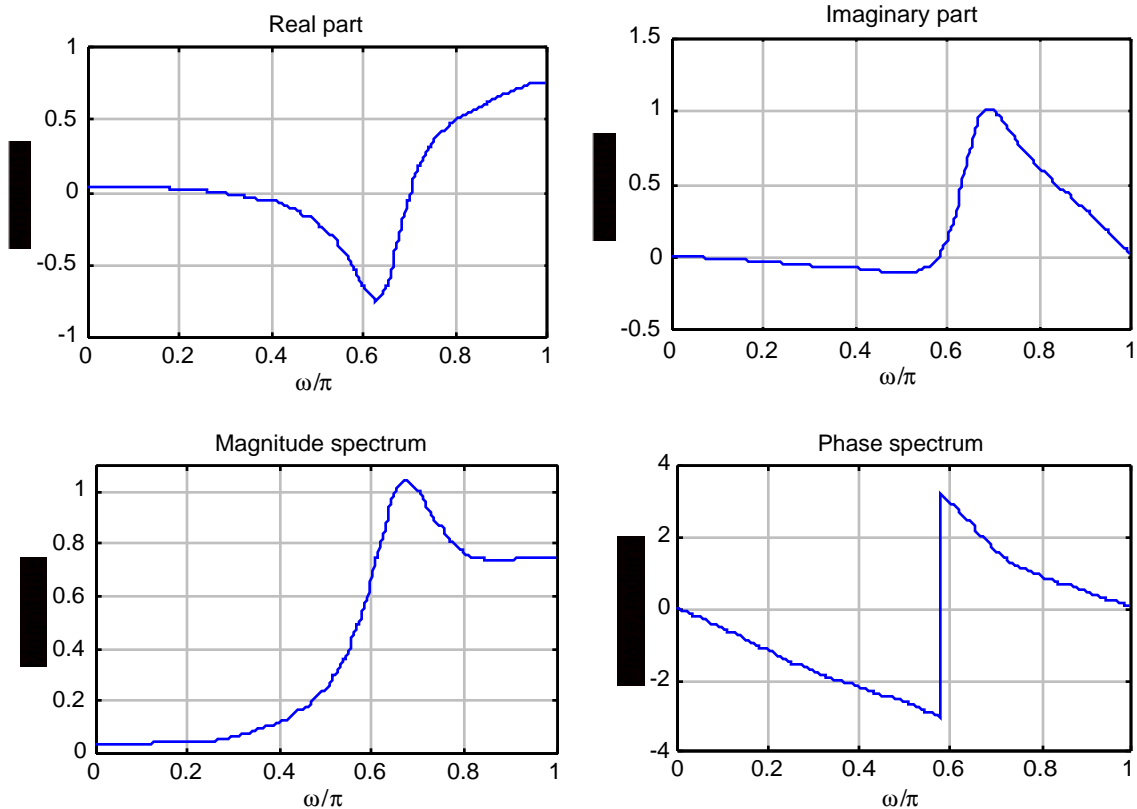
$$\text{For example, for } N = 6, Y_3(e^{j\omega}) = \frac{\sum_{n=-6}^6 \cos\left(\frac{\pi n}{12}\right) e^{-j(6+n)\omega}}{e^{-j6\omega}}.$$



**M3.3 (a)**



**(b)**



```

M3.4 N = input('The length of the sequence = ');
k = 0:N-1;
gamma = -0.5;
g = exp(gamma*k);
% g is an exponential sequence
h = sin(2*pi*k/(N/2));
% h is a sinusoidal sequence with period = N/2
[G,w] = freqz(g,1,512);
[H,w] = freqz(h,1,512);

% Property 1
alpha = 0.5;
beta = 0.25;
y = alpha*g+beta*h;
[Y,w] = freqz(y,1,512);
% Plot Y and alpha*G+beta*H to verify that they are equal

% Property 2
n0 = 12; % S equence shifted by 12 samples
y2 = [zeros([1,n0]) g];
[Y2,w] = freqz(y2,1,512);
G0 = exp(-j*w*n0).*G;
% Plot G0 and Y2 to verify they are equal

% Property 3
w0 = pi/2; % the value of omega0 = pi/2
r=256; %the value of omega0 in terms of number of samples
k = 0:N-1;

```

```

y3 = g.*exp(j*w0*k);
[Y3,w] = freqz(y3,1,512);
k = 0:511;
w = -w0+pi*k/512; % creating G(exp(w-w0))
G1 = freqz(g,1,w');
% Compare G1 and Y3

% Property 4
k = 0:N-1;
y4 = k.*g;
[Y4,w] = freqz(y4,1,512);
% To compute derivative we need sample at pi
y0 = ((-1).^k).*g;
G2 = [G(2:512)' sum(y0)]';
delG = (G2-G)*512/pi;
% Compare Y4, delG

% Property 5
y5 = conv(g,h);
[Y5,w] = freqz(y5,1,512);
% Compare Y5 and G.*H

% Property 6
y6 = g.*h;
[Y6,w] = freqz(y6,1,512,'whole');
[G0,w] = freqz(g,1,512,'whole');
[H0,w] = freqz(h,1,512,'whole');
% Evaluate the sample value at w = pi/2
% and verify with Y6 at pi/2
H1 = [fliplr(H0(1:129)) fliplr(H0(130:512))]'';
val = 1/(512)*sum(G0.*H1);
% Compare val with Y6(129) i.e sample at pi/2
% Can extend this to other points similarly

% Parsevals theorem
val1 = sum(g.*conj(h));
val2 = sum(G0.*conj(H0))/512;
% Compare val1 with val2

M3.5 N = 8; % Number of samples in sequence
gamma = 0.5;
k = 0:N-1;
x = exp(-j*gamma*k);
y = exp(-j*gamma*fliplr(k));
% r = x[-n] then y = r[n-(N-1)]
% so if X1(exp(jw)) is DTFT of x[-n], then
% X1(exp(jw)) = R(exp(jw)) = exp(jw(N-1))Y(exp(jw))
[Y,w] = freqz(y,1,512);
X1 = exp(j*w*(N-1)).*Y;
m = 0:511;
w = -pi*m/512;
X = freqz(x,1,w');
% Verify X = X1

% Property 2
k = 0:N-1;
y = exp(j*gamma*fliplr(k));
[Y,w] = freqz(y,1,512);

```

```

X1 = exp(j*w*(N-1)).*Y;
[X,w] = freqz(x,1,512);
% Verify X1 = conj(X)

% Property 3
y = real(x);
[Y3,w] = freqz(y,1,512);
m = 0:511;
w0 = -pi*m/512;
X1 = freqz(x,1,w0');
[X,w] = freqz(x,1,512);
% Verify Y3 = 0.5*(X+conj(X1))

% Property 4
y = j*imag(x);
[Y4,w] = freqz(y,1,512);
% Verify Y4 = 0.5*(X-conj(X1))

% Property 5
k = 0:N-1;
y = exp(-j*gamma*fliplr(k));
xcs = 0.5*[zeros([1,N-1]) x]+0.5*[conj(y) zeros([1,N-1])];
xacs = 0.5*[zeros([1,N-1]) x]-0.5*[conj(y) zeros([1,N-1])];
[Y5,w] = freqz(xcs,1,512);
[Y6,w] = freqz(xacs,1,512);
Y5 = Y5.*exp(j*w*(N-1));
Y6 = Y6.*exp(j*w*(N-1));
% Verify Y5 = real(X) and Y6 = j*imag(X)

```

```

M3.6 N = 8;
k = 0:N-1;
gamma = 0.5;
x = exp(gamma*k);
y = exp(gamma*fliplr(k));
xev = 0.5*([zeros([1,N-1]) x]+[y zeros([1,N-1])]);
xod = 0.5*([zeros([1,N-1]) x]-[y zeros([1,N-1])]);
[X,w] = freqz(x,1,512);
[Xev,w] = freqz(xev,1,512);
[Xod,w] = freqz(xod,1,512);
Xev = exp(j*w*(N-1)).*Xev;
Xod = exp(j*w*(N-1)).*Xod;
% Verify real(X)= Xev, and imag(X)= Xod

r = 0:511;
w0 = -pi*r/512;
X1 = freqz(x,1,w0');
% Verify X = conj(X1)
% real(X)= real(X1) and imag(X)= -imag(X1)
% abs(X)= abs(X1) and angle(X)= -angle(X1)

```

```

M3.7 N = input('The size of DFT to be computed = ');
for k = 1:N
    for m = 1:N
        D(k,m) = exp(-j*2*pi*(k-1)*(m-1)/N);
    end
end
diff = inv(D)-1/N*conj(D);
% Verify diff is N*N matrix with all elements zero

```

**M3.8 (a)**

```

clf;
N = input('The value of N = ');
k = -N:N;
y1 = ones([1,2*N+1]);
w = 0:2*pi/255:2*pi;
Y1 = freqz(y1, 1, w);
Y1dft = fft(y1);
k = 0:1:2*N;
plot(w/pi,abs(Y1),k*2/(2*N+1),abs(Y1dft),'o');
xlabel('Normalized frequency');ylabel('Amplitude');

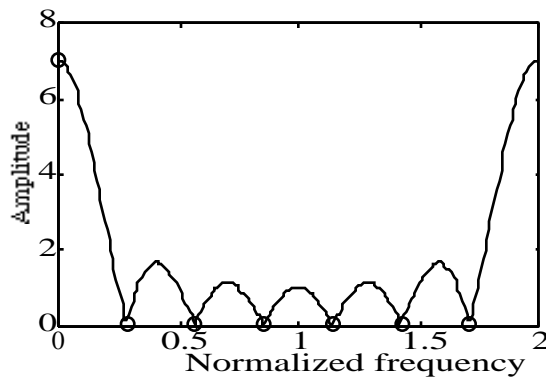
```

**(b)** Add the statement  $y2 = y1 - \text{abs}(k)/N$ ; below the statement  $y1 = \text{ones}([1,2*N+1])$ ; and replace  $y1, Y1,$  and  $Y1dft$  in the program with  $y2, Y2$  and  $Y2dft$ , respectively.

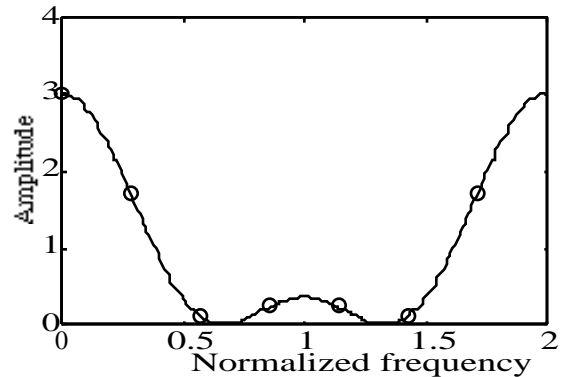
**(c)** Replace the statement  $y1 = \text{ones}([1,2*N+1])$ ; with  $y3 = \cos(\pi*k/(2*N))$ ; , and replace  $y1, Y1,$  and  $Y1dft$  in the program with  $y3, Y3$  and  $Y3dft$ , respectively.

The plots generated for  $N = 3$  is shown below where the circles denote the DFT samples.

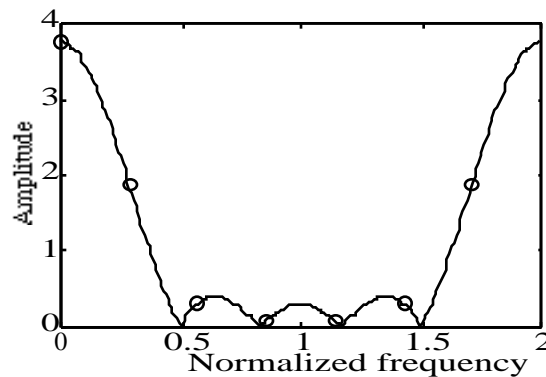
**(a)**



**(b)**



**(c)**



**M3.9**  $g = [3 \ 4 \ -2 \ 0 \ 1 \ -4]$ ;

```

h = [1 -3 0 4 -2 3];
x = [2+j*3 3-j -1+j*2 j*3 2+j*4];
v = [-3-j*2 1+j*4 1+j*2 5+j*3 1+j*2];
k = 0:4;
z = sin(pi*k/2);
y = 2.^k;
G = fft(g); H = fft(h); X = fft(x);
V = fft(v); Z = fft(z); Y = fft(y);
y1 = ifft(G.*H); y2 = ifft(X.*V); y3 = ifft(Z.*Y);

```

```

M3.10 N = 8; % N is length of the sequence(s)
gamma = 0.5;
k = 0:N-1;
g = exp(-gamma*k); h = cos(pi*k/N);
G = fft(g); H=fft(h);

% Property 1
alpha=0.5; beta=0.25;
x1 = alpha*g+beta*h;
X1 = fft(x1);
% Verify X1=alpha*G+beta*H

% Property 2
n0 = N/2; % n0 is the amount of shift
x2 = [g(n0+1:N) g(1:n0)];
X2 = fft(x2);
% Verify X2(k)= exp(-j*k*n0)G(k)

% Property 3
k0 = N/2;
x3 = exp(-j*2*pi*k0*k/N).*g;
X3 = fft(x3);
G3 = [G(k0+1:N) G(1:k0)];
% Verify X3=G3

% Property 4
x4 = G;
X4 = fft(G);
G4 = N*[g(1) g(8:-1:2)]; % This forms N*(g mod(-k))
% Verify X4 = G4;

% Property 5
% To calculate circular convolution between
% g and h use eqn (3.67)
h1 = [h(1) h(N:-1:2)];
T = toeplitz(h',h1);
x5 = T*g';
X5 = fft(x5');
% Verify X5 = G.*H

% Property 6
x6 = g.*h;
X6 = fft(x6);
H1 = [H(1) H(N:-1:2)];
T = toeplitz(H.', H1); % .' is the nonconjugate transpose
G6 = (1/N)*T*G.';
% Verify G6 = X6.'

```

```

M3.11 N = 8; % sequence length
gamma = 0.5;
k = 0:N-1;
x = exp(-gamma*k);
X = fft(x);

% Property 1
X1 = fft(conj(x));
G1 = conj([X(1) X(N:-1:2)]);
% Verify X1 = G1

% Property 2
x2 = conj([x(1) x(N:-1:2)]);
X2 = fft(x2);
% Verify X2 = conj(X)

% Property 3
x3 = real(x);
X3 = fft(x3);
G3 = 0.5*(X+conj([X(1) X(N:-1:2)]));
% Verify X3 = G3

% Property 4
x4 = j*imag(x);
X4 = fft(x4);
G4 = 0.5*(X-conj([X(1) X(N:-1:2)]));
% Verify X4 = G4

% Property 5
x5 = 0.5*(x+conj([x(1) x(N:-1:2)]));
X5 = fft(x5);
% Verify X5 = real(X)

% Property 6
x6 = 0.5*(x-conj([x(1) x(N:-1:2)]));
X6 = fft(x6);
% Verify X6 = j*imag(X)

M3.12 N = 8;
k = 0:N-1;
gamma = 0.5;
x = exp(-gamma*k);
X = fft(x);

% Property 1
xpe = 0.5*(x+[x(1) x(N:-1:2)]);
xpo = 0.5*(x-[x(1) x(N:-1:2)]);
Xpe = fft(xpe);
Xpo = fft(xpo);
% Verify Xpe = real(X) and Xpo = j*imag(X)

% Property 2
X2 = [X(1) X(N:-1:2)];
% Verify X = conj(X2);
% real(X) = real(X2) and imag(X) = -imag(X2)
% abs(X) = abs(X2) and angle(X) = -angle(X2)

```



**M3.13** Using the M-file `fft` we obtain  $X[0]=13$ ,  $X[1]=12.8301+j5.634$ ,  $X[2]=-4+j3.4641$ ,  $X[3]=-8+j7$ ,  $X[4]=13+j1.7321$ ,  $X[5]=4.1699+j7.366$ ,  $X[6]=-13$ ,  $X[7]=4.1699-j7.366$ ,  $X[8]=13-j1.7321$ ,  $X[9]=-8-j7$ ,  $X[10]=-4-j3.4641$ ,  $X[11]=12.8301-j5.634$ .

(a)  $X[0] = 13$ , (b)  $X[6] = -13$ , (c)  $\sum_{k=0}^{11} X[k] = 36$ , (d)  $\sum_{k=0}^{11} e^{-j(4\pi k/6)} X[k] = -48$ ,  
 (e)  $\sum_{k=0}^{11} |X[k]|^2 = 1500$ .

**M3.14** Using the M-file `ifft` we obtain  $x[0]=2.2857$ ,  $x[1]=-0.2643$ ,  $x[2]=0.7713$ ,  $x[3]=-0.4754$ ,  $x[4]=-1.1362$ ,  $x[6]=1.6962$ ,  $x[6]=3.5057$ ,  $x[7]=-0.8571$ ,  $x[8]=2.0763$ ,  $x[9]=-0.6256$ ,  $x[10]=1.9748$ ,  $x[11]=1.0625$ ,  $x[12]=1.5224$ ,  $x[13]=0.4637$ ,

(a)  $x[0] = 2.2857$ , (b)  $x[7] = -0.8571$ , (c)  $\sum_{n=0}^{13} x[n] = 12$ , (d)  $\sum_{n=0}^{13} e^{j(4\pi n/7)} x[n] = -2-j2$ ,  
 (e)  $\sum_{n=0}^{13} |x[n]|^2 = 35.5714$ ,

**M3.15**

```
function y = overlapsave(x,h)
X = length(x);           %Length of longer sequence
M = length(h);          %length of shorter sequence
flops(0);
if (M > X)               %Error condition
    disp('error');
end
%clear all
temp = ceil(log2(M));    %Find length of circular convolution
N = 2^temp;             %zero padding the shorter sequence
if(N > M)
    for i = M+1:N
        h(i) = 0;
    end
end
m = ceil((-N/(N-M+1)));
while (m*(N-M+1) <= X)
    if((N+m*(N-M+1)) <= X)&((m*(N-M+1)) > 0))
        for n = 1:N
            x1(n) = x(n+m*(N-M+1));
        end
    end
    if((m*(N-M+1))<=0)&((N+m*(N-M+1))>=0)) %underflow adjustment
        for n = 1:N
            x1(n) = 0;
        end
        for n = m*(N-M+1):N+m*(N-M+1)
            if(n > 0)
                x1(n-m*(N-M+1)) = x(n);
            end
        end
    end
end
end
```

```

if((N+m*(N-M+1)) > X) %overflow adjustment
    for n = 1:N
        x1(n) = 0;
    end
    for n = 1:(X-m*(N-M+1))
        x1(n) =x (m*(N-M+1)+n);
    end
end

w1 = circonv(h,x1); %circular convolution using DFT
for i = 1:M-1
    y1(i) = 0;
end
for i = M:N
    y1(i) = w1(i);
end
for j = M:N
    if((j+m*(N-M+1)) < (X+M))
        if((j+m*(N-M+1)) > 0)
            yO(j+m*(N-M+1)) = y1(j);
        end
    end
end
end
m = m+1;
end
disp('Number of Floating Point Operations')
flops
%disp('Convolution using Overlap Save:');
y = real(yO);

function y = circonv(x1,x2)
L1 = length(x1); L2 = length(x2);
if L1 ~= L2,
    error('Sequences of unequal lengths'),
end
X1 = fft(x1);
X2 = fft(x2);
X_RES = X1.*X2;
y = ifft(X_RES);

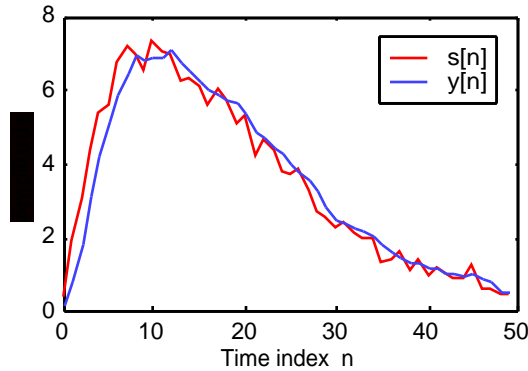
```

The MATLAB program for performing convolution using the overlap-save method is

```

h = [1 1 1]/3;
R = 50;
d = rand(1,R) - 0.5;
m = 0:1:R-1;
s = 2*m.*(0.9.^m);
x = s + d;
%x = [x x x x x x x];
y = overlapsave(x,h);
k = 0:R-1;
plot(k,x,'r-',k,y(1:R),'b--');
xlabel('Time index n');ylabel('Amplitude');
legend('r-', 's[n]','b--','y[n]');

```



**M3.16 (a)** Using the M-file `roots` we determine the zeros and the poles of  $G_1(z)$  which are given by  $z_1 = -3.5616$ ,  $z_2 = -0.4500 + j0.7730$ ,  $z_3 = -0.4500 - j0.7730$ ,  $z_4 = 0.5616$ , **and**  $p_1 = -1.0000 + j1.7321$ ,  $p_2 = -1.0000 - j1.7321$ ,  $p_3 = 0.6000 + j0.3742$ ,  $p_4 = 0.6000 - j0.3742$ .

Next using the M-file `conv` we determine the quadratic factors corresponding to the complex conjugate zeros and poles resulting in the following factored form of  $G_1(z)$ :

$$G_1(z) = \frac{4}{3} \cdot \frac{(1 + 3.5616z^{-1})(1 - 0.5616z^{-1})(1 + 0.9z^{-1} + 0.8z^{-2})}{(1 + 2z^{-1} + 4z^{-2})(1 - 1.2z^{-1} + 0.5z^{-2})}.$$

**(b)** Using the M-file `roots` we determine the zeros and the poles of  $G_2(z)$  which are given by  $z_1 = -2$ ,  $z_2 = -1$ ,  $z_3 = -0.5$ ,  $z_4 = 0.3$ , **and**  $p_1 = -1.0482 + j1.7021$ ,  $p_2 = -1.0482 - j1.7021$ ,  $p_3 = -0.6094$ ,  $p_4 = -0.3942$ .

Next using the M-file `conv` we determine the quadratic factors corresponding to the complex conjugate zeros and poles resulting in the following factored form of  $G_2(z)$ :

$$G_2(z) = \frac{2}{5} \cdot \frac{(1 + 2z^{-1})(1 + z^{-1})(1 + 0.5z^{-1})(1 - 0.3z^{-1})}{(1 + 0.6094z^{-1})(1 + 0.3942z^{-1})(1 + 2.0963z^{-1} + 3.9957z^{-2})}$$

**M3.17** Using Program 3\_9 we arrive at we get

```
Residues
  3.0000  2.0000
Poles
 -1.0000 -0.3333
Constant
 [ ]
```

Hence,  $Y_1(z) = \frac{3}{1+z^{-1}} - \frac{2}{1+\frac{1}{3}z^{-1}}$ , which is same as that determined in Problem 3.98.

**M3.18 (a)** A partial-fraction expansion of  $X_a(z) = \frac{4 - 3z^{-1} + 3z^{-2}}{(z+2)(z-3)^2} = \frac{4z^{-3} - 3z^{-4} + 3z^{-5}}{1 - 4z^{-1} - 3z^{-2} + 18z^{-3}}$ .

using Program 3\_9 we arrive at

$X_a(z) = 0.2361z^{-2} - 0.1389z^{-1} + 0.1667 - \frac{0.1852}{1-3z^{-1}} + \frac{0.0741}{(1-3z^{-1})^2} - \frac{0.125}{1+2z^{-1}}$ . Since the ROC is

given by  $|z| > 3$ , the inverse z-transform is a right-sided sequence and is thus given by  
 $x_a[n] = 0.2361\delta[n-2] - 0.1389\delta[n-1] + 0.1667\delta[n]$   
 $- 0.1852(3)^n\mu[n] + 0.0741n(3)^n\mu[n] - 0.125(-2)^n\mu[n]$ .

A more compact solution is obtained by writing  $X_a(z)$  as  $X_a(z) = z^{-3} \left( \frac{4 - 3z^{-1} + 3z^{-2}}{1 - 4z^{-1} - 3z^{-2} + 18z^{-3}} \right)$ ,  
and making a partial-fraction expansion of the function inside the brackets:

$$X_a(z) = z^{-3} \left( \frac{1}{1-3z^{-1}} + \frac{2}{(1-3z^{-1})^2} + \frac{1}{1+2z^{-1}} \right) = z^{-3} \left( \frac{1}{1-3z^{-1}} + \frac{1}{1+2z^{-1}} \right) + z^{-2} \left( \frac{2z^{-1}}{(1-3z^{-1})^2} \right)$$

The inverse z-transform of the above function is given by

$$x_a[n] = (3)^{n-3}\mu[n-3] + (-2)^{n-3}\mu[n-3] + \frac{2}{3}(n-2)(3)^{n-2}\mu[n-2]$$

$$= [(3)^{n-3}(2n-3) + (-2)^{n-3}]\mu[n-3].$$

(b) Here the ROC is  $|z| < 2$ , hence, the inverse z-transform of

$$X_b(z) = z^{-3} \left( \frac{1}{1-3z^{-1}} + \frac{1}{1+2z^{-1}} \right) + z^{-2} \left( \frac{2z^{-1}}{(1-3z^{-1})^2} \right)$$
 is thus given by

$$x_b[n] = -(3)^{n-3}\mu[-n+2] - (-2)^{n-3}\mu[-n+2] - \frac{2}{3}(n-2)(3)^{n-2}\mu[-n+1]$$

$$= [-(3)^{n-3}(2n-3) - (-2)^{n-3}]\mu[-n+2].$$

(c) In this case, the ROC is given by  $2 < |z| < 3$ . Hence, the inverse z-transform of

$$X_c(z) = z^{-3} \left( \frac{1}{1-3z^{-1}} + \frac{1}{1+2z^{-1}} \right) + z^{-2} \left( \frac{2z^{-1}}{(1-3z^{-1})^2} \right)$$
 is thus given by

$$x_c[n] = -(3)^{n-3}\mu[-n+2] + (-2)^{n-3}\mu[n-3] - \frac{2}{3}(n-2)(3)^{n-2}\mu[-n+1]$$

$$= -(3)^{n-3}(2n-3)\mu[-n+2] + (-2)^{n-3}\mu[n-3].$$

**M3.19 (a)** Using the statement

$$[\text{num}, \text{den}] = \text{residuez}([10/4 \quad -8/2], [-1/4 \quad -1/2], 0);$$
 we get  
 $\text{num} = -3.5 \quad -1.25 \quad -0.25$  and  $\text{den} = 1 \quad 0.75 \quad 0.125$ .

Hence the transfer function is given by  $X_1(z) = -\frac{3.5 + 1.25z^{-1} + 0.25z^{-2}}{1 + 0.75z^{-1} + 0.125z^{-2}}$ .

(b) Using the statement  $[r, p, k] = \text{residuez}([-3 \quad -1], [1 \quad 0 \quad -0.25]);$  we

first obtain a partial-fraction expansion of  $-\frac{3+z^{-1}}{1-0.25z^{-2}}$  which yields

$$r = -2.5 \quad 0.5, \quad p = 0.5 \quad -0.5 \quad \text{and} \quad k = 0.$$

Therefore we can write

$$X_2(z) = 3.5 - \frac{2}{1-0.5z^{-1}} - \frac{2.5}{1-0.5z^{-1}} + \frac{0.5}{1+0.5z^{-1}} = 3.5 - \frac{4.5}{1-0.5z^{-1}} + \frac{0.5}{1+0.5z^{-1}}.$$
 Next we use the

statement  $[\text{num}, \text{den}] = \text{residuez}([-4.5 \quad 0.5], [0.5 \quad -0.5], 3.5);$  which

yields num = -0.5    -2.5    -0.075 and den = 1    0    0.25. Therefore,

$$X_2(z) = -\frac{0.5 + 2.5z^{-1} + 0.875z^{-2}}{1 - 0.25z^{-2}}.$$

(c) We first make a partial-fraction expansion of  $\frac{3}{1+0.81z^{-2}}$  using the statement `[r,p,k] =`

`residuez(3, [1 0 0.81], 0)`; which yields  $\frac{3}{1+0.81z^{-2}} = \frac{1.5}{1+j0.9z^{-1}} + \frac{1.5}{1-j0.9z^{-1}}$ .

Hence, we can write  $X_3(z) = \frac{1.5}{1+j0.9z^{-1}} + \frac{1.5}{1-j0.9z^{-1}} - \frac{4/3}{1+\frac{2}{3}z^{-1}} + \frac{5/9}{\left(1+\frac{2}{3}z^{-1}\right)^2}$ . Using Program

$$3\_10 \text{ we then obtain } X_3(z) = \frac{2.2222 + 3.1111z^{-1} + 0.70333z^{-2} - 0.72z^{-3}}{1 + 1.3333z^{-1} + 1.2544z^{-2} + 1.08z^{-3} + 0.36z^{-4}}.$$

(d) We first make a partial-fraction expansion of  $\frac{\frac{1}{6}z^{-1}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$  using the statement

`[r,p,k] = residuez([0 1 0]/6, [1 5/6 1/6], 0)`; which yields  $-\frac{1}{1+\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}}$ . Hence, we can write  $X_4(z) = 4 + \frac{10/5}{1+\frac{2}{5}z^{-1}} - \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}}$ . Using

$$\text{Program 3\_10 we then arrive at } X_4(z) = \frac{6 + 6.7667z^{-1} + 2.4z^{-2} + 0.2667z^{-3}}{1 + 1.2333z^{-1} + 0.5z^{-2} + 0.0667z^{-3}}.$$

**M3.20 (a)** From Part (a) of M3.19 we observe  $X_1(z) = -2 + \frac{2.5}{1+\frac{1}{4}z^{-1}} - \frac{4}{1+\frac{1}{2}z^{-1}}$ . Hence its inverse

z-transform is given by  $x_1[n] = -2\delta[n] + 2.5(-0.25)^n\mu[n] - 4(-0.5)^n\mu[n]$ . Evaluating for values of  $n = 0, 1, \dots, 9$  we get  $x_1[0] = -3.5$ ,  $x_1[1] = 1.375$ ,  $x_1[2] = -0.8438$ ,  $x_1[3] = 0.4609$ ,  $x_1[4] = -0.2402$ ,  $x_1[5] = 0.1226$ ,  $x_1[6] = -0.0619$ ,  $x_1[7] = 0.0311$ ,  $x_1[8] = -0.0156$ ,  $x_1[9] = 0.0078$

Using Program 3\_11 we get

```

Coefficients of the power series expansion
Columns 1 through 6
-3.5000    1.3750    -0.8438    0.4609    -0.2402    0.1226
Columns 7 through 10
-0.0619    0.0311    -0.0156    0.0078

```

(b) From Part (b) of M3.19 we observe  $X_2(z) = 3.5 - \frac{4.5}{1-0.5z^{-1}} + \frac{0.5}{1+0.5z^{-1}}$ . Hence, its inverse

z-transform is given by  $x_2[n] = 3.5\delta[n] - 4.5(0.5)^n\mu[n] + 0.5(-0.5)^n\mu[n]$ . Evaluating for values of  $n = 0, 1, \dots, 9$  we get  $x_2[0] = -0.5$ ,  $x_2[1] = -2.5$ ,  $x_2[2] = -0.1$ ,  $x_2[3] = -0.625$ ,  $x_2[4] = -0.25$ ,  $x_2[5] = -0.1562$ ,  $x_2[6] = -0.0625$ ,  $x_2[7] = -0.0391$ ,  $x_2[8] = -0.0156$ ,  $x_2[9] = -0.0098$ ,

Using Program 3\_11 we get

Coefficients of the power series expansion  
Columns 1 through 6  
-0.5000    -2.5000    -0.1000    -0.6250    -0.2500    -0.1562

Columns 7 through 10  
-0.0625    -0.0391    -0.0156    -0.0098

(c) From Part (c) of M3.19 we observe  $X_3(z) = \frac{3}{1+0.81z^{-2}} - \frac{4/3}{1+\frac{2}{3}z^{-1}} + \frac{5/9}{\left(1+\frac{2}{3}z^{-1}\right)^2}$ . Hence,

its inverse z-transform is given by

$$x_3[n] = 3(0.9)^n \cos(\pi n / 2) \mu[n+1] - \frac{4}{3} \left(-\frac{2}{3}\right)^n \mu[n] + \frac{5}{9} \left(-\frac{3}{2}\right)(n+1) \left(-\frac{2}{3}\right)^{n+1} \mu[n+1]$$

$$= 3(0.9)^n \cos(\pi n / 2) \mu[n] - \frac{4}{3} \left(-\frac{2}{3}\right)^n \mu[n] + \frac{5}{9}(n+1) \left(-\frac{2}{3}\right)^n \mu[n].$$

Evaluating for values of  $n = 0,$

$1, \dots,$  we get  $x_3[0]=2.2222, x_3[1]=0.14815, x_3[2]=-2.2819, x_3[3]=-0.26337,$   
 $x_3[4]=2.2536, x_3[5]=-0.26337, x_3[6]=-1.37, x_3[7]=-0.18209, x_3[8]=1.4345,$   
 $x_3[9]=-0.10983.$  Using Program 3\_11 we get

Coefficients of the power series expansion  
Columns 1 through 6  
2.2222    0.14815    -2.2819    -0.26337    2.2536    -0.26337

Columns 7 through 10  
-1.37    -0.18209    1.4345    -0.10983

(d) From Part (d) of M3.19 we observe  $X_4(z) = 4 + \frac{10/5}{1+\frac{2}{5}z^{-1}} - \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}}$ . Hence, its

inverse z-transform is given by  $x_4[n] = 4\delta[n] + 2(-2/5)^n \mu[n] - (-1/2)^n \mu[n] + (-1/3)^n \mu[n].$   
Evaluating for values of  $n = 0, 1, \dots,$  we get  $x_4[0]=6, x_4[1]=-0.6333, x_4[2]=0.1811,$   
 $x_4[3]=-0.04, x_4[4]=0.0010, x_4[5]=0.0067, x_4[6]=-0.0061, x_4[7]=0.0041, x_4[8]=-0.0024,$   
 $x_4[9]=0.0014.$  Using Program 3\_11 we get

Coefficients of the power series expansion  
Columns 1 through 6  
6.0000    -0.6331    0.1808    -0.0399    0.0011    0.0066

Columns 8 through 10  
-0.0060    0.0040    -0.0024    0.0014

```
M3.21 % As an example try a sequence x = 0:24;
% calculate the actual uniform dft
% and then use these uniform samples
% with this ndft program to get the
% the original sequence back
% [X,w] = freqz(x,1,25,'whole');
% use freq = X and points = exp(j*w)

freq = input('The sample values = ');
points = input('Frequencies at which samples are taken = ');
L = 1;
```

```

len = length(points);
val = zeros(size(1,len));
L = poly(points);
for k = 1:len
    if(freq(k) ~= 0)
        xx = [1 -points(k)];
        [yy, rr] = deconv(L,xx);
        F(k,:) = yy;
        down = polyval(yy,points(k))*(points(k)^(-len+1));
        F(k,:) = freq(k)/down*yy;
        val = val+F(k,:);
    end
end
coeff = val;

```

## Chapter 4 (2e)

**4.1** If  $u[n] = z^n$  is the input to the LTI discrete-time system, then its output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = z^n H(z),$$

where  $H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$ . Hence  $u[n] = z^n$  is an eigenfunction of the LTI discrete-time

system. If  $v[n] = z^n \mu[n]$  is the input to the system, then its output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]v[n-k] = z^n \sum_{k=-\infty}^{\infty} h[k]\mu[n-k]z^{-k} = z^n \sum_{k=-\infty}^n h[k]z^{-k}.$$

Since in this case the summation depends upon  $n$ ,  $v[n] = z^n \mu[n]$  is not an eigenfunction of an LTI discrete-time system.

**4.2**  $h[n] = \delta[n] - \alpha\delta[n-R]$ . Taking the DTFT of both sides we get  $H(e^{j\omega}) = 1 - \alpha e^{-j\omega R}$ .

Let  $\alpha = |\alpha| e^{j\phi}$ , then the maximum value of  $|H(e^{j\omega})|$  is  $1 + |\alpha|$  and the minimum value of  $|H(e^{j\omega})|$  is  $1 - |\alpha|$ .

**4.3**  $G(e^{j\omega}) = (H(e^{j\omega}))^3 = (1 - \alpha e^{-j\omega R})^3$ .

**4.4**  $G(e^{j\omega}) = \sum_{n=0}^{M-1} \alpha^n e^{-j\omega n} = \frac{1 - \alpha^M e^{-j\omega M}}{1 - \alpha e^{-j\omega}}$ . Note that  $G(e^{j\omega}) = H(e^{j\omega})$ , for  $\alpha = 1$ . In order to have the dc value of the magnitude response equal to unity the impulse response should be multiplied by a factor of  $K$ , where  $K = \left| \frac{1 - \alpha}{1 - \alpha^M} \right|$ .

**4.5** The group delay  $\tau(\omega)$  of an LTI discrete-time system with a frequency response  $H(e^{j\omega}) = |H(e^{j\omega})| e^{j\phi(\omega)}$ , is given by  $\tau(\omega) = -\frac{d\phi(\omega)}{d\omega}$ . Now,

$$\frac{dH(e^{j\omega})}{d\omega} = e^{j\phi(\omega)} \frac{d|H(e^{j\omega})|}{d\omega} + j|H(e^{j\omega})| e^{j\phi(\omega)} \frac{d\phi(\omega)}{d\omega}.$$

Hence,  $-j|H(e^{j\omega})| e^{j\phi(\omega)} \frac{d\phi(\omega)}{d\omega} = e^{j\phi(\omega)} \frac{d|H(e^{j\omega})|}{d\omega} - \frac{dH(e^{j\omega})}{d\omega}$ . Equivalently,

$$-\frac{d\phi(\omega)}{d\omega} = \frac{e^{j\phi(\omega)}}{j|H(e^{j\omega})| e^{j\phi(\omega)}} \frac{d|H(e^{j\omega})|}{d\omega} - \frac{\frac{dH(e^{j\omega})}{d\omega}}{jH(e^{j\omega})} = \frac{1}{j|H(e^{j\omega})|} \frac{d|H(e^{j\omega})|}{d\omega} + j \frac{\frac{dH(e^{j\omega})}{d\omega}}{H(e^{j\omega})}.$$

The first term on the right hand side is purely imaginary. Hence,



$$\tau(\omega) = -\frac{d\phi(\omega)}{d\omega} = \operatorname{Re} \left[ j \frac{d(H(e^{j\omega}))}{d\omega} \frac{1}{H(e^{j\omega})} \right].$$

**4.6**  $H(e^{j\omega}) = [(a_1 + a_5) \cos 2\omega + (a_2 + a_4) \cos \omega + a_3] + j[(a_1 - a_5) \sin 2\omega + (a_2 - a_4) \sin \omega]$ . Hence, the frequency response will have zero phase for  $a_1 = a_5$ , and  $a_2 = a_4$ .

**4.7**  $H(e^{j\omega}) = a_1 + a_2 e^{-j\omega} + a_3 e^{-j2\omega} + a_4 e^{-j3\omega} + a_5 e^{-j4\omega} + a_6 e^{-j5\omega} + a_7 e^{-j6\omega}$   
 $= (a_1 e^{j3\omega} + a_7 e^{-j3\omega}) e^{-j3\omega} + (a_2 e^{j2\omega} + a_6 e^{-j2\omega}) e^{-j3\omega} + (a_3 e^{j\omega} + a_5 e^{-j\omega}) e^{-j3\omega} + a_4 e^{-j3\omega}$ . If  $a_1 = a_7$ ,  $a_2 = a_6$ , and  $a_3 = a_5$ , then we can write  
 $H(e^{j\omega}) = [2(a_1 + a_7) \cos(3\omega) + 2(a_2 + a_6) \cos(2\omega) + 2(a_3 + a_5) \cos(\omega) + a_4] e^{-j3\omega}$ , which is seen to have linear phase.

**4.8** The frequency response of the LTI discrete-time system is given by  
 $H(e^{j\omega}) = a_1 e^{j\omega k} + a_2 e^{j\omega(k-1)} + a_3 e^{j\omega(k-2)} + a_4 e^{j\omega(k-3)} + a_5 e^{j\omega(k-4)}$   
 $= e^{j\omega(k-2)} (a_1 e^{j2\omega} + a_2 e^{j\omega} + a_3 + a_4 e^{-j\omega} + a_5 e^{-j2\omega})$   
 $= e^{j\omega(k-2)} (2a_1 \cos(2\omega) + 2a_2 \cos(\omega) + a_3)$

Hence for  $H(e^{j\omega})$  will be real for  $k = 2$ .

**4.9**  $H_1(e^{j\omega}) = \alpha + e^{-j\omega}$  and  $H_2(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}$ . Thus,  $H(e^{j\omega}) = H_1(e^{j\omega}) H_2(e^{j\omega}) = \frac{\alpha + e^{-j\omega}}{1 - \beta e^{-j\omega}}$ .

$$|H(e^{j\omega})|^2 = \frac{(\alpha + e^{-j\omega})(\alpha + e^{j\omega})}{(1 - \beta e^{-j\omega})(1 - \beta e^{j\omega})} = \frac{\alpha^2 + 2\alpha \cos \omega + 1}{1 - 2\beta \cos \omega + \beta^2}. \quad |H(e^{j\omega})|^2 = 1, \text{ if } \alpha = -\beta.$$

**4.10**  $Y(e^{j\omega}) = |X(e^{j\omega})|^\alpha e^{j \arg X(e^{j\omega})}$ . Hence  $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = |X(e^{j\omega})|^{(\alpha-1)}$ . Since  $H(e^{j\omega})$  is real, it has zero-phase.

**4.11**  $y[n] = x[n] + \alpha y[n - R]$ .  $Y(e^{j\omega}) = X(e^{j\omega}) + \alpha e^{-j\omega R} Y(e^{j\omega})$ . Hence,

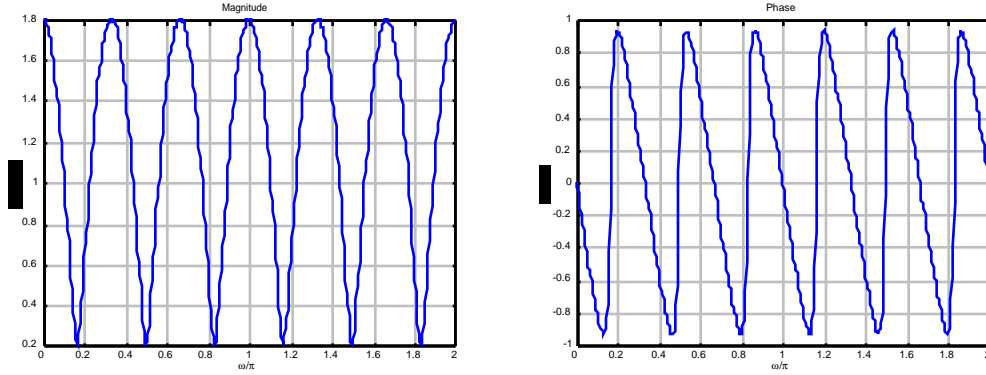
$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \alpha e^{-j\omega R}}. \quad \text{Maximum value of } |H(e^{j\omega})| \text{ is } 1/(1 - |\alpha|), \text{ and the}$$

minimum value is  $1/(1 + |\alpha|)$ . There are  $R$  peaks and dips in the range  $0 \leq \omega < 2\pi$ . The locations of peaks and dips are given by  $1 - \alpha e^{-j\omega R} = 1 \pm |\alpha|$ , or  $e^{-j\omega R} = \pm \frac{|\alpha|}{\alpha}$ . The locations

of peaks are given by  $\omega = \omega_k = \frac{2\pi k}{R}$  and the locations of dips are given by

$$\omega = \omega_k = \frac{(2\pi + 1)k}{R}, \quad 0 \leq k \leq R - 1.$$

Plots of the magnitude and phase responses of  $H(e^{j\omega})$  for  $\alpha = 0.8$  and  $R = 6$  are shown below:



**4.12**  $A(e^{j\omega}) = \frac{b_0 + b_1 e^{-j\omega} + b_2 e^{-j2\omega}}{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}} = \frac{(b_0 e^{j\omega} + b_2 e^{-j\omega}) + b_1}{(e^{j\omega} + a_2 e^{-j\omega}) + a_1}$   
 $= \frac{b_1 + (b_0 + b_2) \cos \omega + j(b_0 - b_2) \sin \omega}{a_1 + (1 + a_2) \cos \omega + j(1 - a_2) \sin \omega}$ . Therefore,  
 $|A(e^{j\omega})|^2 = \frac{[b_1 + (b_0 + b_2) \cos \omega]^2 + (b_0 - b_2)^2 \sin^2 \omega}{[a_1 + (1 + a_2) \cos \omega]^2 + (1 - a_2)^2 \sin^2 \omega} = 1$ . Hence, at  $\omega = 0$ , we have  
 $b_1 + (b_0 + b_2) = \pm[a_1 + (1 + a_2)]$ , and at  $\omega = \pi/2$  we have  $b_0 - b_2 = \pm[1 - a_2]$ .

**Solution #1:** Consider  $b_0 - b_2 = 1 - a_2$ . Choose  $b_0 = 1, -b_2 = 1 - a_2$ . and  $b_2 = a_2$ . Substituting these values in  $b_1 + (b_0 + b_2) = \pm[a_1 + (1 + a_2)]$ , we have  $b_1 = a_1$ . In this case,

$$A(e^{j\omega}) = \frac{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}}{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}} = 1, \text{ a trivial solution.}$$

**Solution #2:** Consider  $b_0 - b_2 = a_2 - 1$ . Choose  $b_0 = a_2$ , and  $b_2 = 1$ . Substituting these values in  $b_1 + (b_0 + b_2) = \pm[a_1 + (1 + a_2)]$ , we have  $b_1 = a_1$ . In this case,

$$A(e^{j\omega}) = \frac{a_2 + a_1 e^{-j\omega} + e^{-j2\omega}}{1 + a_1 e^{-j\omega} + a_2 e^{-j2\omega}}.$$

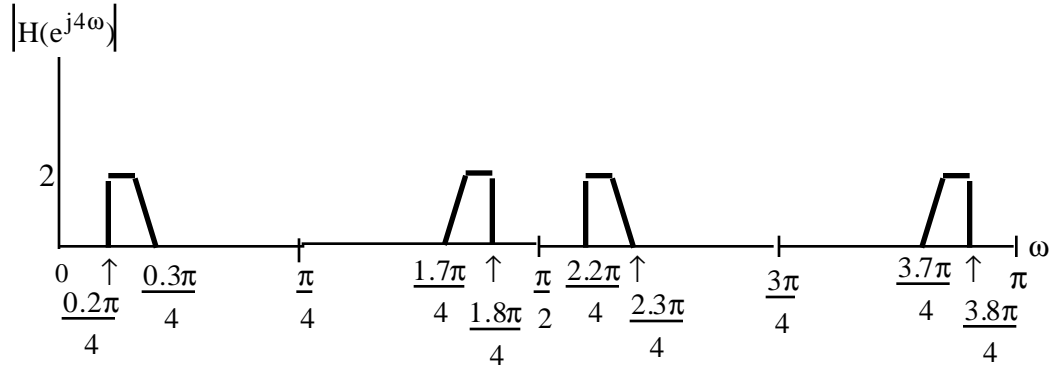
**4.13** From Eq. (2.17), the input-output relation of a factor-of-2 up-sampler is given by

$$y[n] = \begin{cases} x[n/2], & n = 0, \pm 2, \pm 4, \mathbf{K} \\ 0, & \text{otherwise.} \end{cases}$$

The DTFT of the output is therefore given by

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} = \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} x[n/2] e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m] e^{-j2\omega m} = X(e^{j2\omega}) \text{ where } X(e^{j\omega}) \text{ is the DTFT of } x[n].$$

4.14



4.15  $H(e^{j\omega}) = \sum_{n=0}^{\infty} (0.4)^n e^{-j\omega n} = \frac{1}{1-0.4e^{-j\omega}}$ . Thus,  $|H(e^{j\omega})| = \frac{1}{\sqrt{1.16-0.8\cos\omega}}$ , and  $\arg\{H(e^{j\omega})\} = \theta(\omega) = \tan^{-1}\left(\frac{0.4\sin\omega}{1-0.4\cos\omega}\right)$ .  $H(e^{\pm j\pi/4}) = 1.2067 + j0.4759$ . Therefore,  $|H(e^{\pm j\pi/4})| = 1.2972$ , and  $\theta(\pm j\pi/4) = 0.3757$  radians.

Now for an input  $x[n] = \sin(\omega_0 n)\mu[n]$ , the steady-state output is given by

$$y[n] = |H(e^{j\omega_0})| \sin(\omega_0 n + \theta(\omega_0)) \text{ which for } \omega_0 = \pi/4 \text{ reduces to}$$

$$y[n] = |H(e^{j\pi/4})| \sin\left(\frac{\pi}{4}n + \theta(\pi/4)\right) = 1.2972 \sin\left(\frac{\pi}{4}n - 0.3757\right).$$

4.16  $H(e^{j\omega}) = h[0](1 + e^{-j2\omega}) + h[1]e^{-j\omega} = e^{-j\omega}(2h[0]\cos\omega + h[1])$ . Therefore, we require,

$H(e^{j0.2}) = 2h[0]\cos(0.2) + h[1] = 0$ , and  $H(e^{j0.5}) = 2h[0]\cos(0.5) + h[1] = 1$ . Solving these two equations we get  $h[0] = -4.8788$  and  $h[1] = 9.5631$ .

4.17 (a)  $H(e^{j\omega}) = h_0 + h_1e^{-j\omega} - h_1e^{-j3\omega} - h_0e^{-j4\omega} = e^{-j2\omega}[h_0(e^{j2\omega} - e^{-j2\omega}) + h_0(e^{j\omega} - e^{-j\omega})]$

$= je^{-j2\omega}[2h_0\sin(2\omega) + 2h_1\sin(\omega)]$ . This implies,  $|H(e^{j\omega})| = 2h_0\sin(2\omega) + 2h_1\sin(\omega)$ . Thus,

$|H(e^{j\pi/4})| = 2h_0\sin(\pi/2) + 2h_1\sin(\pi/4) = 0.5$ , and

$|H(e^{j\pi/2})| = 2h_0\sin(\pi) + 2h_1\sin(\pi/2) = 1$ . Solving these two equations we get  $h_0 = -0.1036$ , and  $h_1 = 0.5$ .

(b)  $H(e^{j\omega}) = je^{-j2\omega}[-0.2072\sin(2\omega) + \sin(\omega)]$

4.18 (a)  $H(e^{j\omega}) = h_0 + h_1e^{-j\omega} - h_1e^{-j2\omega} - h_0e^{-j3\omega}$

$= e^{-j3\omega/2}[h_0(e^{j3\omega/2} - e^{-j3\omega/2}) + h_0(e^{j\omega/2} - e^{-j\omega/2})]$

$= je^{-j3\omega/2}[2h_0\sin(3\omega/2) + 2h_1\sin(\omega/2)]$ . This implies,

$|H(e^{j\omega})| = 2h_0\sin(3\omega/2) + 2h_1\sin(\omega/2)$ . Thus,  $|H(e^{j\pi/4})| = 2h_0\sin(3\pi/8) + 2h_1\sin(\pi/8) = 1$ ,

and  $\left|H(e^{j\pi/2})\right| = 2h_0 \sin(3\pi/4) + 2h_1 \sin(\pi/4) = 0.5$ . Solving these two equations we get  $h_0 = -0.0381$ , and  $h_1 = 0.7452$ .

(b)  $H(e^{j\omega}) = je^{-j3\omega/2}[-0.0762 \sin(3\omega/2) + 1.4904 \sin(\omega/2)]$

4.19  $H(e^{j\omega}) = h_0 + h_1e^{-j\omega} + h_2e^{-j2\omega} + h_1e^{-j3\omega} + h_0e^{-j4\omega}$   
 $= e^{-j2\omega}[h_0(e^{j2\omega} + e^{-j2\omega}) + h_1(e^{j\omega} + e^{-j\omega}) + h_2] = e^{-j2\omega}[2h_0 \cos(2\omega) + 2h_1 \cos(\omega) + h_2]$ .

This implies  $\left|H(e^{j\omega})\right| = 2h_0 \cos(2\omega) + 2h_1 \cos(\omega) + h_2$ . Thus, we have

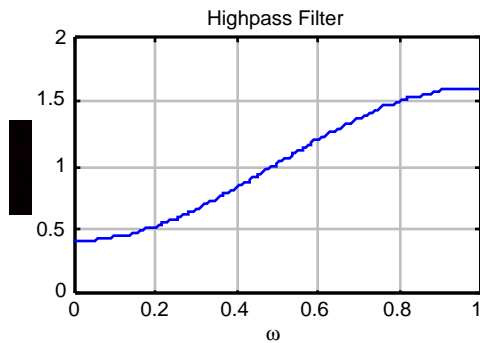
$\left|H(e^{j0.2})\right| = 2h_0 \cos(0.4) + 2h_1 \cos(0.2) + h_2 = 0$ ,  $\left|H(e^{j0.5})\right| = 2h_0 \cos(1) + 2h_1 \cos(0.5) + h_2 = 1$ ,

$\left|H(e^{j0.8})\right| = 2h_0 \cos(1.6) + 2h_1 \cos(0.8) + h_2 = 0$ . Solving these equations we get  $h_0 = -13.4866$ ,  $h_1 = 45.2280$ ,  $h_2 = -63.8089$ .

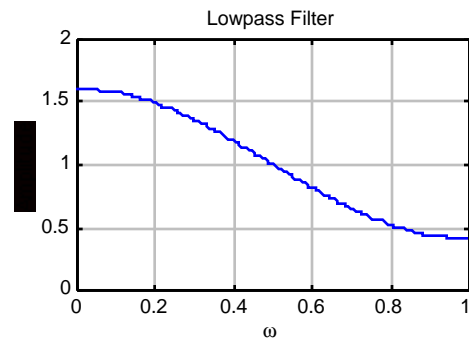
4.20  $H(e^{j0}) = 2$ ,  $H(e^{j\pi/2}) = 7 - j3$ ,  $H(e^{j\pi}) = 0$ . Using the symmetry property of the DTFT of a real sequence, we observe  $H(e^{j3\pi/2}) = H^*(e^{j\pi/2}) = 7 + j3$ . Thus the 4-point DFT  $H[k]$  of the length-4 sequence is given by  $H[k] = [2 \quad 7 - j3 \quad 0 \quad 7 + j3]$  whose 4-point inverse DFT yields the impulse response coefficients  $\{h[n]\} = \{4 \quad 2 \quad -3 \quad -1\}$ ,  $0 \leq n \leq 3$ . Hence,  
 $H(z) = 4 + 2z^{-1} - 3z^{-2} - z^{-3}$ .

4.21 Now, for an anti-symmetric finite-length sequence,  $H(e^{j0}) = 0$ . Also, since the sequence is real,  $H(e^{j3\pi/2}) = H^*(e^{j\pi/2}) = -2 - j2$ . Thus the 4-point DFT  $H[k]$  of the length-4 sequence is given by  $H[k] = [0 \quad -2 + j2 \quad 8 \quad -2 - j2]$  whose 4-point inverse DFT yields the impulse response coefficients  $\{h[n]\} = \{1 \quad -3 \quad 3 \quad -1\}$ ,  $0 \leq n \leq 3$ . Hence,  $H(z) = 1 - 3z^{-1} + 3z^{-2} - z^{-3}$ .

4.22 (a)  $H_A(e^{j\omega}) = 0.3 - e^{-j\omega} + 0.3e^{-2j\omega}$ ,  $H_B(e^{j\omega}) = 0.3 + e^{-j\omega} + 0.3e^{-2j\omega}$ .



(a)  $\left|H_A(e^{j\omega})\right|$



(b)  $\left|H_B(e^{j\omega})\right|$

$H_A(e^{j\omega})$  is a highpass filter, whereas,  $H_B(e^{j\omega})$  a lowpass filter.

(b)  $H_C(e^{j\omega}) = H_B(e^{j\omega}) = H_A(e^{j(\omega+\pi)})$ .

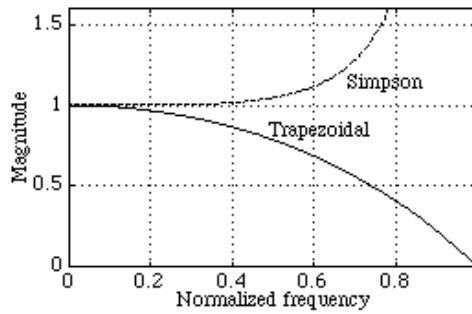
**4.23**  $y[n] = y[n - 1] + 0.5\{x[n] + x[n - 1]\}$ . Hence,

$$Y(e^{j\omega}) = e^{-j\omega}Y(e^{j\omega}) + 0.5\{X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega})\}, \text{ or, } H_{tr}(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 0.5\left(\frac{1 + e^{-j\omega}}{1 - e^{-j\omega}}\right).$$

**4.24**  $y[n] = y[n - 2] + \frac{1}{3}\{x[n] + 4x[n - 1] + x[n - 2]\}$ . Hence,  $H_{sim}(e^{j\omega}) = \frac{1}{3}\left(\frac{1 + 4e^{-j\omega} + e^{-2j\omega}}{1 - e^{-2j\omega}}\right)$ .

Note: To compare the performances of the Trapezoidal numerical integration formula with that of the Simpson's formula, we first observe that if the input is  $x_a(t) = e^{j\omega t}$ , then the result of integration is  $y_a(t) = \frac{1}{j\omega} e^{j\omega t}$ . Thus, the ideal frequency response  $H(e^{j\omega})$  is  $1/j\omega$ .

Hence, we take the ratio of the frequency responses of the approximation to the ideal, and plot the two curves as indicated below.



From the above plot it is evident that the Simpson's formula amplifies high frequencies, whereas, the Trapezoidal formula attenuates them. In the very low frequency range, both formulae yield results close to the ideal. However, Simpson's formula is reasonably accurate for frequencies close to the midband range.

**4.25** A mirror-image polynomial  $A(z)$  of degree  $N$  satisfies the condition  $A(z) = z^{-N}A(z^{-1})$  and an antimirror-image polynomial  $B(z)$  of degree  $N$  satisfies the condition  $B(z) = -z^{-N}B(z^{-1})$ .

**(a)**  $B_{mi}(z) = \frac{1}{2}\{B(z) + z^{-N}B(z^{-1})\}$ . Thus,  $z^{-N}B_{mi}(z^{-1}) = \frac{1}{2}\{z^{-N}B(z^{-1}) + B(z)\} = B_{mi}(z)$ .

Likewise,  $B_{li}(z) = \frac{1}{2}\{B(z) - z^{-N}B(z^{-1})\}$ . Thus,  $z^{-N}B_{li}(z^{-1}) = \frac{1}{2}\{z^{-N}B(z^{-1}) - B(z)\} = -B_{li}(z)$ .

**(b)**  $H(z) = \frac{N_{mi}(z) + N_{ai}(z)}{D_{mi}(z) + D_{ai}(z)}$ . Thus,  $H(z^{-1}) = \frac{N_{mi}(z^{-1}) + N_{ai}(z^{-1})}{D_{mi}(z^{-1}) + D_{ai}(z^{-1})} = \frac{N_{mi}(z) - N_{ai}(z)}{D_{mi}(z) - D_{ai}(z)}$ . Hence,

$$H(z) + H(z^{-1}) = \frac{N_{mi}(z) + N_{ai}(z)}{D_{mi}(z) + D_{ai}(z)} + \frac{N_{mi}(z) - N_{ai}(z)}{D_{mi}(z) - D_{ai}(z)} = 2 \cdot \frac{D_{mi}(z)N_{mi}(z) - D_{ai}(z)N_{ai}(z)}{D_{mi}^2(z) - D_{ai}^2(z)}, \text{ and}$$

$$H(z) - H(z^{-1}) = \frac{N_{mi}(z) + N_{ai}(z)}{D_{mi}(z) + D_{ai}(z)} - \frac{N_{mi}(z) - N_{ai}(z)}{D_{mi}(z) - D_{ai}(z)} = 2 \cdot \frac{D_{mi}(z)N_{ai}(z) - D_{ai}(z)N_{mi}(z)}{D_{mi}^2(z) - D_{ai}^2(z)}.$$

(c) Since  $H(z)$  is a real rational function of  $z^{-1}$ ,  $H(z^{-1})|_{z=e^{j\omega}} = H^*(e^{j\omega})$ . Therefore,

$$H_{\text{re}}(e^{j\omega}) = \frac{1}{2} \{H(e^{j\omega}) + H^*(e^{j\omega})\} = \frac{1}{2} \{H(z) + H(z^{-1})\} \Big|_{z=e^{j\omega}} = H_{\text{ev}}(z) \Big|_{z=e^{j\omega}}, \text{ and}$$

$$H_{\text{im}}(e^{j\omega}) = \frac{1}{2j} \{H(e^{j\omega}) - H^*(e^{j\omega})\} = \frac{1}{2j} \{H(z) - H(z^{-1})\} \Big|_{z=e^{j\omega}} = -jH_{\text{od}}(z) \Big|_{z=e^{j\omega}}.$$

(d)  $H(z) = \frac{3z^2 + 4z + 6}{3z^3 + 2z^2 + z + 1}$  and  $H(z^{-1}) = \frac{3z^{-2} + 4z^{-1} + 6}{3z^{-3} + 2z^{-2} + z^{-1} + 1} = \frac{6z^3 + 4z^2 + 3z}{z^3 + z^2 + 2z + 3}$ . Thus,

$$H_{\text{ev}}(z) = \frac{1}{2} \{H(z) + H(z^{-1})\} = \frac{1}{2} \left\{ \frac{3z^2 + 4z + 6}{3z^3 + 2z^2 + z + 1} + \frac{6z^3 + 4z^2 + 3z}{z^3 + z^2 + 2z + 3} \right\}$$

$$= \frac{9z^6 + 13.5z^5 + 15z^4 + 16z^3 + 15z^2 + 13.5z + 9}{3z^6 + 5z^5 + 9z^4 + 15z^3 + 9z^2 + 5z + 3}, \text{ and}$$

$$H_{\text{od}}(z) = \frac{1}{2} \{H(z) - H(z^{-1})\} = \frac{1}{2} \left\{ \frac{3z^2 + 4z + 6}{3z^3 + 2z^2 + z + 1} - \frac{6z^3 + 4z^2 + 3z}{z^3 + z^2 + 2z + 3} \right\}$$

$$= \frac{-9z^6 - 10.5z^5 - 8z^4 + 8z^2 + 10.5z + 9}{3z^6 + 5z^5 + 9z^4 + 15z^3 + 9z^2 + 5z + 3}. \text{ Therefore,}$$

$$H_{\text{re}}(e^{j\omega}) = H_{\text{ev}}(z) \Big|_{z=e^{j\omega}} = \frac{9\cos(3\omega) + 13.5\cos(2\omega) + 15\cos\omega + 16}{3\cos(3\omega) + 5\cos(2\omega) + 9\cos\omega + 15}, \text{ and}$$

$$H_{\text{im}}(e^{j\omega}) = -jH_{\text{od}}(z) \Big|_{z=e^{j\omega}} = \frac{-9\cos(3\omega) - 10.5\cos(2\omega) - 8\cos\omega}{3\cos(3\omega) + 5\cos(2\omega) + 9\cos\omega + 15}.$$

**4.26** Given the real part of a real, stable transfer function  $H_{\text{re}}(e^{j\omega}) = \frac{\sum_{i=0}^N a_i \cos(i\omega)}{\sum_{i=0}^N b_i \cos(i\omega)} = \frac{A(e^{j\omega})}{B(e^{j\omega})}$ , (1)

the problem is to determine the transfer function:  $H(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^N p_i z^{-i}}{\sum_{i=0}^N d_i z^{-i}}$ .

$$(a) H_{\text{re}}(e^{j\omega}) = \frac{1}{2} [H(e^{j\omega}) + H^*(e^{j\omega})] = \frac{1}{2} [H(e^{j\omega}) + H(e^{-j\omega})] = \frac{1}{2} [H(z) + H(z^{-1})] \Big|_{z=e^{j\omega}}.$$

Substituting  $H(z) = P(z)/D(z)$  in the above we get

$$H_{\text{re}}(e^{j\omega}) = \frac{1}{2} \frac{P(z)D(z^{-1}) + P(z^{-1})D(z)}{D(z)D(z^{-1})} \Big|_{z=e^{j\omega}} \quad (2)$$

which is Eq. (4.241).

$$(b) \text{ Comparing Eqs. (1) and (2) we get } B(e^{j\omega}) = D(z)D(z^{-1}) \Big|_{z=e^{j\omega}}, \quad (3)$$

$$A(e^{j\omega}) = \frac{1}{2} [P(z)D(z^{-1}) + P(z^{-1})D(z)] \Big|_{z=e^{j\omega}}. \quad (4)$$

$$\text{Now } D(z) \text{ is of the form } D(z) = K z^{-N} \prod_{i=1}^N (z - z_i) \quad (5)$$

where the  $z_i$ 's are the roots of  $B(z) = B(e^{j\omega})|_{e^{j\omega}=z}$  inside the unit circle and  $K$  is a scalar constant.

Putting  $\omega = 0$  in Eq. (3) we get  $B(1) = [D(1)]^2$ , or  $\sqrt{B(1)} = K \prod_{i=1}^N (1 - z_i)$ . Hence,

$$K = \sqrt{B(1)} / \prod_{i=1}^N (1 - z_i). \quad (6)$$

(c) By analytic continuation, Eq. (4) yields  $A(z) = \frac{1}{2} [P(z)D(z^{-1}) + P(z^{-1})D(z)]$ . (7)

Substituting  $A(z) = \frac{1}{2} \sum_{i=0}^N a_i (z^i + z^{-i})$  and the polynomial forms of  $P(z)$  and  $D(z)$ , we get

$\sum_{i=0}^N a_i (z^i + z^{-i}) = \left( \sum_{i=0}^N p_i z^{-i} \right) \left( \sum_{i=0}^N d_i z^i \right) + \left( \sum_{i=0}^N p_i z^i \right) \left( \sum_{i=0}^N d_i z^{-i} \right)$  and equating the coefficients of  $(z^i + z^{-i})/2$  on both sides, we arrive at a set of  $N+1$  equations which can be solved for the numerator coefficients  $p_i$  of  $H(z)$ .

For the given example, i.e.,  $H_{re}(e^{j\omega}) = \frac{1 + \cos(\omega) + \cos(2\omega)}{17 - 8\cos(2\omega)}$ , we observe

$$A(z) = 1 + \frac{1}{2}(z + z^{-1}) + \frac{1}{2}(z^2 + z^{-2}). \quad (8)$$

Also,  $B(z) = 17 - 4(z^2 + z^{-2})$ , which has roots at  $z = \pm \frac{1}{2}$  and  $z = \pm 2$ . Hence,

$$D(z) = K z^{-2} (z - \frac{1}{2})(z + \frac{1}{2}) = K(z^2 - \frac{1}{4})z^{-2}.$$

Aldo, from Eq. (6) we have  $K = \sqrt{17-8}/(1 - \frac{1}{4}) = 4$ , so that  $D(z) = 4 - z^{-2}$ . (9)

Substituting Eqs. (8), (9) and  $P(z) = p_0 + p_1 z^{-1} + p_2 z^{-2}$  in Eq. (7) we get

$$1 + \frac{1}{2}(z + z^{-1}) + \frac{1}{2}(z^2 + z^{-2}) = [(p_0 + p_1 z^{-1} + p_2 z^{-2})(4 - z^2) + (p_0 + p_1 z + p_2 z^2)(4 - z^{-2})].$$

Equating coefficients of  $(z^i + z^{-i})/2$ ,  $i = 0, 1, 2$ , on both sides we get  $4 p_0 - p_2 = 1$ ,  $3 p_1 = 1$ ,  $4 p_2 - p_0 = 1$ . Solving these equations we arrive at  $p_0 = p_1 = p_2 = 1/3$ . Therefore,

$$H(z) = \frac{1 + z^{-1} + z^{-2}}{3(4 - z^{-2})}.$$

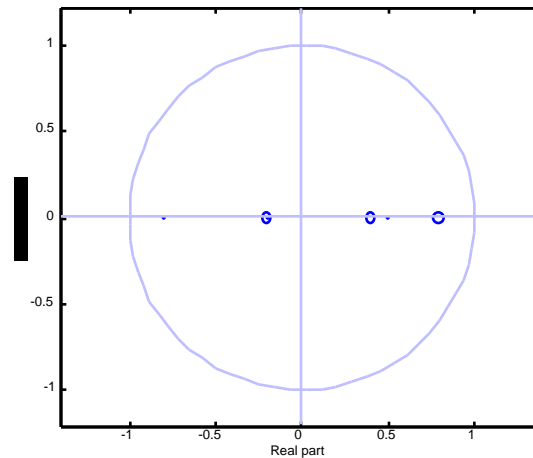
**4.27**  $G(z) = H\left(\frac{1 + \alpha z^{-1}}{\alpha + z^{-1}}\right)$ . Thus,  $G(1) = H\left(\frac{1 + \alpha}{\alpha + 1}\right) = H(1)$ . and  $G(-1) = H\left(\frac{1 - \alpha}{\alpha - 1}\right) = H(-1)$ .

**4.28**  $H(z) = H_1(z)[H_2(z) + H_3(z)] = (3 + 2z^{-1} + 4z^{-2}) \left[ \left(\frac{2}{3} + \frac{4}{3}\right) + \left(\frac{2}{5} + \frac{8}{5}\right)z^{-1} + \left(\frac{4}{7} + \frac{3}{7}\right)z^{-2} \right]$   
 $= (3 + 2z^{-1} + 4z^{-2})(2 + 2z^{-1} + z^{-2}) = 6 + 10z^{-1} + 15z^{-2} + 10z^{-3} + 4z^{-4}$ .

$$4.29 \quad H(z) = \frac{5 - 5z^{-1} + 0.4z^{-2} + 0.32z^{-3}}{1 + 0.5z^{-1} - 0.34z^{-2} - 0.08z^{-3}} = 5 \cdot \frac{(1 - 0.8z^{-1})(1 - 0.4z^{-1})(1 + 0.2z^{-1})}{(1 + 0.8z^{-1})(1 - 0.5z^{-1})(1 + 0.2z^{-1})}$$

$$= 5 \cdot \frac{(1 - 0.8z^{-1})(1 - 0.4z^{-1})}{(1 + 0.8z^{-1})(1 - 0.5z^{-1})}, \text{ obtained using the M-file roots. The pole-zero plot obtained}$$

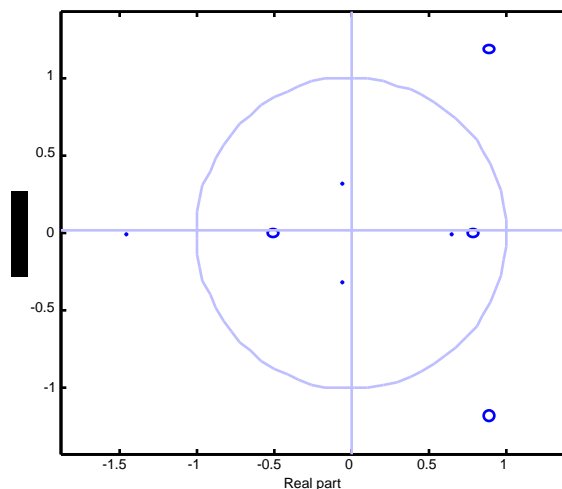
using the M-file `zplane` is shown below. It can be seen from this plot and the factored form of  $H(z)$ , the transfer function is BIBO stable as all poles are inside the unit circle. Note also from the plot the pole-zero cancellation at  $z = -0.2$ .



$$4.30 \quad H(z) = \frac{5 - 10.5z^{-1} + 11.7z^{-2} + 0.3z^{-3} - 4.4z^{-3}}{1 + 0.9z^{-1} - 0.76z^{-2} - 0.016z^{-3} - 0.096z^{-3}}$$

$$= 5 \cdot \frac{(1 + 0.5z^{-1})(1 - 0.8z^{-1})(1 - 1.8z^{-1} + 2.2z^{-2})}{(1 + 1.4486z^{-1})(1 - 0.6515z^{-1})(1 + 0.1029z^{-1} + 0.1017z^{-2})}, \text{ obtained using the M-file}$$

`roots`. The pole-zero plot obtained using the M-file `zplane` is shown below. It can be seen from this plot and the factored form of  $H(z)$ , the transfer function is unstable as there is a pole outside the unit circle at  $z = -1.4486$ .



$$4.31 \quad H(z) = \frac{-0.1z^{-1} + 2.19z^{-2}}{(1 - 0.8z^{-1} + 0.4z^{-2})(1 + 0.3z^{-1})} = \frac{A + Bz^{-1}}{1 - 0.8z^{-1} + 0.4z^{-2}} + \frac{C}{1 + 0.3z^{-1}}, \text{ where}$$



$$C = \left. \frac{-0.1z^{-1} + 2.19z^{-2}}{1 - 0.8z^{-1} + 0.4z^{-2}} \right|_{z^{-1} = -1/0.3} = 3, \text{ Thus,}$$

$$\frac{A + Bz^{-1}}{1 - 0.8z^{-1} + 0.4z^{-2}} = \frac{-0.1z^{-1} + 2.19z^{-2}}{(1 - 0.8z^{-1} + 0.4z^{-2})(1 + 0.3z^{-1})} - \frac{3}{1 + 0.3z^{-1}} = \frac{-3 + 3.2z^{-1}}{1 - 0.8z^{-1} + 0.4z^{-2}} \text{ Hence,}$$

$$H(z) = \frac{-3 + 3.2z^{-1}}{1 - 0.8z^{-1} + 0.4z^{-2}} + \frac{3}{1 + 0.3z^{-1}} = -3 \left[ \frac{1 - 1.0667z^{-1}}{1 - 0.8z^{-1} + 0.4z^{-2}} \right] + \frac{3}{1 + 0.3z^{-1}}. \text{ Now, using Table}$$

3.8 we observe  $r^2 = 0.4$  and  $r \cos \omega_0 = 0.8$ . Therefore,  $r = \sqrt{0.4} = 0.6325$  and  $\cos \omega_0 = \sqrt{0.4}$  or  $\omega_0 = \cos^{-1}(\sqrt{0.4}) = 0.8861$ . Hence  $r \sin \omega_0 = \sqrt{0.4} \sin(0.8861) = 0.4899$ . We can thus write

$$\frac{1 - 1.0667z^{-1}}{1 - 0.8z^{-1} + 0.4z^{-2}} = \frac{1 - 0.4z^{-1}}{1 - 0.8z^{-1} + 0.4z^{-2}} - 1.3609 \left( \frac{0.4899z^{-1}}{1 - 0.8z^{-1} + 0.4z^{-2}} \right). \text{ The inverse z-transform}$$

of this function is thus given by  $(0.6325)^n \cos(0.8861n)\mu[n] - 1.3609(0.6325)^n \sin(0.8861n)\mu[n]$ . Hence, the inverse z-transform of  $H(z)$  is given by

$$h[n] = 3(0.6325)^n \cos(0.8861n)\mu[n] - 4.0827(0.6325)^n \sin(0.8861n)\mu[n] - 3(-0.3)^n \mu[n].$$

**4.32 (a)**  $H(z) = \frac{6 - z^{-1}}{1 + 0.5z^{-1}} + \frac{2}{1 - 0.4z^{-1}} = k + \frac{A}{1 + 0.5z^{-1}} + \frac{2}{1 - 0.4z^{-1}}$  where  $A = (6 - z^{-1}) \Big|_{z^{-1} = -2} = 8$

and  $k = \left. \frac{6 - z^{-1}}{1 + 0.5z^{-1}} \right|_{z^{-1} = \infty} = -2$ . Therefore,  $H(z) = -2 + \frac{8}{1 + 0.5z^{-1}} + \frac{2}{1 - 0.4z^{-1}}$ . Hence, the inverse

z-transform of  $H(z)$  is given by  $h[n] = -2\delta[n] + 8(-0.5)^n \mu[n] + 2(0.4)^n \mu[n]$ .

**(b)**  $x[n] = 1.2(-0.2)^n \mu[n] - 0.2(0.3)^n \mu[n]$ . Its z-transform is thus given by

$$X(z) = \frac{1.2}{1 + 0.2z^{-1}} - \frac{0.2}{1 - 0.3z^{-1}} = \frac{1 - 0.4z^{-1}}{(1 + 0.2z^{-1})(1 - 0.3z^{-1})}, \quad |z| > 0.3. \text{ Therefore,}$$

$$Y(z) = H(z)X(z) = \left[ \frac{6 - z^{-1}}{1 + 0.5z^{-1}} + \frac{2}{1 - 0.4z^{-1}} \right] \frac{1 - 0.4z^{-1}}{(1 + 0.2z^{-1})(1 - 0.3z^{-1})}$$

$$= \left[ \frac{8 - 2.4z^{-1} + 0.4z^{-2}}{(1 + 0.5z^{-1})(1 - 0.4z^{-1})} \right] \frac{1 - 0.4z^{-1}}{(1 + 0.2z^{-1})(1 - 0.3z^{-1})} = \frac{8 - 2.4z^{-1} + 0.4z^{-2}}{(1 + 0.5z^{-1})(1 + 0.2z^{-1})(1 - 0.3z^{-1})}, \quad |z| > 0.5.$$

A partial-fraction expansion of  $Y(z)$  yields  $Y(z) = \frac{15}{1 + 0.5z^{-1}} - \frac{8}{1 + 0.2z^{-1}} + \frac{1}{1 - 0.3z^{-1}}$ . Hence,

the inverse z-transform of  $Y(z)$  is given by  $y[n] = 15(-0.5)^n \mu[n] - 8(-0.2)^n \mu[n] + (0.3)^n \mu[n]$ .

**4.33 (a)**  $h[n] = (0.8)^n \mu[n]$  and  $x[n] = (0.5)^n \mu[n]$ . Their z-transforms are given by

$$H(z) = \frac{1}{1 - 0.8z^{-1}}, \quad |z| > 0.8 \text{ and } X(z) = \frac{1}{1 - 0.5z^{-1}}, \quad |z| > 0.5. \text{ The z-transform } Y(z) \text{ of the output}$$

$y[n]$  is therefore given by  $Y(z) = H(z)X(z) = \frac{1}{(1-0.8z^{-1})(1-0.5z^{-1})}$ ,  $|z| > 0.5$ . Its partial-fraction expansion is given by  $Y(z) = \frac{1}{(1-0.8z^{-1})(1-0.5z^{-1})} = \frac{8/3}{1-0.8z^{-1}} - \frac{5/3}{1-0.5z^{-1}}$ . Hence, the inverse z-transform of  $Y(z)$  is given by  $y[n] = \frac{8}{3}(0.8)^n \mu[n] - \frac{5}{3}(0.5)^n \mu[n]$ .

(b) The z-transform  $Y(z)$  of the output  $y[n]$  is therefore given by

$$Y(z) = H(z)X(z) = \frac{1}{(1-0.5z^{-1})^2}, \quad |z| > 0.5. \quad \text{Now, the inverse z-transform of } \frac{0.5z^{-1}}{(1-0.5z^{-1})^2} \text{ is}$$

given by  $n(0.5)^n \mu[n]$ . Thus, the inverse z-transform of  $\frac{0.5}{(1-0.5z^{-1})^2}$  is given by

$$(n+1)(0.5)^{n+1} \mu[n+1]. \quad \text{Hence, } y[n] = (n+1)(0.5)^n \mu[n+1] = (n+1)(0.5)^n \mu[n].$$

**4.34**  $y[n] = 4(0.75)^n \mu[n]$  and  $x[n] = 3(0.25)^n \mu[n]$ . Their z-transforms are given by

$$Y(z) = \frac{4}{1-0.75z^{-1}}, \quad |z| > 0.75 \quad \text{and} \quad X(z) = \frac{3}{1-0.25z^{-1}}, \quad |z| > 0.25. \quad \text{Thus, the transfer function } H(z)$$

is given by  $H(z) = \frac{Y(z)}{X(z)} = \frac{4}{3} \left( \frac{1-0.25z^{-1}}{1-0.75z^{-1}} \right) = \frac{4}{9} + \frac{8/9}{1-0.75z^{-1}}$ ,  $|z| > 0.75$ . Hence, the impulse

response  $h[n]$  is given by  $h[n] = \frac{4}{9} \delta[n] + \frac{8}{9} (0.75)^n \mu[n]$ .

**4.35 (a)**  $H(z) = \frac{1}{1-0.8z^{-1}+0.15z^{-2}} = \frac{1}{(1-0.5z^{-1})(1-0.3z^{-1})}$ ,  $|z| > 0.8$

(b) A partial-fraction expansion of  $H(z)$  yields  $H(z) = \frac{2.5}{1-0.5z^{-1}} - \frac{1.5}{1-0.3z^{-1}}$ . Hence,

$$h[n] = 2.5(0.5)^n \mu[n] - 1.5(0.3)^n \mu[n].$$

(c) the step response  $s[n]$  is given by the inverse z-transform of

$$\frac{H(z)}{1-z^{-1}} = \frac{1}{(1-0.5z^{-1})(1-0.3z^{-1})(1-z^{-1})}, \quad |z| > 1. \quad \text{A partial-fraction expansion of } \frac{H(z)}{1-z^{-1}} \text{ yields}$$

$$\frac{H(z)}{1-z^{-1}} = \frac{2.8571}{1-z^{-1}} - \frac{2.5}{1-0.5z^{-1}} + \frac{0.6429}{1-0.3z^{-1}}. \quad \text{Hence, the step response is given by}$$

$$s[n] = 2.8571 \mu[n] - 2.5(0.5)^n \mu[n] + 0.6429(0.3)^n \mu[n].$$

**4.36**  $Y(z) = [H_0(z)F_0(z) - H_0(-z)F_0(-z)]X(z)$ . Since the output is a delayed replica of the input,

we must have  $H_0(z)F_0(z) - H_0(-z)F_0(-z) = z^{-r}$ . But  $H_0(z) = 1 + \alpha z^{-1}$ , hence

$$(1 + \alpha z^{-1})F_0(z) - (1 - \alpha z^{-1})F_0(-z) = z^{-r}. \quad \text{Let } F_0(z) = a_0 + a_1 z^{-1}. \quad \text{This implies,}$$

$$2(a_0 \alpha + a_1)z^{-1} = z^{-r}.$$

The solution is therefore,  $r = 1$  and  $2(a_0 \alpha + a_1) = 1$ . One possible solution is thus  $a_0 = 1/2$ ,

$\alpha = 1/2$ , and  $a_1 = 1/4$ . Hence  $F_0(z) = 0.25(1 + z^{-1})$ .

$$\begin{aligned}
\mathbf{4.37} \quad H_0(z)F_0(z) &= (E_0(z^2) + z^{-1}E_1(z^2))(R_0(z^2) + z^{-1}R_1(z^2)) \\
&= E_0(z^2)R_0(z^2) + z^{-1}(E_0(z^2)R_1(z^2) + E_1(z^2)R_0(z^2)) + z^{-2}E_1(z^2)R_1(z^2). \quad \text{Thus,} \\
H_0(-z)F_0(-z) &= (E_0(z^2) - z^{-1}E_1(z^2))(R_0(z^2) - z^{-1}R_1(z^2)) \\
&= E_0(z^2)R_0(z^2) - z^{-1}(E_0(z^2)R_1(z^2) + E_1(z^2)R_0(z^2)) + z^{-2}E_1(z^2)R_1(z^2).
\end{aligned}$$

As a result,  $T(z) = H_0(z)F_0(z) - H_0(-z)F_0(-z) = 2z^{-1}(E_0(z^2)R_1(z^2) + E_1(z^2)R_0(z^2))$ .

Hence the condition to be satisfied by  $E_0(z)$ ,  $E_1(z)$ ,  $R_0(z)$ ,  $R_1(z)$  for the output to be a delayed replica for input is  $E_0(z^2)R_1(z^2) + E_1(z^2)R_0(z^2) = 0.5z^{-r}$ . (A)

(b) In Problem 4.36,  $E_0(z) = 1$ ,  $E_1(z) = 0.5$ ,  $R_0(z) = 0.5$ ,  $R_1(z) = 0.25$ . Thus,

$$E_0(z^2)R_1(z^2) + E_1(z^2)R_0(z^2) = 0.5.$$

Hence the condition of Eq. (A) is satisfied with  $r = 1$ .

(c)  $E_0(z^2) = E_1(z^2) = z^{-2}$ . Let  $R_0(z^2) = R_1(z^2) = 0.25z^{-2}$ .  $r = 4$  in this case. Hence  $H(z) = z^{-2}(1 + z^{-1})$  and  $F(z) = \frac{1}{4}z^{-2}(1 - z^{-1})$ .

**4.38**  $h[n] = h[N - n]$ .

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} = \sum_{n=0}^{N-1} h[N-n]z^{-n} = \sum_{k=0}^{N-1} h[k]z^{-(N-k)} = z^{-N}H(z^{-1}).$$

So if  $z = z_0$  is a root then so is  $z = 1/z_0$ . If  $G(z) = 1/H(z)$  then  $G(z)$  will have poles both inside and outside the unit circle, and will hence be unstable.

**4.39**  $D(z) = (0.5z + 1)(z^2 + z + 0.6) = 0.5(z + 2)(z + 0.5 - j0.5916)(z + 0.5 + j0.5916)$ . Since one of the roots of  $D(z)$  is outside the unit circle at  $z = -2$ ,  $H(z)$  is unstable. To arrive at a stable, transfer function  $G(z)$  such that  $|G(e^{j\omega})| = |H(e^{j\omega})|$ , we multiply  $H(z)$  with an allpass function

$$A(z) = \frac{0.5z + 1}{z + 0.5}. \quad \text{Hence}$$

$$G(z) = H(z)A(z) = \left( \frac{3z^3 + 2z^2 + 5}{(0.5z + 1)(z^2 + z + 0.6)} \right) \left( \frac{0.5z + 1}{z + 0.5} \right) = \frac{3z^3 + 2z^2 + 5}{(z + 0.5)(z^2 + z + 0.6)}.$$

Now,  $H(z) = \frac{3(z + 1.4545)(z^2 - 0.7878z + 1.1459)}{(0.5z + 1)(z^2 + z + 0.6)}$ . Thus, there are 14 other transfer functions with the same magnitude as  $|H(e^{j\omega})|$ .

**4.40**  $D(z) = (z + 3.1238)(z + 0.5762)(z - 0.2 - j0.5568)(z - 0.2 + j0.5568)$ . Since one of the roots of  $D(z)$  is outside the unit circle at  $z = -3.1238$ ,  $H(z)$  is unstable. To arrive at a stable, transfer function  $G(z)$  such that  $|G(e^{j\omega})| = |H(e^{j\omega})|$ , we multiply  $H(z)$  with an allpass function

$$A(z) = \frac{z + 3.1238}{3.1238z + 1}. \text{ Hence}$$

$$G(z) = H(z)A(z) = \left( \frac{(z^2 + 2z - 3)(z^2 - 3z + 5)}{(z + 3.1238)(z + 0.5762)(z^2 - 0.4z + 0.35)} \right) \left( \frac{z + 3.1238}{3.1238z + 1} \right)$$

$$= \frac{(z^2 + 2z - 3)(z^2 - 3z + 5)}{(3.1238z + 1)(z + 0.5762)(z^2 - 0.4z + 0.35)}.$$

Now  $H(z) = \frac{(z + 3)(z - 1)(z^2 - 3z + 5)}{(z + 3.1238)(z + 0.5762)(z^2 - 0.4z + 0.35)}$ . Hence there are ?? other transfer functions with the same magnitude as  $|H(e^{j\omega})|$ .

**4.41** The transfer function of the simplest notch filter is given by

$H(z) = (1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1}) = 1 - 2 \cos \omega_0 z^{-1} + z^{-2}$ . In the steady-state, the output for an input  $x[n] = \cos \omega_0 n$  is given by  $y[n] = |H(e^{j\omega_0})| \cos(\omega_0 n + \theta(\omega_0))$ , (see Eq. (4.18)).

(a) Comparing  $H_1(z) = 1 + z^{-2}$  with  $H(z)$  as given above we conclude  $\cos \omega_0 = 0$  or  $\omega_0 = \pi / 2$ .

Here  $H_1(e^{j\omega_0}) = 1 - e^{-j2\omega_0} = 1 - e^{-j\pi} = 1 - 1 = 0$ . Hence,  $y[n] = |H_1(e^{j\omega_0})| \cos(\omega_0 n + \theta(\omega_0)) = 0$ .

(b) Comparing  $H_2(z) = 1 - \frac{\sqrt{3}}{2} z^{-1} + z^{-2}$  with  $H(z)$  as given above we conclude  $\cos \omega_0 = \sqrt{3} / 4$ .

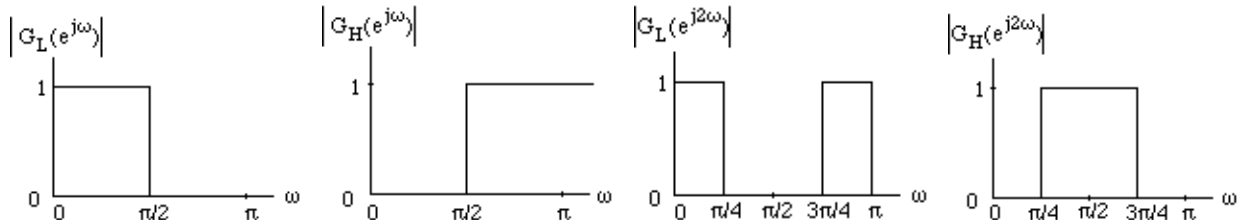
Here now  $H_2(e^{j\omega}) = 1 - \frac{\sqrt{3}}{2} e^{-j\omega} + e^{-j2\omega} = \left( 2 \cos \omega_0 - \frac{\sqrt{3}}{2} \right) e^{-j\omega} = 0$ . Hence  $y[n] = 0$ .

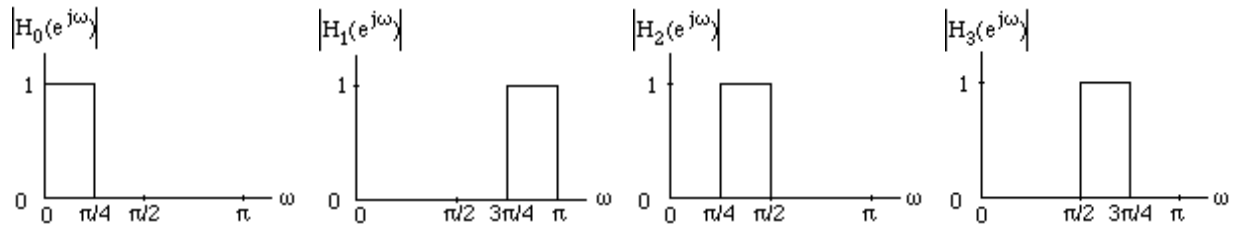
(c) Comparing  $H_3(z) = 1 + \sqrt{2} z^{-1} + z^{-2}$  with  $H(z)$  as given above we conclude  $\cos \omega_0 = -1 / \sqrt{2}$ .

Here now  $H_2(e^{j\omega}) = 1 + \sqrt{2} e^{-j\omega} + e^{-j2\omega} = \left( 2 \cos \omega_0 + \sqrt{2} \right) e^{-j\omega} = 0$ . Hence  $y[n] = 0$ .

**4.42** From the figure  $H_0(z) = \frac{Y_0(z)}{X(z)} = G_L(z)G_L(z^2)$ ,  $H_1(z) = \frac{Y_1(z)}{X(z)} = G_H(z)G_L(z^2)$ ,

$$H_2(z) = \frac{Y_2(z)}{X(z)} = G_L(z)G_H(z^2), \text{ and } H_3(z) = \frac{Y_3(z)}{X(z)} = G_H(z)G_H(z^2),$$





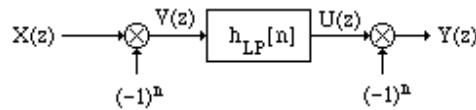
**4.43**  $G(e^{j\omega}) = H_{LP}(e^{j(\pi-\omega)})$ . Since,  $H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq \omega < \omega_c \\ 0, & \omega_c \leq \omega < \pi \end{cases}$ ,  $G(e^{j\omega}) = \begin{cases} 0, & 0 \leq \omega < \pi - \omega_c \\ 1, & \pi - \omega_c \leq \omega < \pi \end{cases}$ .

Hence  $G(e^{j\omega})$  is a highpass filter with a cutoff frequency given by  $\omega_0 = \pi - \omega_c$ . Also, since  $G(z) = H_{LP}(-z)$ , hence  $g[n] = (-1)^n h_{LP}[n]$ .

**4.44**  $G(z) = H_{LP}(e^{j\omega_0} z) + H_{LP}(e^{-j\omega_0} z)$ . Hence,  $g[n] = h_{LP}[n]e^{-j\omega_0 n} + h_{LP}[n]e^{j\omega_0 n} = 2h_{LP}[n]\cos(\omega_0 n)$ . Thus,  $G(z)$  is a real coefficient bandpass filter with a center frequency at  $\omega_0$  and a passband width of  $2\omega_p$ .

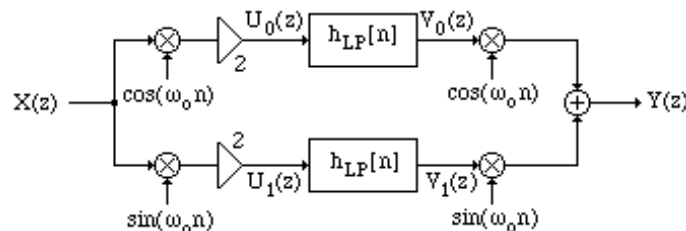
**4.45**  $F(z) = H_{LP}(e^{j\omega_0} z) + H_{LP}(e^{-j\omega_0} z) + H_{LP}(z)$ . Hence,  $f[n] = (1 + 2\cos(\omega_0 n))h_{LP}[n]$ . Thus,  $F(z)$  is a real coefficient bandstop filter with a center frequency at  $\omega_0$  and a stopband width of  $(\pi - 3\omega_p)/2$ .

**4.46**



From the figure,  $V(z) = X(-z)$ ,  $U(z) = H_{LP}(z)X(-z)$ , and  $Y(z) = U(-z) = H_{LP}(-z)X(z)$ . Hence  $H_{eq}(z) = Y(z)/X(z) = H_{LP}(-z)$  which is the highpass filter of Problem 4.41.

**4.47**



$u_0[n] = 2x[n]\cos(\omega_0 n) = x[n]e^{j\omega_0 n} + x[n]e^{-j\omega_0 n}$  or  $U_0(e^{j\omega}) = X(e^{j(\omega+\omega_0)}) + X(e^{j(\omega-\omega_0)})$ .

Likewise,  $U_1(e^{j\omega}) = jX(e^{j(\omega+\omega_0)}) - jX(e^{j(\omega-\omega_0)})$ .

$V_0(z) = H_{LP}(z)X(ze^{j\omega_0}) + H_{LP}(z)X(ze^{-j\omega_0})$ ,

$V_1(z) = jH_{LP}(z)X(ze^{j\omega_0}) - jH_{LP}(z)X(ze^{-j\omega_0})$ ,

$Y(z) = \frac{1}{2}(V_0(ze^{j\omega_0}) + V_0(ze^{-j\omega_0})) + \frac{j}{2}(V_1(ze^{j\omega_0}) - V_1(ze^{-j\omega_0}))$  which after simplification

yields

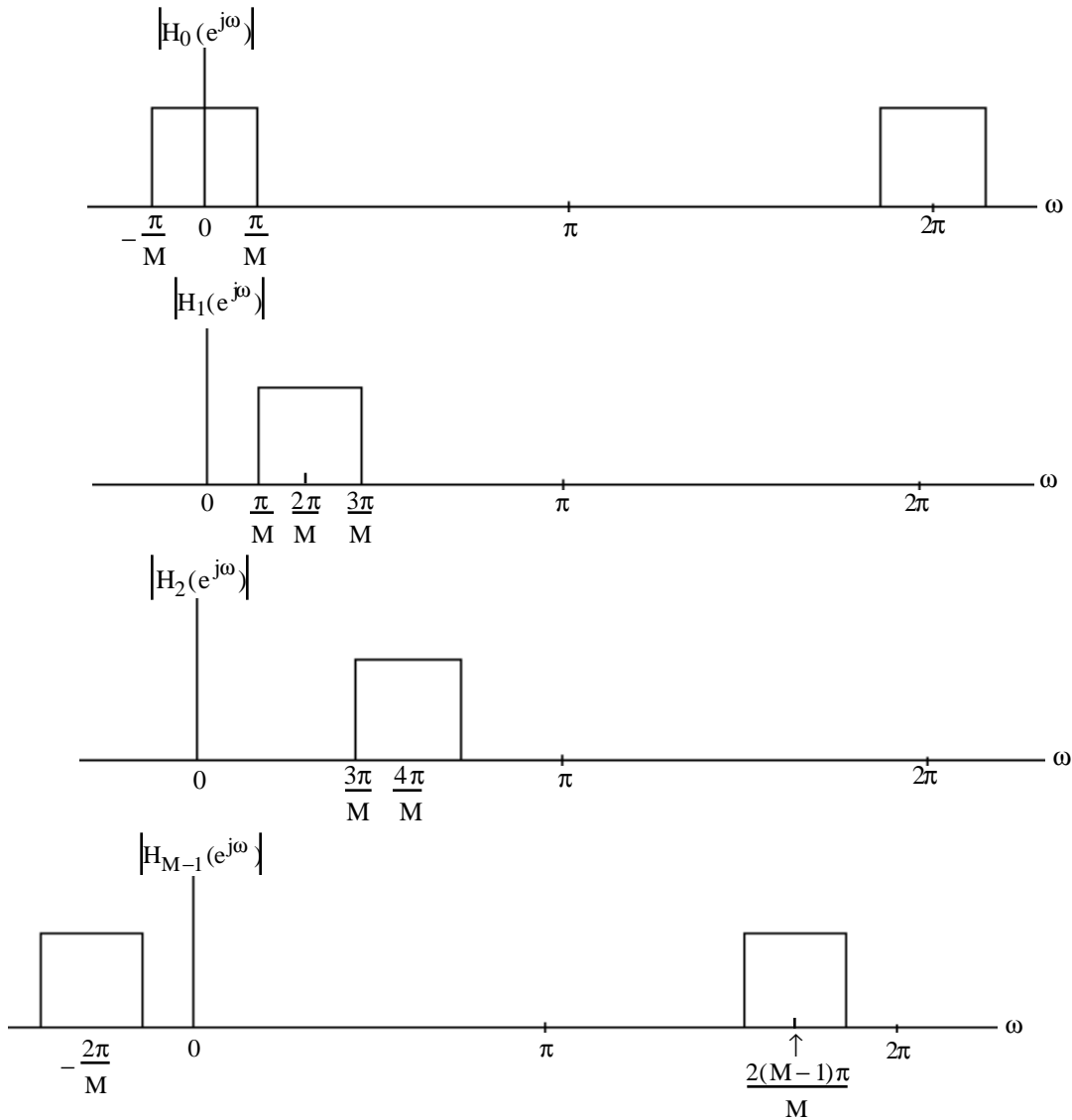
$$Y(z) = \frac{1}{4} \left\{ H_{LP}(ze^{j\omega_0})X(ze^{2j\omega_0}) + H_{LP}(ze^{j\omega_0})X(z) + H_{LP}(ze^{-j\omega_0})X(z) + H_{LP}(ze^{-j\omega_0})X(ze^{-2j\omega_0}) \right\} \\ - \frac{1}{4} \left\{ H_{LP}(ze^{j\omega_0})X(ze^{2j\omega_0}) - H_{LP}(ze^{j\omega_0})X(z) - H_{LP}(ze^{-j\omega_0})X(z) + H_{LP}(ze^{-j\omega_0})X(ze^{-2j\omega_0}) \right\}$$

Hence  $Y(z) = \frac{1}{2} \left\{ H_{LP}(ze^{j\omega_0}) + H_{LP}(ze^{-j\omega_0}) \right\} X(z)$ , Therefore

$$H_{eq}(z) = \frac{Y(z)}{X(z)} = \frac{1}{2} \left\{ H_{LP}(ze^{j\omega_0}) + H_{LP}(ze^{-j\omega_0}) \right\}.$$

Thus the structure shown in Figure P4.6 implements the bandpass filter of Problem 4.42.

#### 4.48



The output of the  $k$ -th filter is a bandpass signals occupying a bandwidth of  $2\pi/M$  and centered at  $\omega = k\pi/M$ . In general, the  $k$ -th filter  $H_k(z)$  has a complex impulse response

generating a complex output sequence. To realize a real coefficient bandpass filter, one can combine the outputs of  $H_k(z)$  and  $H_{M-k}(z)$ .

**4.49**  $H_0(z) = \frac{1}{2}(1+z^{-1})$ . Thus,  $|H_0(e^{j\omega})| = \cos(\omega/2)$ . Now,  $G(z) = (H_0(z))^M$ . Hence,  $|G(e^{j\omega})|^2 = |H_0(e^{j\omega})|^{2M} = (\cos(\omega/2))^{2M}$ . The 3-dB cutoff frequency  $\omega_c$  of  $G(z)$  is thus given by  $(\cos(\omega_c/2))^{2M} = \frac{1}{2}$ . Hence,  $\omega_c = 2 \cos^{-1}(2^{-1/2M})$ .

**4.50**  $H_1(z) = \frac{1}{2}(1-z^{-1})$ . Thus,  $|H_1(e^{j\omega})|^2 = \sin^2(\omega/2)$ . Let  $F(z) = (H_1(z))^M$ . Then  $|F(e^{j\omega})|^2 = (\sin(\omega/2))^{2M}$ . The 3-dB cutoff frequency  $\omega_c$  of  $F(z)$  is thus given by  $(\sin(\omega_c/2))^{2M} = \frac{1}{2}$ , which yields  $\omega_c = 2 \sin^{-1}(2^{-1/2M})$ .

**4.51**  $H_{LP}(z) = \frac{1-\alpha}{2} \left( \frac{1+z^{-1}}{1-\alpha z^{-1}} \right)$ . Note that  $H_{LP}(z)$  is stable if  $|\alpha| < 1$ . Now, 
$$\alpha = \frac{1 - \sin(\omega_c)}{\cos(\omega_c)} = \frac{\cos^2(\omega_c/2) + \sin^2(\omega_c/2) - 2\sin(\omega_c/2)\cos(\omega_c/2)}{\cos^2(\omega_c/2) - \sin^2(\omega_c/2)}$$

$$= \frac{\cos(\omega_c/2) - \sin(\omega_c/2)}{\cos(\omega_c/2) + \sin(\omega_c/2)} = \frac{1 - \tan(\omega_c/2)}{1 + \tan(\omega_c/2)}. \quad (\text{B})$$

If  $0 \leq \omega < \pi$  then  $\tan(\omega_c/2) \geq 0$  hence  $|\alpha| < 1$ .

**4.52** From Eq. (B),  $\alpha = \frac{1 - \tan(\omega_c/2)}{1 + \tan(\omega_c/2)}$ , hence  $\tan(\omega_c/2) = \frac{1-\alpha}{1+\alpha}$ .

**4.53 (a)** From Eq. (111b) we get  $\alpha = \frac{1 - \sin(0.4)}{\cos(0.4)} = 0.6629$ . Substituting this value of  $\alpha$  in Eq.

$$(4.109) \text{ we arrive at } H_{LP}(z) = \frac{0.1685(1+z^{-1})}{1-0.6629z^{-1}}.$$

**(b)** From Eq. (111b) we get  $\alpha = \frac{1 - \sin(0.3\pi)}{\cos(0.3\pi)} = 0.3249$ . Substituting this value of  $\alpha$  in Eq.

$$(4.109) \text{ we arrive at } H_{LP}(z) = \frac{0.3375(1+z^{-1})}{1+0.3249z^{-1}}.$$

**4.54**  $H_{HP}(z) = \frac{1+\alpha}{2} \left( \frac{1-z^{-1}}{1-\alpha z^{-1}} \right)$ . Thus,  $H_{HP}(e^{j\omega}) = \frac{1+\alpha}{2} \left( \frac{1-e^{-j\omega}}{1-\alpha e^{-j\omega}} \right)$ .

$$\left|H_{\text{HP}}(e^{j\omega})\right|^2 = \left[\frac{1+\alpha}{2}\right]^2 \left[\frac{2-2\cos(\omega)}{1+\alpha^2-2\alpha\cos(\omega)}\right]. \text{ At 3-dB cutoff frequency } \omega_c, \left|H_{\text{HP}}(e^{j\omega_c})\right|^2 = \frac{1}{2}.$$

$$\text{Hence, } \left[\frac{1+\alpha}{2}\right]^2 \left[\frac{2-2\cos(\omega_c)}{1+\alpha^2-2\alpha\cos(\omega_c)}\right] = \frac{1}{2} \text{ which yields } \cos(\omega_c) = \frac{2\alpha}{1+\alpha^2}.$$

**4.55 (a)** From Eq. (111b) we get  $\alpha = \frac{1-\sin(0.25)}{\cos(0.25)} = 0.7767$ . Substituting this value of  $\alpha$  in Eq.

$$(4.112) \text{ we arrive at } H_{\text{HP}}(z) = \frac{0.8884(1-z^{-1})}{1-0.7767z^{-1}}.$$

**(b)** From Eq. (111b) we get  $\alpha = \frac{1-\sin(0.4\pi)}{\cos(0.4\pi)} = 0.1584$ . Substituting this value of  $\alpha$  in Eq.

$$(4.112) \text{ we arrive at } H_{\text{HP}}(z) = \frac{0.5792(1-z^{-1})}{1-0.1584z^{-1}}.$$

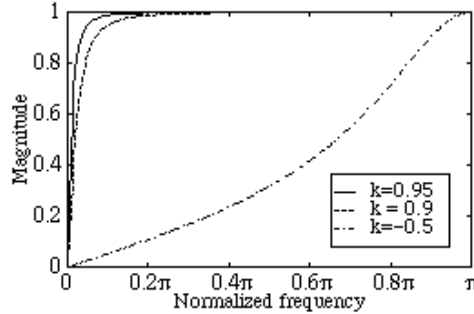
$$\mathbf{4.56} \quad H(z) = \frac{1-z^{-1}}{1-kz^{-1}}. \text{ Hence, } \left|H(e^{j\omega})\right|^2 = \frac{(1-\cos\omega)^2 + \sin^2\omega}{(1-k\cos\omega)^2 + k^2\sin^2\omega}.$$

$$\left|H(e^{j\omega})\right|^2 = \frac{(1-\cos(\omega))^2 + \sin^2(\omega)}{(1-k\cos(\omega))^2 + k^2\sin^2(\omega)} = \frac{2-2\cos\omega}{1+k^2-2k\cos\omega}.$$

Now  $\left|H(e^{j\pi})\right|^2 = \frac{4}{(1+k)^2}$ . Thus, the scaled transfer function is given by

$$H(z) = \frac{1+k}{2} \left( \frac{1-z^{-1}}{1-kz^{-1}} \right). \text{ A plot of the magnitude responses of the scaled transfer function for}$$

$k = 0.95, 0.9$  and  $-0.5$  are given below.



$$\mathbf{4.57} \quad H_{\text{BP}}(z) = \frac{1-\alpha}{2} \left( \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \right). \text{ Thus, } H_{\text{BP}}(e^{j\omega}) = \frac{1-\alpha}{2} \left( \frac{1-e^{-2j\omega}}{1-\beta(1+\alpha)e^{-j\omega}+\alpha e^{-2j\omega}} \right),$$

$$\left|H_{\text{BP}}(e^{j\omega})\right|^2 = \left(\frac{1-\alpha}{2}\right)^2 \frac{2(1-\cos 2\omega)}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos 2\omega-2\beta(1+\alpha)^2\cos\omega}, \text{ which can be}$$

$$\text{simplified as } \left|H_{\text{BP}}(e^{j\omega})\right|^2 = \frac{(1-\alpha)^2 \sin^2\omega}{(1+\alpha)^2(\cos\omega-\beta)^2+(1-\alpha)^2\sin^2\omega}.$$

At the center frequency  $\omega_0$ ,  $\left|H_{\text{BP}}(e^{j\omega_0})\right|^2 = 1$ . Hence,  $(\cos\omega_0-\beta)^2 = 0$  or  $\cos\omega_0 = \beta$ .



At the 3-dB bandedges  $\omega_1$  and  $\omega_2$ ,  $|\mathbf{H}_{\text{BP}}(e^{j\omega_i})|^2 = \frac{1}{2}$ ,  $i = 1, 2$ . This implies

$$(1+\alpha)^2(\cos\omega_i - \beta)^2 = (1-\alpha)^2 \sin^2 \omega_i, \quad (\text{C})$$

or  $\sin\omega_i = \pm \left(\frac{1+\alpha}{1-\alpha}\right)(\cos\omega_i - \beta)$ ,  $i = 1, 2$ . Since  $\omega_1 < \omega_0 < \omega_2$ ,  $\sin\omega_1$  must have positive sign whereas,  $\sin\omega_2$  must have negative sign because otherwise,  $\sin\omega_2 < 0$  for  $\omega_2$  in  $(0, \pi)$ .

Now, Eq. (C) can be rewritten as  $2(1+\alpha^2)\cos^2\omega_i - 2\beta(1+\alpha)^2\cos\omega_i + \beta^2(1+\alpha)^2 - (1-\alpha)^2 = 0$ .

$$\text{Hence, } \cos\omega_1 + \cos\omega_2 = \beta \frac{(1+\alpha)^2}{1+\alpha^2}, \text{ and } (\cos\omega_1)(\cos\omega_2) = \frac{\beta^2(1+\alpha)^2 - (1-\alpha)^2}{2(1+\alpha^2)}.$$

Denote  $\omega_{3\text{dB}} = \omega_2 - \omega_1$ . Then  $\cos(\omega_{3\text{dB}}) = \cos\omega_2 \cos\omega_1 + \sin\omega_2 \sin\omega_1$

$$= \cos\omega_2 \cos\omega_1 - \left(\frac{1+\alpha}{1-\alpha}\right)^2 (\cos\omega_2 \cos\omega_1 + \beta^2 - \beta(\cos\omega_2 + \cos\omega_1)) = \frac{2\alpha}{1+\alpha^2}.$$

**4.58 (a)** Using Eq. (4.115) we get  $\beta = \cos(0.45\pi) = 0.1564$ . Next from Eq. (4.116) we get

$$\frac{2\alpha}{1+\alpha^2} = \cos(0.2\pi) = 0.8090 \text{ or, equivalently } \alpha^2 - 2.4721\alpha + 1 = 0. \text{ Solution of this quadratic}$$

equation yields  $\alpha = 1.9626$  and  $\alpha = 0.5095$ . Substituting  $\alpha = 1.9626$  and  $\beta = 0.1564$  in Eq. (4.113) we arrive at the denominator polynomial of the transfer function  $\mathbf{H}_{\text{BP}}(z)$  as

$$D(z) = (1 - 0.4635z^{-1} + 1.9626z^{-2}) = (1 - (0.2317 + j1.3816)z^{-1})(1 - (0.2317 - j1.3816)z^{-1}) \text{ which}$$

has roots outside the unit circle making  $\mathbf{H}_{\text{BP}}(z)$  an unstable transfer function.

Next, substituting  $\alpha = 0.5095$  and  $\beta = 0.1564$  in Eq. (4.113) we arrive at the denominator polynomial of the transfer function  $\mathbf{H}_{\text{BP}}(z)$  as

$$D(z) = (1 - 0.2361z^{-1} + 0.5095z^{-2}) = (1 - (0.1181 + j0.0704)z^{-1})(1 - (0.1181 - j0.0704)z^{-1}) \text{ which}$$

has roots inside the unit circle making  $\mathbf{H}_{\text{BP}}(z)$  a stable transfer function. The desired transfer function is therefore  $\mathbf{H}_{\text{BP}}(z) = \frac{0.2452(1 - z^{-2})}{1 - 0.2361z^{-1} + 0.5095z^{-2}}$ .

**(b)** Using Eq. (4.115) we get  $\beta = \cos(0.6\pi) = -0.3090$ . Next from Eq. (4.116) we get

$$\frac{2\alpha}{1+\alpha^2} = \cos(0.15\pi) = 0.8910 \text{ or, equivalently } \alpha^2 - 2.2447\alpha + 1 = 0. \text{ Solution of this quadratic}$$

equation yields  $\alpha = 1.6319$  and  $\alpha = 0.6128$ . Substituting  $\alpha = 1.6319$  and  $\beta = -0.3090$  in Eq. (4.113) we arrive at the denominator polynomial of the transfer function  $\mathbf{H}_{\text{BP}}(z)$  as

$$D(z) = (1 + 0.8133z^{-1} + 1.6319z^{-2}) = (1 + (0.4066 + j1.2110)z^{-1})(1 + (0.4066 - j1.2110)z^{-1}) \text{ which}$$

has roots outside the unit circle making  $\mathbf{H}_{\text{BP}}(z)$  an unstable transfer function.

Next, substituting  $\alpha = 0.6128$  and  $\beta = -0.3090$  in Eq. (4.113) we arrive at the denominator polynomial of the transfer function  $\mathbf{H}_{\text{BP}}(z)$  as

$$D(z) = (1 + 0.4984z^{-1} + 0.6128z^{-2}) = (1 + (0.1181 + j0.7040)z^{-1})(1 + (0.1181 - j0.7040)z^{-1}) \text{ which}$$

has roots inside the unit circle making  $H_{BP}(z)$  a stable transfer function. The desired transfer function is therefore  $H_{BP}(z) = \frac{0.1936(1 - z^{-2})}{1 + 0.4984z^{-1} + 6128z^{-2}}$ .

$$4.59 \quad H_{BS}(z) = \left( \frac{1 + \alpha}{2} \right) \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1 + \alpha)z^{-1} + \alpha z^{-2}}.$$

$$\left| H_{BS}(e^{j\omega}) \right|^2 = \left( \frac{1 + \alpha}{2} \right)^2 \frac{2 + 4\beta^2 - 8\beta \cos \omega + 2 \cos 2\omega}{1 + \beta^2(1 + \alpha)^2 + \alpha^2 + 2\alpha \cos 2\omega - 2\beta(1 + \alpha)^2 \cos \omega}, \text{ which can be}$$

$$\text{simplified as } \left| H_{BS}(e^{j\omega}) \right|^2 = \frac{(1 + \alpha)^2 (\cos \omega - \beta)^2}{(1 + \alpha)^2 (\cos \omega - \beta)^2 + (1 - \alpha)^2 \sin^2 \omega}.$$

At the center frequency  $\omega_0$ ,  $\left| H_{BS}(e^{j\omega_0}) \right|^2 = 0$ . Hence,  $(\cos \omega_0 - \beta)^2 = 0$  or  $\cos \omega_0 = \beta$ .

At the 3-dB bandedges  $\omega_1$  and  $\omega_2$ ,  $\left| H_{BS}(e^{j\omega_i}) \right|^2 = \frac{1}{2}$ ,  $i = 1, 2$ . This leads to Eq. (C) given earlier. Hence, as in the solution of the previous problem  $\omega_{3dB} = \frac{2\alpha}{1 + \alpha^2}$ .

**4.60 (a)** Using Eq. (4.115) we get  $\beta = \cos(0.6\pi) = 0.3090$ . Next from Eq. (4.116) we get

$\frac{2\alpha}{1 + \alpha^2} = \cos(0.15\pi) = 0.8910$  or, equivalently  $\alpha^2 - 2.2447\alpha + 1 = 0$ . Solution of this quadratic equation yields  $\alpha = 1.6319$  and  $\alpha = 0.6128$ . Substituting  $\alpha = 1.6319$  and  $\beta = 0.3090$  in Eq. (4.118) we arrive at the denominator polynomial of the transfer function  $H_{BS}(z)$  as  $D(z) = (1 + 0.8133z^{-1} + 1.6319z^{-2}) = (1 + (0.4066 + j1.2110)z^{-1})(1 + (0.4066 - j1.2110)z^{-1})$  which has roots outside the unit circle making  $H_{BS}(z)$  an unstable transfer function.

Next, substituting  $\alpha = 0.6128$  and  $\beta = 0.3090$  in Eq. (4.113) we arrive at the denominator polynomial of the transfer function  $H_{BS}(z)$  as

$D(z) = (1 - 0.4984z^{-1} + 0.6128z^{-2}) = (1 + (0.2492 + j0.7421)z^{-1})(1 + (0.2492 - j0.7421)z^{-1})$  which has roots inside the unit circle making  $H_{BS}(z)$  a stable transfer function. The desired transfer function is therefore  $H_{BS}(z) = \frac{0.8064(1 - 0.4984z^{-1} + z^{-2})}{1 - 0.4984z^{-1} + 6128z^{-2}}$ .

**(b)** Using Eq. (4.115) we get  $\beta = \cos(0.55\pi) = -0.1564$ . Next from Eq. (4.116) we get

$\frac{2\alpha}{1 + \alpha^2} = \cos(0.25\pi) = 0.7071$  or, equivalently  $\alpha^2 - 2.8284\alpha + 1 = 0$ . Solution of this quadratic equation yields  $\alpha = 2.4142$  and  $\alpha = 0.4142$ . Substituting  $\alpha = 2.4142$  and  $\beta = -0.1564$  in Eq. (4.118) we arrive at the denominator polynomial of the transfer function  $H_{BS}(z)$  as

$D(z) = (1 + 0.5341z^{-1} + 2.4142z^{-2}) = (1 + (0.2671 + j1.5307)z^{-1})(1 + (0.2671 - j1.5307)z^{-1})$  which has roots outside the unit circle making  $H_{BS}(z)$  an unstable transfer function.

Next, substituting  $\alpha = 0.4142$  and  $\beta = 0.3090$  in Eq. (4.113) we arrive at the denominator polynomial of the transfer function  $H_{BS}(z)$  as

$D(z) = (1 + 0.2212z^{-1} + 0.4142z^{-2}) = (1 + (0.1106 + j0.6344)z^{-1})(1 + (0.1106 - j0.6344)z^{-1})$  which has roots inside the unit circle making  $H_{BS}(z)$  a stable transfer function. The desired transfer

function is therefore  $H_{BS}(z) = \frac{0.7071(1 + 0.3129z^{-1} + z^{-2})}{1 + 0.2212z^{-1} + 0.4142z^{-2}}$ .

**4.61**  $\frac{(1-\alpha)^2(1+\cos\omega_c)}{2(1+\alpha^2-2\alpha\cos\omega_c)} = 2^{-1/K}$ . Let  $B = 2^{(K-1)/K}$ . Simplifying this equation we get

$$\alpha^2(\cos\omega_c + 1 - B) - 2\alpha(1 + \cos\omega_c - B\cos\omega_c) + 1 + \cos\omega_c - B = 0$$

Solving the above quadratic equation for  $\alpha$  we obtain

$$\alpha = \frac{2(1+(1-B)\cos\omega_c) \pm \sqrt{4(1+(1-B)\cos\omega_c)^2 - 4(1+\cos\omega_c-B)^2}}{2(1+\cos\omega_c-B)}$$

$$= \frac{(1+(1-B)\cos\omega_c) \pm \sqrt{(2+2\cos\omega_c-B-B\cos\omega_c)(B(1-\cos\omega_c))}}{(1+\cos\omega_c-B)}$$

$$= \frac{1+(1-B)\cos\omega_c \pm \sin\omega_c\sqrt{2B-B^2}}{(1+\cos\omega_c-B)}. \text{ For stability we require } |\alpha| < 1, \text{ hence the desired}$$

$$\text{solution is } \alpha = \frac{1+(1-B)\cos\omega_c - \sin\omega_c\sqrt{2B-B^2}}{(1+\cos\omega_c-B)}.$$

$$\begin{aligned} \mathbf{4.62} \quad H_{HP}(z) &= \frac{1+\alpha}{2} \left( \frac{1-z^{-1}}{1-\alpha z^{-1}} \right). \quad |H_{HP}(e^{j\omega})|^2 = \left[ \frac{1+\alpha}{2} \right]^2 \left| \frac{1-e^{-j\omega}}{1-\alpha e^{-j\omega}} \right|^2 \\ |H_{HP}(e^{j\omega})|^{2K} &= \left[ \frac{1+\alpha}{2} \right]^{2K} \left| \frac{1-e^{-j\omega}}{1-\alpha e^{-j\omega}} \right|^{2K} = \left[ \frac{1+\alpha}{2} \right]^{2K} \frac{2^K(1-\cos\omega)^K}{(1+\alpha^2-2\alpha\cos\omega)^K} \end{aligned}$$

At the 3-dB cut off frequency  $\omega_c$ ,  $|H_{HP}(e^{j\omega_c})|^{2K} = \frac{1}{2}$ . Let  $B = 2^{(K-1)/K}$ . Simplifying the

above equation we get  $\alpha^2(1-\cos\omega_c-B) + 2\alpha(1-\cos\omega_c+B\cos\omega_c) + 1-\cos\omega_c-B = 0$ .

$$\text{Hence, } \alpha = \frac{-2(1-\cos\omega_c+B\cos\omega_c) \pm 2\sqrt{(1-\cos\omega_c+B\cos\omega_c)^2 - (1-\cos\omega_c-B)^2}}{2(1-\cos\omega_c-B)}$$

For  $H_{HP}(z)$  to be stable, we require  $|\alpha| < 1$ , hence the desired solution is

$$\alpha = \frac{\sin\omega_c\sqrt{2B-B^2} - (1-\cos\omega_c+B\cos\omega_c)}{(1-\cos\omega_c-B)}.$$

**4.63**  $H_1(z) = H(-z)$ .  $H_1(e^{j\omega}) = H(e^{(\pi-\omega)})$ . If  $H(e^{j\omega})$  is a lowpass filter with a passband edge at  $\omega_p$ , then  $H_1(e^{j\omega}) \approx \begin{cases} 1, & \text{if } \pi - \omega_p \leq \omega < \pi, \\ 0, & \text{if } 0 \leq \omega < \pi - \omega_p, \end{cases}$  hence  $H_1(e^{j\omega})$  is a highpass filter.

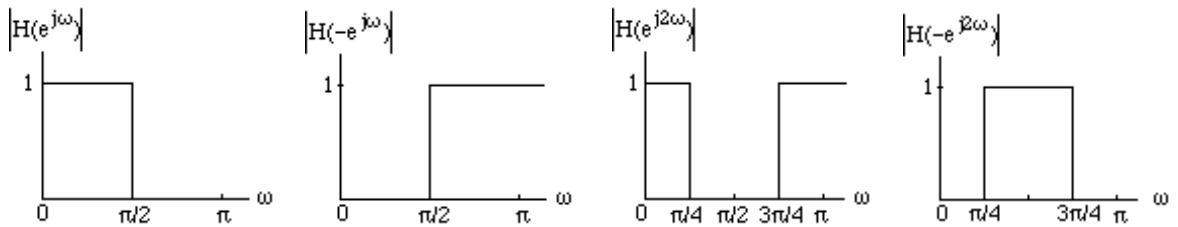
Now,  $h_1[n] = (-1)^n h[n]$ . The passband edge of the highpass filter is given by  $\omega_{p,HP} = \pi - \omega_{p,LP}$ , and the stopband edge is given by  $\omega_{s,HP} = \pi - \omega_{s,LP}$ .

**4.64**  $H_{LP}(z) = \frac{1-\alpha}{2} \left( \frac{1+z^{-1}}{1-\alpha z^{-1}} \right)$ ,  $G_{HP}(z) = \frac{1-\alpha}{2} \left( \frac{1-z^{-1}}{1+\alpha z^{-1}} \right)$ . Let  $\beta = -\alpha$ , hence

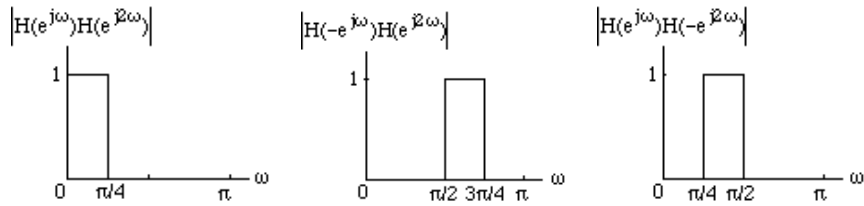
$$G_{HP}(z) = \frac{1+\beta}{2} \left( \frac{1-z^{-1}}{1-\beta z^{-1}} \right).$$

Therefore  $\omega_c = \cos^{-1}(\beta) = \cos^{-1}(-\alpha)$ .

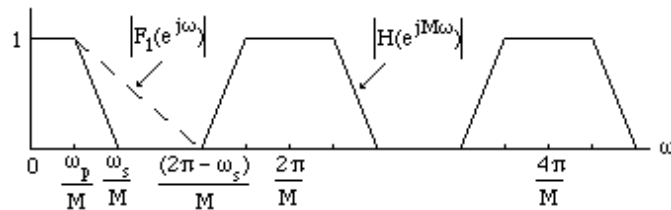
**4.65** The magnitude responses of  $H(z)$ ,  $H(-z)$ ,  $H(z^2)$  and  $H(-z^2)$  are shown below.



The magnitude responses of  $H(-z)H(z^2)$ ,  $H(-z)H(z^2)$ , and  $H(z)H(-z^2)$  are shown below.

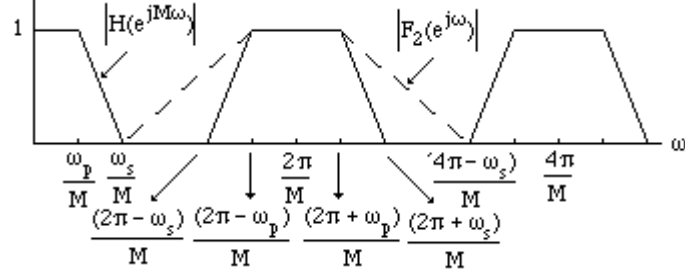


**4.66** The magnitude responses of  $H(z^M)$  (solid line) and  $F_1(z)$  (dashed line) are shown below:



Hence  $G_1(z) = H(z^M)F_1(z)$  is a lowpass filter with unity passband magnitude and passband edge at  $\omega_p/M$ , and a stopband edge at  $\omega_s/M$ .

The magnitude responses of  $H(z^M)$  (solid line) and  $F_2(z)$  (dashed line) are shown below:



Hence  $G_2(z) = H(z^M)F_2(z)$  is a bandpass filter with unity passband magnitude and passband edges at  $(2\pi - \omega_p)/M$  and  $(2\pi + \omega_p)/M$ , and a stopband edges at  $(2\pi - \omega_s)/M$  and  $(2\pi + \omega_s)/M$ .

**4.67**  $v[n] = x[-n] \circledast h[n]$ , and  $u[n] = v[-n] = x[n] \circledast h[-n]$ . Hence  $y[n] = (h[n] + h[-n]) \circledast x[n]$ .  
Therefore  $G(e^{j\omega}) = H(e^{j\omega}) + H^*(e^{j\omega})$ . Hence the equivalent transfer function  $G(e^{j\omega})$  is real and has zero phase.

**4.68** From Eq. (4.81) we observe that the amplitude response  $\hat{H}(\omega)$  of a Type 1 FIR transfer function is a function of  $\cos(\omega n)$ . Thus,  $\hat{H}(\omega + 2\pi k)$  will be a function of  $\cos[(\omega + 2\pi k)n] = \cos(\omega n + 2\pi kn) = \cos(\omega n)\cos(2\pi kn) - \sin(\omega n)\sin(2\pi kn) = \cos(\omega n)$ , and hence  $\hat{H}(\omega)$  is a periodic function in  $\omega$  with a period  $2\pi$ .

Likewise, from Eq. (4.91) we note that the amplitude response  $\hat{H}(\omega)$  of a Type 3 FIR transfer function is a function of  $\sin(\omega n)$ . Thus,  $\hat{H}(\omega + 2\pi k)$  will be a function of  $\sin[(\omega + 2\pi k)n] = \sin(\omega n + 2\pi kn) = \sin(\omega n)\cos(2\pi kn) + \cos(\omega n)\sin(2\pi kn) = \sin(\omega n)$ , and hence  $\hat{H}(\omega)$  is a periodic function in  $\omega$  with a period  $2\pi$ .

Next, from Eq. (4.86) we observe that the amplitude response  $\hat{H}(\omega)$  of a Type 2 FIR transfer function is a function of  $\cos\left(\omega\left(n - \frac{1}{2}\right)\right)$ . Thus,  $\hat{H}(\omega + 4\pi k)$  will be a function of  $\cos\left((\omega + 4\pi k)\left(n - \frac{1}{2}\right)\right) = \cos\left(\omega\left(n - \frac{1}{2}\right) + 4\pi k\left(n - \frac{1}{2}\right)\right) = \cos\left(\omega\left(n - \frac{1}{2}\right)\right)\cos\left(4\pi k\left(n - \frac{1}{2}\right)\right) - \sin\left(\omega\left(n - \frac{1}{2}\right)\right)\sin\left(4\pi k\left(n - \frac{1}{2}\right)\right) = \cos\left(\omega\left(n - \frac{1}{2}\right)\right)$  as  $\cos\left(4\pi k\left(n - \frac{1}{2}\right)\right) = 1$  and  $\sin\left(4\pi k\left(n - \frac{1}{2}\right)\right) = 0$ . Hence  $\hat{H}(\omega)$  is a periodic function in  $\omega$  with a period  $4\pi$ .

Finally, from Eq. (4.96) we observe that the amplitude response  $\hat{H}(\omega)$  of a Type 4 FIR transfer function is a function of  $\sin\left(\omega\left(n - \frac{1}{2}\right)\right)$ . Thus,  $\hat{H}(\omega + 4\pi k)$  will be a function of  $\sin\left((\omega + 4\pi k)\left(n - \frac{1}{2}\right)\right) = \sin\left(\omega\left(n - \frac{1}{2}\right) + 4\pi k\left(n - \frac{1}{2}\right)\right) = \sin\left(\omega\left(n - \frac{1}{2}\right)\right)\cos\left(4\pi k\left(n - \frac{1}{2}\right)\right) + \cos\left(\omega\left(n - \frac{1}{2}\right)\right)\sin\left(4\pi k\left(n - \frac{1}{2}\right)\right) = \sin\left(\omega\left(n - \frac{1}{2}\right)\right)$  as

$\cos\left(4\pi k\left(n - \frac{1}{2}\right)\right) = 1$  and  $\sin\left(4\pi k\left(n - \frac{1}{2}\right)\right) = 0$ . Hence  $H(\omega)$  is a periodic function in  $\omega$  with a period  $4\pi$ .

**4.69** The zeros given are at  $z_1 = -0.5$ ,  $z_2 = 0.3 + j0.5$ ,  $z_3 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$ . The remaining zeros are at  $z_4 = 1/z_1 = -2$ ,  $z_5 = z_2^* = 0.3 - j0.5$ ,  $z_6 = 1/z_5 = 0.12 - j0.1993$ ,  $z_7 = z_6^* = 0.12 + j0.1993$ , and  $z_8 = z_3^* = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$ . Hence,  $H(z) = \prod_{i=1}^8 (1 - z_i z^{-1})$   
 $= 1 - 1.1353z^{-1} + 0.5635z^{-2} + 5.6841z^{-3} + 4.9771z^{-4} + 5.6841z^{-5} + 0.5635z^{-6} - 1.1353z^{-7} + z^{-8}$ .

**4.70** The zeros given are at  $z_1 = 3$ ,  $z_2 = j0.8$ ,  $z_3 = j$ . The remaining zeros are at  $z_4 = 1/z_1 = 1/3$ ,  $z_5 = z_2^* = -j0.8$ ,  $z_6 = z_3^* = -j$ ,  $z_7 = 1/z_2 = -j0.125$ ,  $z_8 = z_7^* = j0.125$ , and  $z_9 = -1$ . Hence,  $H(z) = \prod_{i=1}^9 (1 - z_i z^{-1})$   
 $= 1 - 2.3333z^{-1} + 0.8692z^{-2} - 6.4725z^{-3} - 4.27z^{-4} - 4.27z^{-5} - 6.4725z^{-6}$   
 $+ 0.8692z^{-7} - 2.3333z^{-8} + z^{-9}$ .

**4.71** The zeros given are at  $z_1 = -0.3 + j0.5$ ;  $z_2 = j0.8$ ; and  $z_3 = -0.3$ . The remaining zeros are at  $z_4 = z_1^* = -0.3 - j0.5$ ;  $z_5 = 1/z_1 = -0.8824 - j1.4786$ ;  $z_6 = z_5^* = -0.8824 + j1.4786$ ;  $z_7 = z_2^* = -j0.8$ ;  $z_8 = 1/z_7 = -j1.25$ ;  $z_9 = z_8^* = j1.25$ ;  $z_{10} = 1/z_3 = -3.3333$ ;  $z_{11} = 1$ ; and  $z_{12} = -1$ . Hence,  $H(z) = \prod_{i=1}^{12} (1 - z_i z^{-1}) = 1 + 5.998z^{-1} + 15.1343z^{-2} + 27.7107z^{-3}$   
 $+ 29.4822z^{-4} + 23.4343z^{-5} - 23.4343z^{-7} - 23.4343z^{-8} - 29.4822z^{-9}$   
 $- 27.7107z^{-10} - 15.1343z^{-11} - 5.998z^{-12} - z^{-13}$ .

**4.72** The zeros given are at  $z_1 = -1.2 + j1.4$ ;  $z_2 = \frac{1}{2} + j\frac{\sqrt{3}}{2}$ ;  $z_3 = \frac{1}{4} + j\frac{\sqrt{15}}{4}$ . The remaining zeros are at  $z_4 = z_1^* = -1.2 - j1.4$ ;  $z_5 = 1/z_1 = -0.3529 - j0.4181$ ;  $z_6 = z_5^* = -0.3529 + j0.4181$ ;  $z_7 = z_2^* = \frac{1}{2} - j\frac{\sqrt{3}}{2}$ ;  $z_8 = z_3^* = \frac{1}{4} - j\frac{\sqrt{15}}{4}$ , and  $z_9 = 1$ . Hence,  $H(z) = \prod_{i=1}^9 (1 - z_i z^{-1})$   
 $= 1 + 0.6059z^{-1} + 1.6235z^{-2} - 1.9412z^{-3} + 4.8647z^{-4} - 4.8647z^{-5} + 1.9412z^{-6}$   
 $- 1.6235z^{-7} - 0.6059z^{-8} - z^{-9}$ .

**4.73**  $H(z) = \sum_{n=0}^N h[n]z^{-n}$ , and  $H(e^{j\omega}) = \sum_{n=0}^N h[n]e^{-j\omega n}$ . The frequency response will exhibit generalized linear phase if it can be expressed in the form  $H(e^{j\omega}) = H(\omega)e^{-j\alpha\omega}e^{-j\beta}$ , where  $H(\omega)$ , the amplitude function, is a real function of  $\omega$ , and  $\alpha$  and  $\beta$  are constants.

We need to examine the case when the order  $N$  is even and when  $N$  is odd separately. Without

any loss of generality, assume first  $N = 5$ . Then  $H(z) = \sum_{n=0}^5 h[n]z^{-n}$ , and

$$\begin{aligned} H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} + h[5]e^{-j5\omega} \\ &= e^{-j5\omega/2} \left[ h[0]e^{j5\omega/2} + h[5]e^{-j5\omega/2} + h[1]e^{j3\omega/2} + h[4]e^{-j3\omega/2} + h[2]e^{j\omega/2} + h[3]e^{-j\omega/2} \right] \\ &= e^{-j5\omega/2} \left[ (h[0] + h[5])\cos(5\omega/2) + (h[1] + h[4])\cos(3\omega/2) + (h[2] + h[3])\cos(\omega/2) \right] \\ &\quad + je^{-j5\omega/2} \left[ (h[0] - h[5])\sin(5\omega/2) + (h[1] - h[4])\sin(3\omega/2) + (h[2] - h[3])\sin(\omega/2) \right]. \end{aligned}$$

Note from the above expression that if  $h[n] = h[5 - n]$ ,  $0 \leq n \leq 5$ , then we have

$$H(e^{j\omega}) = e^{-j5\omega/2} \tilde{H}(\omega), \text{ where } \tilde{H}(\omega) = 2h[0]\cos(5\omega/2) + 2h[1]\cos(3\omega/2) + 2h[2]\cos(\omega/2),$$

which is a real function of  $\omega$ , and as a result,  $H(e^{j\omega})$  has linear phase. Alternately, if

$h[n] = -h[5 - n]$ ,  $0 \leq n \leq 5$ , then we have  $H(e^{j\omega}) = je^{-j5\omega/2} \tilde{H}(\omega) = e^{-j5\omega/2} e^{j\pi/2} \tilde{H}(\omega)$ , where

$$\tilde{H}(\omega) = 2h[0]\sin(5\omega/2) + 2h[1]\sin(3\omega/2) + 2h[2]\sin(\omega/2),$$

which is a real function of  $\omega$ , and as a result,  $H(e^{j\omega})$  has generalized linear phase.

Next, we assume first  $N = 6$ . Then  $H(z) = \sum_{n=0}^6 h[n]z^{-n}$ , and

$$\begin{aligned} H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega} + h[3]e^{-j3\omega} + h[4]e^{-j4\omega} + h[5]e^{-j5\omega} + h[6]e^{-j6\omega} \\ &= e^{-j3\omega} \left[ h[0]e^{j3\omega} + h[6]e^{-j3\omega} + h[1]e^{j2\omega} + h[5]e^{-j2\omega} + h[2]e^{j\omega} + h[4]e^{-j\omega} + h[3] \right] \\ &= e^{-j3\omega} \left[ (h[0] + h[6])\cos(3\omega) + (h[1] + h[5])\cos(2\omega) + (h[2] + h[4])\cos(\omega) + h[3] \right] \\ &\quad + e^{-j3\omega} \left[ (h[0] - h[6])\sin(3\omega) + (h[1] - h[5])\sin(2\omega) + (h[2] - h[4])\sin(\omega) \right]. \end{aligned}$$

Note from the above expression that if  $h[n] = h[6 - n]$ ,  $0 \leq n \leq 6$ , then we have

$$H(e^{j\omega}) = e^{-j3\omega} \tilde{H}(\omega), \text{ where } \tilde{H}(\omega) = 2h[0]\cos(3\omega) + 2h[1]\cos(2\omega) + 2h[2]\cos(\omega) + h[3]$$

which is a real function of  $\omega$ , and as a result,  $H(e^{j\omega})$  has linear phase. Alternately, if

$h[n] = -h[6 - n]$ ,  $0 \leq n \leq 6$ , then we have  $H(e^{j\omega}) = je^{-j3\omega} \tilde{H}(\omega) = e^{-j3\omega} e^{j\pi/2} \tilde{H}(\omega)$ , where

$$\tilde{H}(\omega) = 2h[0]\sin(3\omega) + 2h[1]\sin(2\omega) + 2h[2]\sin(\omega)$$

which is a real function of  $\omega$ , and as a result,  $H(e^{j\omega})$  has generalized linear phase.

**4.74 (a)**  $H_2(z)$  is linear-phase (also can be seen from the symmetric impulse response).

**(b)**  $H_3(z)$  is minimum-phase as all its zeros are inside the unit circle.

**(c)**  $H_5(z)$  is maximum-phase as all its zeros are outside the unit circle.

**4.75**  $G_1(z) = (6 - z^{-1} - 12z^{-2})(2 + 5z^{-1}) = 30(1 - \frac{3}{2}z^{-1})(1 + \frac{4}{3}z^{-1})(\frac{2}{5} + z^{-1})$ .

**(a)** The other transfer functions having the same magnitude response are:

(i)  $H_1(z) = 30(-\frac{3}{2} + z^{-1})(1 + \frac{4}{3}z^{-1})(\frac{2}{5} + z^{-1})$ , (ii)  $H_2(z) = 30(1 - \frac{3}{2}z^{-1})(\frac{4}{3} + z^{-1})(\frac{2}{5} + z^{-1})$ ,  
 (iii)  $H_3(z) = 30(1 - \frac{3}{2}z^{-1})(\frac{4}{3} + z^{-1})(1 + \frac{2}{5}z^{-1})$ , (iv)  $H_4(z) = 30(-\frac{3}{2} + z^{-1})(\frac{4}{3} + z^{-1})(1 + \frac{2}{5}z^{-1})$ ,  
 (v)  $H_5(z) = 30(-\frac{3}{2} + z^{-1})(1 + \frac{4}{3}z^{-1})(1 + \frac{2}{5}z^{-1})$ , (vi)  $H_6(z) = 30(1 - \frac{3}{2}z^{-1})(1 + \frac{4}{3}z^{-1})(1 + \frac{2}{5}z^{-1})$ ,  
 (vii)  $H_7(z) = 30(-\frac{3}{2} + z^{-1})(\frac{4}{3} + z^{-1})(1 + \frac{2}{5}z^{-1})$ .

(b)  $H_7(z)$  has a minimum phase response, and  $G_1(z)$  has a maximum phase response.

(c) The partial energy of the impulse responses of each of the above transfer functions for different values of  $k$  are given by

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$G_1(z)$	144	928	1769	5369	5369
$H_1(z)$	324	3573	3769	5369	5369
$H_2(z)$	256	1040	3344	5369	5369
$H_3(z)$	900	949	4793	5369	5369
$H_4(z)$	576	4420	4469	5369	5369
$H_5(z)$	2025	4329	5113	5369	5369
$H_6(z)$	1600	1796	5045	5369	5369
$H_7(z)$	3600	4441	5225	5369	5369

The partial energy remains the same for values of  $k > 2$ . From the table given above it can be

seen that  $\sum_{m=0}^k |h_i[m]|^2 \leq \sum_{m=0}^k |h_7[m]|^2$ , and  $\sum_{m=0}^{\infty} |h_i[m]|^2 = \sum_{m=0}^{\infty} |h_7[m]|^2$ ,  $i = 1, 2, \dots, 7$ .

**4.76** A maximum-phase FIR transfer function has all zeros outside the unit circle, and hence, the product of the roots is necessarily greater than 1. This implies that only those FIR transfer functions which have the coefficient of the highest power in  $z^{-1}$  ( $z^{-6}$  in the present case) greater than 1 can have maximum phase. Thus only  $H_1(z)$  and  $H_3(z)$  can be maximum-phase transfer functions. Also, maximum-phase transfer functions will have minimum partial-energy (as indicated in the solution of Problem 4.65). Hence,  $H_1(z)$  is a maximum-phase transfer function since it has the smallest constant term in comparison with that of  $H_3(z)$ .

Likewise, a minimum-phase FIR transfer function is characterized by: (1) largest constant term, and (2) the value of the coefficient of the highest power of  $z^{-1}$  being less than 1. In the present problem, it can be seen that  $H_2(z)$  satisfies these two conditions and, is hence, a minimum-phase transfer function.

Total no. of length-7 sequences having the same magnitude response is 27. Thus, there exist 123 other sequences with the same magnitude responses as those given here.

**4.77 (a)** Type 1:  $\{h[n]\} = \{a \ b \ c \ d \ c \ b \ a\}$ ,

(b) Type 2:  $\{h[n]\} = \{a \ b \ c \ d \ d \ c \ b \ a\}$ ,



(c) Type 3:  $\{h[n]\} = \{a \ b \ c \ d \ 0 \ -d \ -c \ -b \ -a\}$ ,

(d) Type 4:  $\{h[n]\} = \{a \ b \ c \ d \ -d \ -c \ -b \ -a\}$ .

**4.78 (a)** Type 1:  $H(z) = -1 - 2z^{-1} - 4z^{-2} + 5z^{-3} + 6z^{-4} + 5z^{-5} - 4z^{-6} - 2z^{-7} - z^{-8}$ . From the zero plot obtained using the M-file `zplane` it can be seen that complex-conjugate zero pairs inside and outside the unit circle appear in mirror-image symmetry, complex conjugate zero-pair on the unit circle appear singly and the zero at  $z = 1$  is of order 2, verifying the observations of Section 4.4.4 of text.

(b) Type 2:  $H(z) = -1 - 2z^{-1} - 4z^{-2} + 5z^{-3} + 6z^{-4} + 6z^{-5} + 5z^{-6} - 4z^{-7} - 2z^{-8} - z^{-9}$ . From the zero plot obtained using the M-file `zplane` it can be seen that complex-conjugate zero pair inside and outside the unit circle appear in mirror-image symmetry, complex conjugate zero-pair on the unit circle appear singly, there is a zero at  $z = -1$ , and the zeros on the real axis exhibit mirror-image symmetry, verifying the observations of Section 4.4.4 of text.

(c) Type 3:  $H(z) = -1 - 2z^{-1} - 4z^{-2} + 5z^{-3} + 6z^{-4} - 6z^{-6} - 5z^{-7} + 4z^{-8} + 2z^{-9} + z^{-10}$ . From the zero plot obtained using the M-file `zplane` it can be seen that complex-conjugate zero pair inside and outside the unit circle appear in mirror-image symmetry, complex conjugate zero-pair on the unit circle appear singly, there is a zero at  $z = -1$  and at  $z = 1$ , verifying the observations of Section 4.4.4 of text.

(d) Type 4:  $H(z) = -1 - 2z^{-1} - 4z^{-2} + 5z^{-3} + 6z^{-4} - 6z^{-5} - 5z^{-6} + 4z^{-7} + 2z^{-8} + z^{-9}$ . From the zero plot obtained using the M-file `zplane` it can be seen that complex-conjugate zero pair inside and outside the unit circle appear in mirror-image symmetry, complex conjugate zero-pair on the unit circle appear singly, and there is a zero at  $z = 1$ , verifying the observations of Section 4.4.4 of text.

**4.79**  $H_1(z)$  is of Type 1 and hence, it has a symmetric impulse response of odd length  $2n+1$ . Let  $\alpha$  be the constant term of  $H_1(z)$ . Then, the coefficient of the highest power of  $z^{-1}$  of  $H_1(z)$  is also  $\alpha$ .

$H_2(z)$  is of Type 2 and hence, it has a symmetric impulse response of even length  $2m$ . Let  $\beta$  be the constant term of  $H_2(z)$ . Then, the coefficient of the highest power of  $z^{-1}$  of  $H_2(z)$  is also  $\beta$ .

$H_3(z)$  is of Type 3 and hence, it has an anti-symmetric impulse response of odd length  $2r+1$ . Let  $\gamma$  be the constant term of  $H_3(z)$ . Then, the coefficient of the highest power of  $z^{-1}$  of  $H_3(z)$  is  $-\gamma$ .

$H_4(z)$  is of Type 4 and hence, it has an anti-symmetric impulse response of even length  $2k$ . Let  $\delta$  be the constant term of  $H_4(z)$ . Then, the coefficient of the highest power of  $z^{-1}$  of  $H_4(z)$  is  $-\delta$ .

**(a)** The length of  $H_1(z)H_1(z)$  is  $(2n+1)+(2n+1) - 1 = 4n + 1$  which is odd. The constant term of  $H_1(z)H_1(z)$  is  $\alpha^2$  and the coefficient of the highest power of  $z^{-1}$  of  $H_1(z)H_1(z)$  is also  $\alpha^2$ . Hence,  $H_1(z)H_1(z)$  is of Type 1.

**(b)** The length of  $H_1(z)H_2(z)$  is  $(2n+1)+(2m) - 1 = 2(n + m)$  which is even. The constant term of  $H_1(z)H_2(z)$  is  $\alpha\beta$  and the coefficient of the highest power of  $z^{-1}$  of  $H_1(z)H_2(z)$  is also  $\alpha\beta$ . Hence,  $H_1(z)H_2(z)$  is of Type 2.

**(c)** The length of  $H_1(z)H_3(z)$  is  $(2n+1)+(2r+1) - 1 = 2(n + r) + 1$  which is odd. The constant term of  $H_1(z)H_3(z)$  is  $\alpha\gamma$  and the coefficient of the highest power of  $z^{-1}$  of  $H_1(z)H_3(z)$  is also  $-\alpha\gamma$ . Hence,  $H_1(z)H_3(z)$  is of Type 3.

**(d)** The length of  $H_1(z)H_4(z)$  is  $(2n+1)+(2k) - 1 = 2(n + k)$  which is even. The constant term of  $H_1(z)H_4(z)$  is  $\alpha\delta$  and the coefficient of the highest power of  $z^{-1}$  of  $H_1(z)H_4(z)$  is also  $-\alpha\delta$ . Hence,  $H_1(z)H_4(z)$  is of Type 4.

**(e)** The length of  $H_2(z)H_2(z)$  is  $(2m)+(2m) - 1 = 4m - 1$  which is odd. The constant term of  $H_2(z)H_2(z)$  is  $\beta^2$  and the coefficient of the highest power of  $z^{-1}$  of  $H_2(z)H_2(z)$  is also  $\beta^2$ . Hence,  $H_2(z)H_2(z)$  is of Type 1.

**(f)** The length of  $H_3(z)H_3(z)$  is  $(2r+1)+(2r+1) - 1 = 4r + 1$  which is odd. The constant term of  $H_3(z)H_3(z)$  is  $\gamma^2$  and the coefficient of the highest power of  $z^{-1}$  of  $H_3(z)H_3(z)$  is also  $\gamma^2$ . Hence,  $H_3(z)H_3(z)$  is of Type 1.

**(g)** The length of  $H_4(z)H_4(z)$  is  $(2k)+(2k) - 1 = 4k - 1$  which is odd. The constant term of  $H_4(z)H_4(z)$  is  $\delta^2$  and the coefficient of the highest power of  $z^{-1}$  of  $H_4(z)H_4(z)$  is also  $\delta^2$ . Hence,  $H_4(z)H_4(z)$  is of Type 1.

**(h)** The length of  $H_2(z)H_3(z)$  is  $(2m)+(2r+1) - 1 = 2(m + r)$  which is even. The constant term of  $H_2(z)H_3(z)$  is  $\beta\gamma$  and the coefficient of the highest power of  $z^{-1}$  of  $H_2(z)H_3(z)$  is  $-\beta\gamma$ . Hence,  $H_2(z)H_3(z)$  is of Type 4.

**(i)** The length of  $H_3(z)H_4(z)$  is  $(2r+1)+(2k) - 1 = 2(r + k)$  which is even. The constant term of  $H_3(z)H_4(z)$  is  $\gamma\delta$  and the coefficient of the highest power of  $z^{-1}$  of  $H_3(z)H_4(z)$  is also  $\gamma\delta$ . Hence,  $H_3(z)H_4(z)$  is of Type 2.

**4.80 (a)**  $G(\omega) = H(\omega) + \delta$ .  $H(\omega) = \sum_{n=0}^{N/2} a(n) \cos(\omega n)$  where  $a[0] = h[N/2]$ . Hence

$$g[n] = \begin{cases} h[n], & \forall n \text{ except } n = N/2, \\ h[N/2] + \delta, & n = N/2. \end{cases} \quad \text{Thus } m = N/2 \text{ and } \alpha = \delta.$$

**(b)** Since  $G(\omega)$  is real and positive hence it can be expressed as  $G(\omega) = F(e^{j\omega})F^*(e^{j\omega})$ . As  $H(z)$  is a linear phase filter so is  $G(z)$ . Therefore  $G(z)$  will have roots at  $z = z_0$  and at  $z = 1/z_0$ . This means that  $G(z)$  will have roots inside the unit circle with reciprocal roots outside the unit circle. Hence  $F(z)$  can be obtained by assigning to it all the roots that are inside the unit circle. Then  $F(z^{-1})$  is automatically assigned the reciprocal roots outside the unit circle.

**(c)** No,  $H(\omega)$  can not be expressed as the square magnitude of a minimum-phase FIR filter because  $H(\omega)$  takes on negative values too.

**4.81**  $\sum_{n=0}^K |h[n]|^2 = 0.95 \sum_{n=0}^{\infty} |h[n]|^2$ . Since  $H(z) = 1/(1 + \alpha z^{-1})$ , hence  $h[n] = (-\alpha)^n \mu[n]$ . Thus

$$\frac{1 - |\alpha|^{2K}}{1 - |\alpha|^2} = \frac{0.95}{1 - |\alpha|^2}. \quad \text{Solving this equation for } K \text{ we get } K = 0.5 \frac{\log(0.05)}{\log(|\alpha|)}.$$

**4.82 (a)**  $F_1(z) = 1 + 2z^{-1} + 3z^{-2}$ .  $F_1(z)$  has roots at  $z = -1 \pm j\sqrt{2}$ . Since  $H(z)$  is a linear phase FIR transfer function, its roots must exist in reciprocal pairs. Hence if  $H(z)$  has roots at  $z = -1 \pm j\sqrt{2}$ , then it should also have roots at  $z = \frac{1}{-1 \pm j\sqrt{2}} = -\frac{1}{3} \pm j\frac{\sqrt{2}}{3}$ . Therefore  $H(z)$  should atleast have another factor with roots at  $z = -\frac{1}{3} \pm j\frac{\sqrt{2}}{3}$ . Hence  $F_2(z) = 3 + 2z^{-1} + z^{-2}$ , which is the mirrorimage polynomial to  $F_1(z)$  and  $H(z) = F_1(z)F_2(z)$

$$= (1 + 2z^{-1} + 3z^{-2})(3 + 2z^{-1} + z^{-2}) = 3 + 8z^{-1} + 14z^{-2} + 8z^{-3} + 3z^{-4}.$$

**(b)**  $F_1(z) = 3 + 5z^{-1} - 4z^{-2} - 2z^{-3}$ . Its mirror-image polynomial is given by  $F_2(z) = -2 - 4z^{-1} + 5z^{-2} + 3z^{-3}$ . Therefore,  $H(z) = F_1(z)F_2(z)$

$$= (3 + 5z^{-1} - 4z^{-2} - 2z^{-3})(-2 - 4z^{-1} + 5z^{-2} + 3z^{-3})$$

$$= -6 - 22z^{-1} + 3z^{-2} + 54z^{-3} + 3z^{-4} - 22z^{-5} - 6z^{-6}.$$

**4.83**  $A_1(z) = \frac{1 - d_1^* z}{z - d_1}$ .  $|A_1(z)|^2 = \frac{(1 - d_1^* z)(1 - d_1 z^*)}{(z - d_1)(z^* - d_1^*)}$ . Thus,

$$1 - |A_1(z)|^2 = \frac{(z - d_1)(z^* - d_1^*) - (1 - d_1^* z)(1 - d_1 z^*)}{(z - d_1)(z^* - d_1^*)}$$

$$= \frac{|z|^2 + |d_1|^2 - d_1 z^* - z d_1^* - 1 - |d_1|^2 |z|^2 + d_1 z^* + d_1^* z}{(z - d_1)(z^* - d_1^*)} = \frac{(|z|^2 - 1)(1 - |d_1|^2)}{|z - d_1|^2}.$$

$$\text{Hence, } 1 - |A_1(z)|^2 \begin{cases} > 0, & \text{if } |z| > 1, \\ = 0, & \text{if } |z| = 1, \\ < 0, & \text{if } |z| < 1. \end{cases} \quad \text{Thus, } |A_1(z)|^2 \begin{cases} < 1, & \text{if } |z| > 1, \\ = 1, & \text{if } |z| = 1, \\ > 1, & \text{if } |z| < 1. \end{cases}$$

Thus Eq. (4.132) holds for any first order allpass function. If the allpass is of higher order it can be factored into a product of first order allpass functions. Since Eq. (4.132) holds true for each of these factors individually hence it also holds true for the product.

**4.84** An  $m$ -th order stable, real allpass transfer function  $A(z)$  can be expressed as a product of first-

order allpass transfer functions of the form  $A_i(z) = \frac{1 - d_i^* z}{z - d_i}$ . If  $d_i$  is complex, then  $A(z)$  has

another factor of the form  $A_i'(z) = \frac{1 - d_i z}{z - d_i^*}$ . Now,

$$A_i(e^{j\omega}) = \frac{1 - d_i^* e^{j\omega}}{e^{j\omega} - d_i} = e^{-j\omega} \frac{(1 - d_i^* e^{j\omega})(1 - d_i e^{j\omega})}{(1 - d_i e^{-j\omega})(1 - d_i^* e^{j\omega})}. \quad \text{Let } d_i = |d_i| e^{j\theta} = \alpha e^{j\theta}. \quad \text{Then,}$$

$$A_i(e^{j\omega}) = e^{-j\omega} \frac{(1 - \alpha e^{-j\theta} e^{j\omega})^2}{(1 + \alpha^2 - 2\alpha \cos(\theta - \omega))}. \quad \text{Therefore,}$$

$$\begin{aligned} \arg\{A_i(e^{j\omega})\} &= -\omega + \arg\left\{(1 - \alpha e^{-j\theta} e^{j\omega})^2\right\} = -\omega + 2 \arg\left\{(1 - \alpha e^{-j\theta} e^{j\omega})\right\} \\ &= -\omega + 2 \tan^{-1} \left[ \frac{\alpha \sin(\theta - \omega)}{1 - \alpha \cos(\theta - \omega)} \right]. \end{aligned}$$

$$\text{Similarly, } \arg\{A_i'(e^{j\omega})\} = -\omega + 2 \tan^{-1} \left[ \frac{-\alpha \sin(\theta + \omega)}{1 - \alpha \cos(\theta + \omega)} \right].$$

$$\text{If } d_i = \alpha \text{ is real, then } \arg\{A_i(e^{j\omega})\} = -\omega + 2 \tan^{-1} \left[ \frac{-\alpha \sin \omega}{1 - \alpha \cos \omega} \right].$$

$$\text{Now, for real } d_i, \arg\{A_i(e^{j0})\} - \arg\{A_i(e^{j\pi})\} = -0 + 2 \tan^{-1}(-0) - \{-\pi + 2 \tan^{-1}(-0)\} = \pi.$$

$$\text{For complex } d_i, \arg\{A_i(e^{j0})\} + \arg\{A_i'(e^{j0})\} - \arg\{A_i(e^{j\pi})\} - \arg\{A_i'(e^{j\pi})\}$$

$$\begin{aligned} &= -0 + 2 \tan^{-1} \left[ \frac{\alpha \sin \theta}{1 - \alpha \cos \theta} \right] - 0 + 2 \tan^{-1} \left[ \frac{-\alpha \sin \theta}{1 - \alpha \cos \theta} \right] \\ &\quad + \pi - 2 \tan^{-1} \left[ \frac{-\alpha \sin \theta}{1 + \alpha \cos \theta} \right] + \pi - 2 \tan^{-1} \left[ \frac{\alpha \sin \theta}{1 + \alpha \cos \theta} \right] = 2\pi. \end{aligned}$$

$$\text{Now, } \tau(\omega) = -\frac{d}{d\omega} \left( \arg\{A(e^{j\omega})\} \right).$$

$$\text{Therefore, } \int_0^\pi \tau(\omega) d\omega = -\int_0^\pi d \left[ \arg\{A(e^{j\omega})\} \right] = \arg\{A(e^{j0})\} - \arg\{A(e^{j\pi})\}.$$

Since  $\arg\{A(e^{j\omega})\} = \sum_{i=1}^m \arg\{A_i(e^{j\omega})\}$ , it follows that  $\int_0^\pi \tau(\omega) d\omega = m\pi$ .

**4.85 (a)**  $A_1(z) = \frac{d_1 + z^{-1}}{1 + d_1 z^{-1}}$ . Thus,  $A_1(e^{j\omega}) = \frac{d_1 + e^{-j\omega}}{1 + d_1 e^{-j\omega}} = \frac{d_1 e^{j\omega/2} + e^{-j\omega/2}}{e^{j\omega/2} + d_1 e^{-j\omega/2}} = \frac{\alpha e^{j\beta}}{\alpha e^{-j\beta}} = e^{j2\beta}$ ,

where  $\alpha e^{j\beta} = d_1 e^{j\omega/2} + e^{-j\omega/2} = (d_1 + 1) \cos(\omega/2) + j(d_1 - 1) \sin(\omega/2)$ . Therefore, phase is given by  $\theta(\omega) = 2\beta = -2 \tan^{-1}\left(\frac{1-d_1}{1+d_1} \tan(\omega/2)\right)$ . For small values of  $x$ ,  $\tan(x) \approx x$  and

$\tan^{-1}(x) \approx x$ . Hence, the approximate expression for the phase at low frequencies is given by  $\theta(\omega) \approx -2 \left(\frac{1-d_1}{1+d_1}\right) \frac{\omega}{2} = -\left(\frac{1-d_1}{1+d_1}\right) \omega$ . Therefore, the approximate expression for the phase delay

is given by  $\tau_{ph}(\omega) = -\frac{\theta(\omega)}{\omega} = \delta \approx \frac{1-d_1}{1+d_1}$  samples.

**(b)** For  $\delta = 0.5$  samples,  $d_1 = \frac{1-\delta}{1+\delta} = \frac{0.5}{1.5} = \frac{1}{3}$ . Then,  $A_1(z) = \frac{\frac{1}{3} + z^{-1}}{1 + \frac{1}{3} z^{-1}}$ . Thus, the exact phase

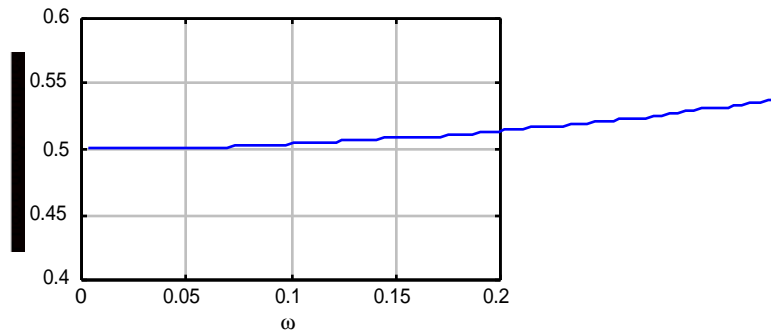
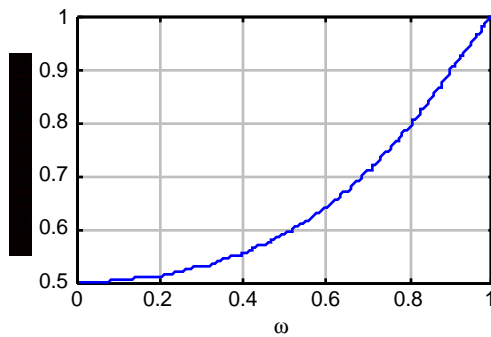
delay is given by  $\tau_{ph}(\omega) = -\frac{\theta(\omega)}{\omega} = \frac{2}{\omega} \left(\frac{1-d_1}{1+d_1} \tan(\omega/2)\right) = \frac{2}{\omega} (0.5 \tan(\omega/2))$ .

For a sampling rate of 20 kHz, the normalized angular frequency equivalent of 1 kHz is

$\omega_o = \frac{10^3}{20 \times 10^3} = \frac{1}{20} = 0.05$ . The exact phase delay at  $\omega_o$  is thus

$\tau_{ph}(\omega_o) = \frac{2}{\omega_o} (0.5 \tan(\omega_o/2)) = \frac{2}{0.05} (0.5 \tan(0.025)) = 0.500078$  samples, which is seen to be

very close to the desired phase delay of 0.5 samples.



$$\begin{aligned}
4.86 \quad A_2(e^{j\omega}) &= \frac{d_2 + d_1 e^{-j\omega} + e^{-j2\omega}}{1 + d_1 e^{-j\omega} + d_2 e^{-j2\omega}} = \frac{d_2 e^{j\omega} + d_1 + e^{-j\omega}}{e^{j\omega} + d_1 + d_2 e^{-j\omega}} \\
&= \frac{(d_1 + d_2 \cos \omega + \cos \omega) + j(d_2 \sin \omega - \sin \omega)}{(d_1 + d_2 \cos \omega + \cos \omega) + j(\sin \omega - d_2 \sin \omega)} = \frac{(d_1 + (d_2 + 1) \cos \omega) + j(d_2 - 1) \sin \omega}{(d_1 + (d_2 + 1) \cos \omega) - j(d_2 - 1) \sin \omega}. \text{ Therefore} \\
\theta(\omega) &= 2 \tan^{-1} \left( \frac{(d_2 - 1) \sin \omega}{d_1 + (d_2 + 1) \cos \omega} \right). \text{ Now, } \tau_p(\omega) = -\frac{\theta(\omega)}{\omega} = -\frac{2}{\omega} \tan^{-1} \left( \frac{(d_2 - 1) \sin \omega}{d_1 + (d_2 + 1) \cos \omega} \right).
\end{aligned}$$

For  $\omega \cong 0$ ,  $\sin \omega = \omega$  and  $\cos \omega = 1$ . Then  $\tau_p(\omega) = -\frac{2}{\omega} \tan^{-1} \left( \frac{(d_2 - 1)\omega}{d_1 + (d_2 + 1)} \right)$ . Again, for  $x \cong 0$ ,  $\tan^{-1} x \cong x$ . Hence,  $\tau_p(\omega) = -\frac{2}{\omega} \frac{(d_2 - 1)\omega}{d_1 + (d_2 + 1)} = -\frac{2(d_2 - 1)}{d_1 + d_2 + 1} = \delta$ .

Now, substituting  $d_1 = 2 \left( \frac{2 - \delta}{1 + \delta} \right)$  and  $d_2 = \frac{(2 - \delta)(1 - \delta)}{(2 + \delta)(1 + \delta)}$  we can show easily that  $-\frac{2(d_2 - 1)}{d_1 + d_2 + 1} = \delta$ .

**4.87** Since  $G(z)$  is non-minimum phase but causal, it will have some zeros outside the unit circle. Let  $z = \alpha$  be one such zero. We can then write  $G(z) = P(z)(1 - \alpha z^{-1}) = P(z)(-\alpha^* + z^{-1}) \left( \frac{1 - \alpha z^{-1}}{-\alpha^* + z^{-1}} \right)$ ,

Note that  $\left( \frac{1 - \alpha z^{-1}}{-\alpha^* + z^{-1}} \right)$  is a stable first order allpass function. If we carry out this operation for all zeros of  $G(z)$  that are outside the unit circle we can write  $G(z) = H(z)A(z)$  where  $H(z)$  will have all zeros inside the unit circle and will thus be a minimum phase function and  $A(z)$  will be a product of first order allpass functions, and hence an allpass function.

**4.88**  $H(z) = \frac{(z+1.4)(z^2+2z+4)}{(z+0.8)(z-0.6)}$ . In order to correct for magnitude distortion we require the

transfer function  $G(z)$  to satisfy the following property  $|G(e^{j\omega})| = \frac{1}{|H(e^{j\omega})|}$ . Hence, one possible

solution is  $G_d(z) = \frac{1}{H(z)} = \frac{(z+0.8)(z-0.6)}{(z+1.4)(z^2+2z+4)}$ . Since  $G_d(z)$  has poles outside the unit circle, it is not stable. Therefore we require a stable transfer function with magnitude response same as  $G_d(z)$ . Using the technique of the previous problem we thus get:

$G(z) = \frac{(z+0.8)(z-0.6)}{(z+1.4)(z^2+2z+4)} \frac{(z^2+2z+4)}{(4z^2+2z+1)} \frac{(z+1.4)}{(1.4z+1)} = \frac{(z+0.8)(z-0.6)}{(4z^2+2z+1)(1.4z+1)}$  is the desired stable solution such that  $|G(e^{j\omega})||H(e^{j\omega})| = 1$ .

**4.89 (a)**  $G(z) = H(z)A(z)$  where  $A(z)$  is an allpass function. Then,  $g[0] = \lim_{z \rightarrow \infty} G(z)$ .

$$|g[0]| = \left| \lim_{z \rightarrow \infty} G(z) \right| = \left| \lim_{z \rightarrow \infty} H(z)A(z) \right| = \left| \lim_{z \rightarrow \infty} H(z) \right| \left| \lim_{z \rightarrow \infty} A(z) \right|$$

$$\leq \left| \lim_{z \rightarrow \infty} H(z) \right| \quad \text{because} \quad \left| \lim_{z \rightarrow \infty} A(z) \right| < 1 \quad (\text{see Problem 4.71})$$

$$\leq |h[0]|$$

(b) If  $\lambda_1$  is a zero of  $H(z)$ , then  $|\lambda_1| < 1$ , since  $H(z)$  is a minimum-phase causal stable transfer function which has all zeros inside the unit circle. We can express  $H(z) = B(z)(1 - \lambda_1 z^{-1})$ . It follows that  $B(z)$  is also a minimum-phase causal transfer function.

Now consider the transfer function  $F(z) = B(z)(\lambda_1^* - z^{-1}) = H(z) \frac{(\lambda_1^* - z^{-1})}{(1 - \lambda_1 z^{-1})}$ . If  $h[n]$ ,  $b[n]$ , and  $f[n]$  denote, respectively, the inverse  $z$ -transforms of  $H(z)$ ,  $B(z)$ , and  $F(z)$ , then we get

$$h[n] = \begin{cases} b[0], & n = 0 \\ b[n] - \lambda_1 b[n-1], & n \geq 1, \end{cases}$$

and

$$f[n] = \begin{cases} \lambda_1^* b[0], & n = 0, \\ \lambda_1^* b[n] - b[n-1], & n \geq 1. \end{cases}$$

$$\text{Consider } \varepsilon = \sum_{n=0}^m |h[n]|^2 - \sum_{n=0}^m |f[n]|^2 = |b[0]|^2 - |\lambda_1^*|^2 |b[0]|^2 + \sum_{n=1}^m |h[n]|^2 - \sum_{n=1}^m |f[n]|^2.$$

Now  $|h[n]|^2 = |b[n]|^2 + |\lambda_1|^2 |b[n-1]|^2 - \lambda_1 b[n-1] b^*[n] - \lambda_1^* b^*[n-1] b[n]$ , and

$$|f[n]|^2 = |\lambda_1|^2 |b[n]|^2 + |b[n-1]|^2 - \lambda_1 b[n-1] b^*[n] - \lambda_1^* b^*[n-1] b[n],$$

$$\text{Hence, } \varepsilon = |b[0]|^2 - |\lambda_1|^2 |b[0]|^2 + \sum_{n=1}^m \left( |b[n]|^2 + |\lambda_1|^2 |b[n-1]|^2 \right) - \sum_{n=1}^m \left( |\lambda_1|^2 |b[n]|^2 - |b[n-1]|^2 \right)$$

$$= \left( 1 - |\lambda_1|^2 \right) |b[m]|^2.$$

Since  $|\lambda_1| < 1$ ,  $\varepsilon > 0$ , i.e.,  $\sum_{n=0}^m |h[n]|^2 > \sum_{n=0}^m |f[n]|^2$ . Hence,  $\sum_{n=0}^m |h[n]|^2 \geq \sum_{n=0}^m |g[n]|^2$ .

**4.90**  $H(z) = \frac{(z+3)(z-2)}{(z-0.25)(z+0.5)} = G(z)A(z) = \frac{(3z+1)(1-2z)}{(z-0.25)(z+0.5)} \frac{(z+3)(z-2)}{(3z+1)(1-2z)}$

Therefore  $G(z) = \frac{(3z+1)(1-2z)}{(z-0.25)(z+0.5)}$ . The inverse  $z$ -transforms of these causal transfer

functions are given by  $h[n] = \{1, 0.75, -6.0625, 1.6093, -1.16015, \mathbf{L}\}$ , and

$g[n] = \{-6, 2.5, -0.375, 0.40625, -0.1484, 0.0879, \mathbf{L}\}$ , respectively. It can be

seen that  $\sum_{n=0}^m |g[n]|^2$  is bigger than  $\sum_{n=0}^m |h[n]|^2$  for all values of  $m$ .

**4.91** See Example 4.13.

(a)  $H_{BS}(z) = \frac{1}{4}(1+z^{-2})^2$ . Thus,  $H_{BP}(z) = z^{-2} - \frac{1}{4}(1+z^{-2})^2 = -\frac{1}{4}(1-z^{-2})^2$ .

(b)  $H_{BS}(z) = \frac{1}{16}(1+z^{-2})(-1+6z^{-2}-z^{-4})$ . Thus,  $H_{BP}(z) = z^{-4} - \frac{1}{16}(1+z^{-2})(-1+6z^{-2}-z^{-4})$

$$= \frac{1}{16} \{1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8}\} = \frac{1}{16} (1 - z^{-2})^4.$$

(c)  $H_{BS}(z) = \frac{1}{32} (1 + z^{-2})^2 (-3 + 14z^{-2} - 3z^{-4})$ . Thus,

$$\begin{aligned} H_{BP}(z) &= z^{-4} - \frac{1}{32} (1 + z^{-2})^2 (-3 + 14z^{-2} - 3z^{-4}) = \frac{1}{32} \{3 - 8z^{-2} + 10z^{-4} - 8z^{-6} + 3z^{-8}\} \\ &= \frac{1}{32} \{3 - 8z^{-2} + 10z^{-4} - 8z^{-6} + 3z^{-8}\}. \end{aligned}$$

**4.92**  $H_0(z) = A_0(z) + A_1(z)$ , and  $H_1(z) = A_0(z) - A_1(z)$ , where  $A_0(z)$  and  $A_1(z)$  are allpass functions of orders  $M$  and  $N$ , respectively, with no common factors. Hence, the orders of  $H_0(z)$

and  $H_1(z)$  are  $M+N$ . Now, we can write  $A_0(z) = \frac{z^{-M}D_0(z^{-1})}{D_0(z)}$ , and  $A_1(z) = \frac{z^{-N}D_1(z^{-1})}{D_1(z)}$ .

Then,  $H_0(z) = \frac{P(z)}{D(z)} = \frac{z^{-M}D_0(z^{-1})D_1(z) + z^{-N}D_0(z)D_1(z^{-1})}{D_0(z)D_1(z)}$ , and

$$H_1(z) = \frac{Q(z)}{D(z)} = \frac{z^{-M}D_0(z^{-1})D_1(z) - z^{-N}D_0(z)D_1(z^{-1})}{D_0(z)D_1(z)}.$$

Since  $P(z)$  is of degree  $M+N$  and  $z^{-(M+N)}P(z^{-1}) = z^{-(M+N)}(z^M D_0(z)D_1(z^{-1}) + z^N D_0(z^{-1})D_1(z))$

$= z^{-N}D_0(z)D_1(z^{-1}) + z^{-M}D_0(z^{-1})D_1(z) = P(z)$ . Hence  $P(z)$  is symmetric. Similarly one can prove that  $Q(z)$  is anti-symmetric.

**4.93**  $H_0(z) = \frac{1}{2}[A_0(z) + A_1(z)]$ ,  $H_1(z) = \frac{1}{2}[A_0(z) - A_1(z)]$ . Thus,  $H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1})$

$$\begin{aligned} &= \frac{1}{4}[A_0(z) + A_1(z)][A_0(z^{-1}) + A_1(z^{-1})] + \frac{1}{4}[A_0(z) - A_1(z)][A_0(z^{-1}) - A_1(z^{-1})] \\ &= \frac{1}{4}[A_0(z)A_0(z^{-1}) + A_0(z)A_1(z^{-1}) + A_1(z)A_0(z^{-1}) + A_1(z)A_1(z^{-1})] \\ &\quad + \frac{1}{4}[A_0(z)A_0(z^{-1}) - A_0(z)A_1(z^{-1}) - A_1(z)A_0(z^{-1}) + A_1(z)A_1(z^{-1})] \\ &= \frac{1}{2}[A_0(z)A_0(z^{-1}) + A_1(z)A_1(z^{-1})] = 1. \end{aligned}$$

Thus,  $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1$  implying that

$H_0(z)$  and  $H_1(z)$  form a power complementary pair.

**4.94**  $|H_0(e^{j\omega})|^2 = \frac{1}{4} \{A_0(e^{j\omega})A_0^*(e^{j\omega}) + A_1(e^{j\omega})A_0^*(e^{j\omega}) + A_0(e^{j\omega})A_1^*(e^{j\omega}) + A_1(e^{j\omega})A_1^*(e^{j\omega})\}$ .

Since  $A_0(z)$  and  $A_1(z)$  are allpass functions,  $A_0(e^{j\omega}) = e^{j\phi_0(\omega)}$  and  $A_1(e^{j\omega}) = e^{j\phi_1(\omega)}$ .

Therefore  $|H_0(e^{j\omega})|^2 = \frac{1}{4} \{2 + e^{j(\phi_0(\omega) - \phi_1(\omega))} + e^{-j(\phi_0(\omega) - \phi_1(\omega))}\} \leq 1$  as maximum values of

$e^{j(\phi_0(\omega) - \phi_1(\omega))}$  and  $e^{-j(\phi_0(\omega) - \phi_1(\omega))}$  are 1.  $H_0(z)$  is stable since  $A_0(z)$  and  $A_1(z)$  are stable transfer functions. Hence,  $H_0(z)$  is BR.



**4.95**  $H(z) = \frac{1}{M} \sum_{k=0}^{M-1} A_k(z)$ . Thus,  $H(z)H(z^{-1}) = \frac{1}{M^2} \sum_{r=0}^{M-1} \sum_{k=0}^{M-1} A_k(z)A_r(z^{-1})$ . Hence,

$$\left|H(e^{j\omega})\right|^2 = \frac{1}{M^2} \sum_{r=0}^{M-1} \sum_{k=0}^{M-1} e^{j(\phi_k(\omega) - \phi_r(\omega))} \leq 1. \text{ Again } H(z) \text{ is stable since } \{A_k(z)\} \text{ are stable}$$

transfer functions. Hence,  $H(z)$  is BR.

**4.96**  $H_{BP}(z) = \frac{1-\alpha}{2} \left( \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \right)$  and  $H_{BS}(z) = \frac{1-\alpha}{2} \frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$ .

$$H_{BS}(z) + H_{BP}(z) = \frac{1-\alpha-z^{-2}+\alpha z^{-2}+1-2\beta z^{-1}+z^{-2}+\alpha-2\alpha\beta z^{-1}+\alpha z^{-2}}{2(1-\beta(1+\alpha)z^{-1}+\alpha z^{-2})} = 1.$$

Hence  $H_{BP}(z)$  and  $H_{BS}(z)$  are allpass complementary.

**4.97**  $H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = K$ . On the unit circle, this reduces to

$$H(e^{j\omega})H(e^{-j\omega}) + H(-e^{j\omega})H(-e^{-j\omega}) = K, \text{ or equivalently, } \left|H(e^{j\omega})\right|^2 + \left|H(-e^{j\omega})\right|^2 = K, \text{ as } H(z) \text{ is a}$$

real-coefficient transfer function. Now,  $\left|H(-e^{j\omega})\right|^2 = \left|H(e^{j(\pi+\omega)})\right|^2$ . Hence, for  $\omega = \pi/2$ , the

power-symmetric condition reduces to

$$\left|H(e^{j\pi/2})\right|^2 + \left|H(e^{j(\pi+\pi/2)})\right|^2 = \left|H(e^{j\pi/2})\right|^2 + \left|H(e^{j(2\pi-\pi/2)})\right|^2 = K. \text{ Since } H(z) \text{ is a real-coefficient}$$

transfer function,  $\left|H(e^{j\omega})\right|^2$  is an even function of  $\omega$ , and thus,  $\left|H(e^{j\pi/2})\right|^2 = \left|H(e^{j(2\pi-\pi/2)})\right|^2$ .

As a result,  $2\left|H(e^{j\pi/2})\right|^2 = K$ , from which we obtain  $10 \log_{10} 2 + 20 \log_{10} \left|H(e^{j\pi/2})\right| = 10 \log_{10} K$ ,

or  $20 \log_{10} \left|H(e^{j\pi/2})\right| = G(\omega) \Big|_{\omega=\pi/2} = 10 \log_{10} K - 3$ .

**4.98**  $H(z) = A_0(z^2) + z^{-1}A_1(z^2)$ . Therefore,

$$\begin{aligned} H(z)H(z^{-1}) + H(-z)H(-z^{-1}) &= \left[ A_0(z^2) + z^{-1}A_1(z^2) \right] \left[ A_0(z^{-2}) + zA_1(z^{-2}) \right] \\ &\quad + \left[ A_0((-z)^2) - z^{-1}A_1((-z)^2) \right] \left[ A_0((-z)^{-2}) - zA_1((-z)^{-2}) \right] \\ &= A_0(z^2)A_0(z^{-2}) + z^{-1}A_1(z^2)A_0(z^{-2}) + zA_0(z^2)A_1(z^{-2}) + A_1(z^2)A_1(z^{-2}) \\ &\quad + A_0(z^2)A_0(z^{-2}) - z^{-1}A_1(z^2)A_0(z^{-2}) - zA_0(z^2)A_1(z^{-2}) + A_1(z^2)A_1(z^{-2}) = 4, \text{ as} \\ A_0(z^2)A_0(z^{-2}) &= A_1(z^2)A_1(z^{-2}) = 1. \end{aligned}$$

**4.99 (a)**  $H_a(z) = \frac{1}{2} - z^{-1} + \frac{21}{2}z^{-2} - \frac{27}{2}z^{-3} - 5z^{-4} - \frac{5}{2}z^{-5}$ . Thus,

$$H_a(z)H_a(z^{-1}) = -1.25z^5 - 28z^3 - 72,75z + 325 - 72,75z^{-1} - 28z^{-3} - 1.25z^{-5} \text{ and}$$

$$H_a(-z)H_a(-z^{-1}) = 1.25z^5 + 28z^3 + 72,75z + 325 + 72,75z^{-1} + 28z^{-3} + 1.25z^{-5}. \text{ Hence,}$$

$$H_a(z)H_a(z^{-1}) + H_a(-z)H_a(-z^{-1}) = 650.$$

(b)  $H_b(z) = 1 + 3z^{-1} + 14z^{-2} + 22z^{-3} - 12z^{-4} + 4z^{-5}$ . Thus,  
 $H_b(z)H_b(z^{-1}) = 4z^5 + 42z^3 + 41z + 850 + 41z^{-1} + 42z^{-3} + 4z^{-5}$  and  
 $H_b(-z)H_b(-z^{-1}) = -4z^5 - 42z^3 - 41z + 850 - 41z^{-1} - 42z^{-3} - 4z^{-5}$ . Hence,  
 $H_b(z)H_b(z^{-1}) + H_b(-z)H_b(-z^{-1}) = 1700$ .

**4.100**  $H(z)H(z^{-1}) = a^2(1 + bz^{-1})(1 + bz) = a^2bz + a^2(1 + b^2) + a^2bz^{-1}$ . Thus,  $c = a^2b$  and  
 $d = a^2(1 + b^2)$ . Now,  $H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = cz + d + cz^{-1} - cz + d - cz^{-1} = 2d$ . Therefore,  
 $2d = 2a^2(1 + b^2) = 1$ . This condition is satisfied by  $a = 1/\sqrt{2(1 + b^2)}$ . For  $b = 1$ , then  $a = \frac{1}{2}$ .

Other solutions include  $b = -1$  and  $a = \frac{1}{2}$ ,  $b = 2$  and  $a = \frac{1}{\sqrt{10}}$ .

Since  $H(z)$  is a first-order FIR causal FIR transfer function,  $G(z) = -z^{-1}H(-z^{-1})$  is also a first-order FIR causal FIR transfer function. Now,

$$\begin{aligned} H(z)H(z^{-1}) + G(z)G(z^{-1}) &= H(z)H(z^{-1}) + [-z^{-1}H(-z^{-1})][-zH(-z)] \\ &= H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1. \end{aligned}$$

Hence  $H(z)$  and  $G(z)$  are power-complementary.

**4.101**  $H(z)H(z^{-1}) = (cz + d + cz^{-1})[d_2z^2 + d_1(1 + d_2)z + (1 + d_1^2 + d_2^2) + d_1(1 + d_2)z^{-1} + d_2z^{-2}]$ .

Thus,  $H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2[cd_1(1 + d_2)z^2 + 2cd_1(1 + d_2) + dd_2z^2 + d(1 + d_1^2 + d_2^2) + dd_2z^{-2} + cd_1(1 + d_2)z^{-2}] = 1$ . Hence, we require

$dd_2 + cd_1(1 + d_2) = 0$ , and  $2cd_1(1 + d_2) + d(1 + d_1^2 + d_2^2) = 1$ . Solving these two equations we arrive at  $c = \frac{d_2}{d_2(1 + d_2)(2d_2 - 1 - d_1^2 - d_2^2)}$  and  $d = -\frac{1}{2d_2 - 1 - d_1^2 - d_2^2}$ . For  $d_1 = d_2 = 1$ , we get  $c = -\frac{1}{2}$  and  $d = 1$ .

Since  $H(z)$  is a third-order FIR causal FIR transfer function,  $G(z) = -z^{-3}H(-z^{-1})$  is also a third-order FIR causal FIR transfer function. Now,

$$\begin{aligned} H(z)H(z^{-1}) + G(z)G(z^{-1}) &= H(z)H(z^{-1}) + [-z^{-3}H(-z^{-1})][-z^3H(-z)] \\ &= H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1. \end{aligned}$$

Hence  $H(z)$  and  $G(z)$  are power-complementary.

**4.102**  $H(z) = \frac{0.1 + 0.5z^{-1} + 0.45z^{-2} + 0.45z^{-3} + 0.5z^{-4} + 0.1z^{-5}}{1 + 0.9z^{-2} + 0.2z^{-4}} = \frac{1}{2}[A(z^2) + z^{-1}]$ , where

$A(z) = \frac{0.2 + 0.9z^{-1} + z^{-2}}{1 + 0.9z^{-1} + 0.2z^{-2}}$  is a stable allpass function. Thus,

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = \frac{1}{4}[A(z^2) + z^{-1}][A(z^{-2}) + z] + \frac{1}{4}[A(z^2) - z^{-1}][A(z^{-2}) - z] = 1.$$

**4.103 (a)**  $H_1(z) = \frac{1}{1 + \alpha} (1 + \alpha z^{-1})$ .

$$\left|H_1(e^{j\omega})\right|^2 = \frac{1}{(1+\alpha)^2} \left\{ (1+\alpha\cos\omega)^2 + (\alpha\sin\omega)^2 \right\} = \frac{1+\alpha^2+2\alpha\cos\omega}{(1+\alpha)^2}. \text{ Thus,}$$

$$\frac{d\left|H_1(e^{j\omega})\right|^2}{d\omega} = \frac{-2\alpha\sin\omega}{(1+\alpha)^2} < 0, \text{ for } \alpha > 0. \text{ The maximum value of } \left|H_1(e^{j\omega})\right| = 1 \text{ at } \omega = 0, \text{ and the}$$

minimum value is at  $\omega = \pi$ . On the other hand, if  $\alpha < 0$ , then  $\frac{d\left|H_1(e^{j\omega})\right|^2}{d\omega} > 0$ . In this case the

maximum value of  $\left|H_1(e^{j\omega})\right| = (1-\alpha)^2 / (1+\alpha)^2 > 1$  at  $\omega = \pi$ , and the minimum value is at  $\omega = 0$ . Hence,  $H_1(z)$  is BR only for  $\alpha > 0$ .

$$\text{(b) } H_2(z) = \frac{1}{1+\beta}(1-\beta z^{-1}). \quad \left|H_2(e^{j\omega})\right|^2 = \frac{1+\beta^2-2\beta\cos\omega}{(1+\beta)^2}. \text{ Thus,}$$

$$\frac{d\left|H_2(e^{j\omega})\right|^2}{d\omega} = \frac{2\beta\sin\omega}{(1+\beta)^2} > 0, \text{ for } \beta > 0. \text{ The maximum value of } \left|H_2(e^{j\omega})\right| = 1 \text{ at } \omega = \pi, \text{ and the}$$

minimum value is at  $\omega = 0$ . On the other hand, if  $\beta < 0$ , then  $\frac{d\left|H_2(e^{j\omega})\right|^2}{d\omega} < 0$ . In this case the

maximum value of  $\left|H_2(e^{j\omega})\right| = (1-\beta)^2 / (1+\beta)^2 > 1$  at  $\omega = 0$ , and the minimum value is at  $\omega = \pi$ . Hence,  $H_2(z)$  is BR only for  $\beta > 0$ .

**(c)**  $H_3(z) = \frac{(1+\alpha z^{-1})(1-\beta z^{-1})}{(1+\alpha)(1+\beta)}$ . From the results of Parts (a) and (b) it follows that  $H_3(z)$  is BR only for  $\alpha > 0$  and  $\beta > 0$ .

**(d)**  $H_4(z) = \frac{(1+0.4z^{-1})(1+0.5z^{-1})(1+0.6z^{-1})}{3.36} = \left( \frac{1+0.4z^{-1}}{1.4} \right) \left( \frac{1+0.5z^{-1}}{1.5} \right) \left( \frac{1+0.6z^{-1}}{1.6} \right)$ . Since each individual factors on the right-hand side is BR,  $H_4(z)$  is BR.

**4.104 (a)**  $H_1(z) = \frac{2+2z^{-1}}{3+z^{-1}} = \frac{1}{2} \left( 1 + \frac{1+3z^{-1}}{3+z^{-1}} \right) = \frac{1}{2} (A_0(z) + A_1(z))$ , where  $A_0(z) = 1$  and

$A_1(z) = \frac{1+3z^{-1}}{3+z^{-1}}$  are stable allpass functions. In view of Problem 4.80,  $H_1(z)$  is BR.

**(b)**  $H_2(z) = \frac{1-z^{-1}}{4+2z^{-1}} = \frac{1}{2} \left( 1 - \frac{2+4z^{-1}}{4+2z^{-1}} \right) = \frac{1}{2} (A_0(z) - A_1(z))$ , where  $A_0(z) = 1$  and

$A_1(z) = \frac{2+4z^{-1}}{4+2z^{-1}}$  are stable allpass functions. In view of Problem 4.79,  $H_1(z)$  is BR.

(c)  $H_3(z) = \frac{1-z^{-2}}{4+2z^{-1}+2z^{-2}} = \frac{1}{2} \left( 1 - \frac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}} \right)$ , where  $A_1(z) = \frac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}}$  is a stable allpass function. Hence,  $H_3(z)$  is BR.

(d)  $H_4(z) = \frac{3+6z^{-1}+3z^{-2}}{6+5z^{-1}+z^{-2}} = \left[ \frac{1}{2} \left( 1 + \frac{1+3z^{-1}}{3+z^{-1}} \right) \right] \left[ \frac{1}{2} \left( 1 + \frac{1+2z^{-1}}{2+z^{-1}} \right) \right]$  which is seen to be a product of two BR functions. Hence,  $H_4(z)$  is BR.

(e)  $H_5(z) = \frac{3+2z^{-1}+3z^{-2}}{4+2z^{-1}+2z^{-2}} = \frac{1}{2} \left( 1 + \frac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}} \right) = \frac{1}{2} (1 + A_1(z))$  where  $A_1(z) = \frac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}}$  is a stable allpass function. Hence,  $H_5(z)$  is BR.

(f)  $H_6(z) = \frac{3+9z^{-1}+9z^{-2}+3z^{-3}}{12+10z^{-1}+2z^{-2}} = \left[ \frac{1}{2} \left( 1 + \frac{1+3z^{-1}}{3+z^{-1}} \right) \right] \left[ \frac{1}{2} \left( 1 + \frac{1+2z^{-1}}{2+z^{-1}} \right) \right] \left[ \frac{1+z^{-1}}{2} \right]$  which is seen to be a product of three BR functions. Hence,  $H_6(z)$  is BR.

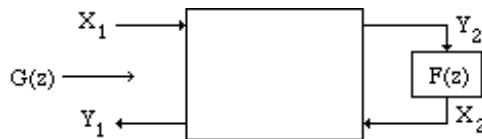
**4.105** Since  $A_1(z)$  and  $A_2(z)$  are LBR,  $|A_1(e^{j\omega})| = 1$  and  $|A_2(e^{j\omega})| = 1$ . Thus,  $A_1(e^{j\omega}) = e^{j\phi_1(\omega)}$ , and  $A_2(e^{j\omega}) = e^{j\phi_2(\omega)}$ . Now,  $A_1\left(\frac{1}{A_2(e^{j\omega})}\right) = A_1(e^{-j\phi_2(\omega)})$ . Thus,  $|A_1(e^{-j\phi_2(\omega)})| = 1$ . Hence,  $A_1\left(\frac{1}{A_2(z)}\right)$  is LBR.

**4.106**  $F(z) = z \left( \frac{G(z) + \alpha}{1 + \alpha G(z)} \right)$ . Thus,  $F(e^{j\omega}) = e^{j\omega} \left( \frac{G(e^{j\omega}) + \alpha}{1 + \alpha G(e^{j\omega})} \right) = e^{j\omega} \left( \frac{e^{j\phi(\omega)} + \alpha}{1 + \alpha e^{j\phi(\omega)}} \right)$  since  $G(z)$  is LBR.

$$|F(e^{j\omega})|^2 = \left| \frac{e^{j\phi(\omega)} + \alpha}{1 + \alpha e^{j\phi(\omega)}} \right|^2 = \frac{(\cos\phi(\omega) + \alpha)^2 + (\sin\phi(\omega))^2}{(1 + \alpha \cos\phi(\omega))^2 + (\alpha \sin\phi(\omega))^2} = \frac{1 + 2\alpha \cos\phi(\omega) + \alpha^2}{1 + 2\alpha \cos\phi(\omega) + \alpha^2} = 1.$$

Let  $z = 1$  be a pole of  $F(z)$ . This implies,  $G(1) = -1/\alpha$  or  $|G(\lambda)| = |\lambda/\alpha|$ . If  $|\alpha| < 1$ , then  $|G(\lambda)| > 1$ , which is satisfied by the LBR  $G(z)$  if  $|\lambda| < 1$ . Hence,  $F(z)$  is LBR. The order of  $F(z)$  is same as that of  $G(z)$ .

$G(z)$  can be realized in the form of a two-pair constrained by the transfer function  $F(z)$ .



To this end, we express  $G(z)$  in terms of  $F(z)$  arriving at  $G(z) = \frac{-\alpha + z^{-1}F(z)}{1 - \alpha z^{-1}F(z)} = \frac{C + DF(z)}{A + BF(z)}$ ,

where  $A, B, C,$  and  $D,$  are the chain parameters of the two-pair. Comparing the two expressions on the right-hand side we get  $A = 1, B = -\alpha z^{-1}, C = -\alpha,$  and  $D = z^{-1}$ . The corresponding transfer parameters are given by  $t_{11} = -\alpha, t_{21} = 1, t_{12} = (1 - \alpha^2)z^{-1},$  and  $t_{22} = \alpha z^{-1}$ .

**4.107** Let  $F(z) = G\left(\frac{1}{A(z)}\right)$ . Now  $A(z)$  being LBR,  $A(e^{j\omega}) = e^{j\phi(\omega)}$ . Thus,

$$F(e^{j\omega}) = G\left(\frac{1}{A(e^{j\omega})}\right) = G(e^{-j\phi(\omega)}). \text{ Since } G(z) \text{ is a BR function, } |G(e^{-j\phi(\omega)})| \leq 1. \text{ Hence,}$$

$$|F(e^{j\omega})| = \left|G\left(\frac{1}{A(e^{j\omega})}\right)\right| \leq 1.$$

Let  $z = \xi$  be a pole of  $F(z)$ . Hence,  $F(z)$  will be a BR function if  $|\xi| < 1$ . Let  $z = \lambda$  be a pole of  $G(z)$ . Then this pole is mapped to the location  $z = \xi$  of  $F(z)$  by the relation

$$\frac{1}{A(z)}\Big|_{z=\xi} = \lambda, \text{ or } A(\xi) = \frac{1}{\lambda}. \text{ Hence, } |A(\xi)| = \frac{1}{|\lambda|} > 1 \text{ because of Eq. (4.129). This implies,}$$

$$|\lambda| < 1. \text{ Thus } G\left(\frac{1}{A(z)}\right) \text{ is a BR function.}$$

**4.108 (a)**  $H(z) = \frac{2+2z^{-1}}{3+z^{-1}}, G(z) = \frac{1-z^{-1}}{3+z^{-1}}$ . Now,  $H(z) + G(z) = \frac{2+2z^{-1}}{3+z^{-1}} + \frac{1-z^{-1}}{3+z^{-1}} = \frac{3+z^{-1}}{3+z^{-1}} = 1$ .

$$\text{Next, } H(z)H(z^{-1}) + G(z)G(z^{-1}) = \frac{2+2z^{-1}}{3+z^{-1}} \frac{2+2z}{3+z} + \frac{1-z^{-1}}{3+z^{-1}} \frac{1+z}{3+z} =$$

$$\frac{4+4z^{-1}+4z+4+1+1-z^{-1}-z}{(3+z^{-1})(3+z)} = 1. \text{ Thus, } |H(e^{j\omega})|^2 + |G(e^{j\omega})|^2 = 1$$

Hence  $H(z)$  and  $G(z)$  are both allpass-complementary and power complementary. As a result, they are doubly complementary.

**(b)**  $H(z) = \frac{-1+z^{-2}}{4+2z^{-1}+2z^{-2}}, G(z) = \frac{3+2z^{-1}+3z^{-2}}{4+2z^{-1}+2z^{-2}}$ . Note  $H(z) + G(z) = \frac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}}$

$$\text{implying that } H(z) \text{ and } G(z) \text{ are allpass complementary. Next, } H(z)H(z^{-1}) + G(z)G(z^{-1})$$

$$= \frac{-1+z^{-2}}{4+2z^{-1}+2z^{-2}} \frac{-1+z^2}{4+2z+2z^2} + \frac{3+2z^{-1}+3z^{-2}}{4+2z^{-1}+2z^{-2}} \frac{3+2z+3z^2}{4+2z+2z^2}$$

$$= \frac{1+1-z^{-2}-z^2+9+6z+9z^2+6z^{-1}+4+6z+9z^{-2}+6z^{-1}+9}{(4+2z^{-1}+2z^{-2})(4+2z+2z^2)} = 1. \text{ Hence, } H(z)$$

and  $G(z)$  are also power complementary. As a result, they are doubly-complementary.

**4.109 (a)**  $H_a(z) = \frac{2(1+z^{-1}+z^{-2})}{3+2z^{-1}+z^{-2}} = \frac{1}{2} \left( 1 + \frac{1+2z^{-1}+3z^{-2}}{3+2z^{-1}+z^{-2}} \right)$ . Its power-complementary transfer

function therefore is given by  $G_a(z) = \frac{1}{2} \left( 1 - \frac{1+2z^{-1}+3z^{-2}}{3+2z^{-1}+z^{-2}} \right) = \frac{1-z^{-2}}{3+2z^{-1}+z^{-2}}$ .

**(b)**  $H_b(z) = \frac{3(1.5+6.5z^{-1}+6.5z^{-2}+1.5z^{-3})}{18+21z^{-1}+8z^{-2}+z^{-3}} = \frac{3(1.5+6.5z^{-1}+6.5z^{-2}+1.5z^{-3})}{(2+z^{-1})(3+z^{-1})(3+z^{-1})}$   
 $= \frac{1}{2} \left[ \frac{(1+2z^{-1})(1+3z^{-1})}{(2+z^{-1})(3+z^{-1})} + \frac{1+3z^{-1}}{3+z^{-1}} \right]$ . Its power-complementary transfer

function therefore is given by

$$G_b(z) = \frac{1}{2} \left[ \frac{(1+2z^{-1})(1+3z^{-1})}{(2+z^{-1})(3+z^{-1})} - \frac{1+3z^{-1}}{3+z^{-1}} \right] = \frac{-1.5-3.5z^{-1}+3.5z^{-2}+1.5z^{-3}}{18+21z^{-1}+8z^{-2}+z^{-3}}.$$

**4.110**  $\left| G_0(e^{j\omega}) \right|^2 + \left| G_1(e^{j\omega}) \right|^2 = \frac{1}{4} \left\{ \left| A_0(e^{j\omega}) + A_1(e^{j\omega}) \right|^2 + \left| A_0(e^{j\omega}) - A_1(e^{j\omega}) \right|^2 \right\}$   
 $= \frac{1}{4} \left\{ (A_0(e^{j\omega}) + A_1(e^{j\omega})) (A_0^*(e^{j\omega}) + A_1^*(e^{j\omega})) + (A_0(e^{j\omega}) - A_1(e^{j\omega})) (A_0^*(e^{j\omega}) - A_1^*(e^{j\omega})) \right\}$   
 $= \frac{1}{4} \{ A_0(e^{j\omega})A_0^*(e^{j\omega}) + A_1(e^{j\omega})A_1^*(e^{j\omega}) + A_0(e^{j\omega})A_1^*(e^{j\omega}) + A_1(e^{j\omega})A_0^*(e^{j\omega})$   
 $\quad + A_0(e^{j\omega})A_0^*(e^{j\omega}) + A_1(e^{j\omega})A_1^*(e^{j\omega}) - A_0(e^{j\omega})A_1^*(e^{j\omega}) - A_1(e^{j\omega})A_0^*(e^{j\omega}) \}$   
 $= \frac{1}{4} \left\{ 2|A_0(e^{j\omega})|^2 + 2|A_1(e^{j\omega})|^2 \right\} = 1.$

**4.111** Let the output of the predictor in Figure P4.11(a) be denoted by  $E(z)$ . Then analyzing this figure we get  $E(z) = P(z)[U(z) + E(z)]$  and  $U(z) = X(z) - E(z)$ . From the first equation we have  $E(z) = \frac{P(z)}{1-P(z)}U(z)$  which when substituted in the second equation yields

$$H(z) = \frac{U(z)}{X(z)} = 1 - P(z).$$

Analyzing Figure P4.11(b) we arrive at  $Y(z) = V(z) + P(z)Y(z)$  which yields

$$G(z) = \frac{Y(z)}{V(z)} = \frac{1}{1-P(z)}, \text{ which is seen to be the inverse of } H(z).$$

For  $P(z) = h_1z^{-1}$ ,  $H(z) = 1 - h_1z^{-1}$  and  $G(z) = \frac{1}{1-h_1z^{-1}}$ . Similarly, for  $P(z) = h_1z^{-1} + h_2z^{-2}$ ,

$$H(z) = 1 - h_1z^{-1} - h_2z^{-2} \text{ and } G(z) = \frac{1}{1-h_1z^{-1}-h_2z^{-2}}.$$

**4.112** The z-transform of  $h_1[n]$  is given by  $H_1(z) = \frac{5/3}{1-0.5z^{-1}} + \frac{1/12}{1+0.2z^{-1}} - 1$ . Using the M-file

residuez we arrive at  $H_1(z) = \frac{0.75 + 0.5917z^{-1} + 0.1z^{-2}}{1 - 0.3z^{-1} - 0.1z^{-2}} = \frac{0.75(1 + 0.5436z^{-1})(1 + 0.2453z^{-1})}{(1 - 0.5z^{-1})(1 + 0.2z^{-1})}$ .

Since the zeros of the numerator of  $H_1(z)$  are inside the unit circle, the inverse system is causal and stable. Hence, the transfer function of the inverse system is given by

$$H_2(z) = \frac{1 - 0.3z^{-1} - 0.1z^{-2}}{0.75 + 0.5917z^{-1} + 0.1z^{-2}} = \frac{1.3333 - 0.4z^{-1} - 0.1333z^{-2}}{1 + 0.7889z^{-1} + 0.1333z^{-2}}$$

A partial-fraction expansion of

$$H_2(z) \text{ using the M-file residuez yields } H_2(z) = \frac{2.9474}{1 + 0.5437z^{-1}} - \frac{0.6141}{1 + 0.2452z^{-1}} - 1$$

whose inverse z-transform is given by  $H_2(z) = 2.9474(-0.5437)^n \mu[n] - 0.6141(-0.2452)^n \mu[n] - \delta[n]$ .

**4.113**  $X_1 = A Y_2 + B X_2$ ,  $Y_1 = C Y_2 + D X_2$ . From the first equation,  $Y_2 = \frac{1}{A} X_1 - \frac{B}{A} X_2$ .

Substituting this in the second equation we get

$$Y_1 = C \left( \frac{1}{A} X_1 - \frac{B}{A} X_2 \right) + D X_2 = \frac{C}{A} X_1 + \frac{AD - BC}{A} X_2, \quad \text{Comparing the last two equations with}$$

$$\text{Eq. (4.146) we arrive at } t_{11} = \frac{C}{A}, \quad t_{12} = \frac{AD - BC}{A}, \quad t_{21} = \frac{1}{A}, \quad t_{22} = -\frac{B}{A}.$$

Next,  $Y_1 = t_{22} X_1 + t_{12} X_2$ ,  $Y_2 = t_{21} X_1 + t_{22} X_2$ . From the second equation we get

$$X_1 = -\frac{t_{22}}{t_{21}} X_2 + \frac{1}{t_{21}} Y_2. \quad \text{Substituting this expression in the first equation we get,}$$

$$Y_1 = t_{11} \left( -\frac{t_{22}}{t_{21}} X_2 + \frac{1}{t_{21}} Y_2 \right) + t_{12} X_2 = \frac{t_{11}}{t_{21}} Y_2 + \frac{t_{12} t_{21} - t_{11} t_{22}}{t_{21}} X_2. \quad \text{Comparing the last two}$$

$$\text{equations with Eq. (4.149) we arrive at } A = \frac{1}{t_{21}}, \quad B = -\frac{t_{22}}{t_{21}}, \quad C = \frac{t_{11}}{t_{21}}, \quad D = \frac{t_{12} t_{21} - t_{11} t_{22}}{t_{21}}.$$

**4.114** From Eq. (4.176a) we note  $t_{12} = \frac{AD - BC}{A}$  and  $t_{21} = \frac{1}{A}$ . Hence,  $t_{12} = t_{21}$  imply

$$AD - BC = 1.$$

**4.115**  $\begin{bmatrix} Y_1' \\ Y_2' \end{bmatrix} = \begin{bmatrix} k_1 & (1 - k_1^2)z^{-1} \\ 1 & -k_1 z^{-1} \end{bmatrix} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix}$ ,  $\begin{bmatrix} Y_1'' \\ Y_2'' \end{bmatrix} = \begin{bmatrix} k_2 & (1 - k_2^2)z^{-1} \\ 1 & -k_2 z^{-1} \end{bmatrix} \begin{bmatrix} X_1'' \\ X_2'' \end{bmatrix}$ , where  $\begin{bmatrix} X_1'' \\ X_2'' \end{bmatrix} = \begin{bmatrix} Y_1' \\ Y_2' \end{bmatrix}$ .

Thus, the transfer matrices of the two two-pairs are given by  $\tau_1 = \begin{bmatrix} k_1 & (1 - k_1^2)z^{-1} \\ 1 & -k_1 z^{-1} \end{bmatrix}$ , and

$\tau_2 = \begin{bmatrix} k_2 & (1 - k_2^2)z^{-1} \\ 1 & -k_2 z^{-1} \end{bmatrix}$ . The corresponding chain matrices are obtained using Eq. (4.76b) and

are given by  $\Gamma_1 = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix}$ , and  $\Gamma_2 = \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix}$ . Therefore, the chain matrix of the  $\Gamma$ -

cascade is given by  $\Gamma_1 \Gamma_2 = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix} = \begin{bmatrix} 1 + k_1 k_2 z^{-1} & k_2 z^{-1} + k_1 z^{-2} \\ k_1 + k_2 z^{-1} & k_1 k_2 z^{-1} + z^{-2} \end{bmatrix}$ .

Hence using Eq. (4.176a) we arrive at the transfer matrix of the  $\Gamma$ -cascade as

$$\tau = \begin{bmatrix} \frac{k_1 + k_2 z^{-1}}{1 + k_1 k_2 z^{-1}} & \frac{z^{-2}(1 - k_1^2)(1 - k_2^2)}{1 + k_1 k_2 z^{-1}} \\ 1 & z^{-1}(k_2 + k_1 z^{-1}) \\ \frac{1}{1 + k_1 k_2 z^{-1}} & -\frac{1}{1 + k_1 k_2 z^{-1}} \end{bmatrix}.$$

**4.116**  $\begin{bmatrix} X_1' \\ Y_1' \end{bmatrix} = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \begin{bmatrix} Y_2' \\ X_2' \end{bmatrix}$ ,  $\begin{bmatrix} X_1'' \\ Y_1'' \end{bmatrix} = \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix} \begin{bmatrix} Y_2'' \\ X_2'' \end{bmatrix}$ , where  $\begin{bmatrix} Y_2' \\ X_2' \end{bmatrix} = \begin{bmatrix} X_1'' \\ Y_1'' \end{bmatrix}$ . Therefore,

the chain matrices of the two two-pairs are given by  $\Gamma_1 = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix}$ , and  $\Gamma_2 = \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix}$ .

The corresponding transfer matrices are obtained using Eq. (4.151a) and are given by

$$\tau_1 = \begin{bmatrix} k_1 & (1 - k_1^2)z^{-1} \\ 1 & -k_1 z^{-1} \end{bmatrix}, \text{ and } \tau_2 = \begin{bmatrix} k_2 & (1 - k_2^2)z^{-1} \\ 1 & -k_2 z^{-1} \end{bmatrix}. \text{ The transfer matrix of the t-cascade is}$$

$$\begin{aligned} \text{therefore given by } \tau &= \tau_2 \tau_1 = \begin{bmatrix} k_2 & (1 - k_2^2)z^{-1} \\ 1 & -k_2 z^{-1} \end{bmatrix} \begin{bmatrix} k_1 & (1 - k_1^2)z^{-1} \\ 1 & -k_1 z^{-1} \end{bmatrix} \\ &= \begin{bmatrix} k_1 k_2 + z^{-1}(1 - k_2^2) & z^{-1}k_2(1 - k_1^2) - z^{-2}k_1(1 - k_2^2) \\ k_1 - k_2 z^{-1} & z^{-1}(1 - k_1^2) + k_1 k_2 z^{-2} \end{bmatrix}. \end{aligned}$$

Using Eq. (4.151b) we thus arrive at the chain matrix of the t-cascade:

$$\Gamma = \begin{bmatrix} \frac{1}{k_1 - k_2 z^{-1}} & \frac{-(k_1 k_2 z^{-1} + 1 - k_1^2)z^{-1}}{k_1 - k_2 z^{-1}} \\ \frac{k_1 k_2 + z^{-1}(1 - k_2^2)}{k_1 - k_2 z^{-1}} & \frac{z^{-2}}{k_1 - k_2 z^{-1}} \end{bmatrix}.$$

**4.117 (a)** Analyzing Figure P4.13(a) we obtain  $Y_2 = X_1 - k_m z^{-1} X_2$  and  $Y_1 = k_m Y_2 + z^{-1} X_2$   
 $= k_m (X_1 - k_m z^{-1} X_2) + z^{-1} X_2 = k_m X_1 + (1 - k_m^2) z^{-1} X_2$ . Hence, the transfer parameters are given  
 by  $\tau = \begin{bmatrix} k_m & (1 - k_m^2)z^{-1} \\ 1 & -k_m z^{-1} \end{bmatrix}$ . Using Eq. (4.176b) we arrive at the chain parameters given by

$$\Gamma = \begin{bmatrix} 1 & k_m z^{-1} \\ k_m & z^{-1} \end{bmatrix}.$$

**(b)** Analyzing Figure P4.13(a) we obtain  $V_1 = k_m (X_1 - z^{-1} X_2)$ ,  
 $Y_1 = V_1 + X_1 = (1 + k_m) X_1 - k_m z^{-1} X_2$ , and  $Y_2 = V_1 + z^{-1} X_2 = k_m X_1 + (1 - k_m) z^{-1} X_2$ . Hence, the  
 transfer parameters are given by  $\tau = \begin{bmatrix} 1 + k_m & -k_m z^{-1} \\ k_m & (1 - k_m) z^{-1} \end{bmatrix}$ . Using Eq. (4.176b) we arrive at the

$$\text{chain parameters given by } \Gamma = \begin{bmatrix} \frac{1}{k_m} & -\frac{(1 - k_m) z^{-1}}{k_m} \\ \frac{1 - k_m}{k_m} & -\frac{1}{k_m} \end{bmatrix}.$$



**4.118** For the constrained two pair  $H(z) = \frac{C+DG(z)}{A+BG(z)}$ . Hence,  $C = k_m$ ,  $D = z^{-1}$ ,  $A = 1$ ,  $B = k_m z^{-1}$ .

Substituting these values of the chain parameters in Eq. (4.151a) we get

$$t_{11} = \frac{C}{A} = k_m, \quad t_{12} = z^{-1}(1 - k_m^2), \quad t_{21} = 1, \quad t_{22} = -k_m z^{-1}.$$

**4.119** From the results of Problem 4.117, Part (a), we observe that the chain matrix of the  $i$ -th lattice

two-pair is given by  $\Gamma_i = \begin{bmatrix} 1 & k_i z^{-1} \\ k_i & z^{-1} \end{bmatrix}$ ,  $i = 1, 2, 3$ . Thus, the chain matrix of the cascade of three

lattice two-pairs is given by  $\Gamma_{\text{cascade}} = \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_3 z^{-1} \\ k_3 & z^{-1} \end{bmatrix}$

$$\begin{aligned} &= \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 + k_2 k_3 z^{-1} & k_3 z^{-1} + k_2 z^{-2} \\ k_2 + k_3 z^{-1} & k_2 k_3 z^{-1} + z^{-2} \end{bmatrix} \\ &= \begin{bmatrix} 1 + k_2 k_3 z^{-1} + k_1 z^{-1}(k_2 + k_3 z^{-1}) & k_3 z^{-1} + k_2 z^{-2} + k_1 z^{-1}(k_2 k_3 + z^{-2}) \\ k_1(1 + k_2 k_3 z^{-1}) + z^{-1}(k_2 + k_3 z^{-1}) & k_1(k_3 z^{-1} + k_2 z^{-2}) + z^{-1}(k_2 k_3 + z^{-2}) \end{bmatrix}. \end{aligned}$$

From Eq. (4.181a)

we obtain  $A_3(z) = \frac{C+D}{A+B}$

$$\begin{aligned} &= \frac{1 + k_2 k_3 z^{-1} + k_1 z^{-1}(k_2 + k_3 z^{-1}) + k_3 z^{-1} + k_2 z^{-2} + k_1 z^{-1}(k_2 k_3 + z^{-2})}{k_1(1 + k_2 k_3 z^{-1}) + z^{-1}(k_2 + k_3 z^{-1}) + k_1(k_3 z^{-1} + k_2 z^{-2}) + z^{-1}(k_2 k_3 + z^{-2})} \\ &= \frac{k_1 + (k_2 k_3 + k_1 k_2 + k_3)z^{-1} + (k_1 k_3 + k_2 + k_1 k_2 k_3)z^{-2} + k_1 z^{-3}}{k_1 + (k_1 k_3 + k_2 + k_1 k_2 k_3)z^{-1} + (k_2 k_3 + k_1 k_2 + k_3)z^{-2} + z^{-3}} \end{aligned}$$

which is seen to be an allpass function.

**4.120** Let  $D(z) = 1 + d_1 z^{-1} + d_2 z^{-2} = (1 - \lambda_1 z^{-1})(1 - \lambda_2 z^{-1})$ . Thus,  $d_2 = \lambda_1 \lambda_2$  and  $d_1 = -(\lambda_1 + \lambda_2)$ .

For stability,  $|\lambda_i| < 1$ ,  $i = 1, 2$ . As a result,  $|d_2| = |\lambda_1 \lambda_2| < 1$ .

Case 1: Complex poles:  $d_2 > 0$ . In this case,  $\lambda_2 = \lambda_1^*$ . Now,

$$\lambda_1, \lambda_2 = \frac{-d_1 \pm \sqrt{d_1^2 - 4d_2}}{2}. \text{ Hence, } \lambda_1 \text{ and } \lambda_2 \text{ will be complex, if } d_1^2 < 4d_2. \text{ In this case,}$$

$$\lambda_1 = -\frac{d_1}{2} + \frac{j}{2} \sqrt{4d_2 - d_1^2}. \text{ Thus, } |\lambda_1|^2 = \frac{1}{4}(d_1^2 + 4d_2 - d_1^2) = d_2 < 1. \text{ Consequently, if the poles are complex and } d_2 < 1, \text{ then they are inside the unit circle.}$$

Case 2: Real poles. In this case we get  $-1 < \lambda_i < 1$ ,  $i = 1, 2$ . Since,  $|\lambda_i| < 1$ , it follows then

$$|d_1| < |\lambda_1| + |\lambda_2| < 2. \text{ Now, } -1 < \frac{-d_1 \pm \sqrt{d_1^2 - 4d_2}}{2} < 1, \text{ or } \pm \sqrt{d_1^2 - 4d_2} < 2 + d_1.$$

It is not possible to satisfy the inequality on the right hand side with a minus sign in front of the square root as it would imply then  $d_1 < -2$ . Therefore,

$$\sqrt{d_1^2 - 4d_2} < 2 + d_1, \text{ or } d_1^2 - 4d_2 < 4 + d_1^2 + 4d_1, \text{ or } -d_1 < 1 + d_2. \quad (14)$$

Similarly,  $\frac{-d_1 \pm \sqrt{d_1^2 - 4d_2}}{2} < -1$ , or  $\pm\sqrt{d_1^2 - 4d_2} > -2 + d_1$ . Again it is not possible to satisfy the inequality on the right hand side with a plus sign in front of the square root as it would imply then  $d_1 > 2$ . Therefore,  $-\sqrt{d_1^2 - 4d_2} > -2 + d_1$ , or  $\sqrt{d_1^2 - 4d_2} < 2 - d_1$ , or  $d_1^2 - 4d_2 < 4 + d_1^2 - 4d_1$ , or equivalently,  $d_1 < 1 + d_2$ . (15)

Combining Eqs. (14) and (15) we get  $|d_1| < 1 + d_2$ .

**4.121 (a)**  $|d_1| = 0.92$  and  $1 + d_2 = 1.1995$ . Since  $|d_1| < 1 + d_2$  and  $|d_2| < 1$ , both roots are inside the unit circle.

**(b)**  $|d_1| = 0.2$  and  $1 + d_2 = -0.43$ . Since  $|d_2| > 1$  and  $|d_1| > 1 + d_2$ , all roots are not inside the unit circle.

**(c)**  $|d_1| = 1.4562$  and  $1 + d_2 = 1.81$ . Since  $|d_1| < 1 + d_2$  and  $|d_2| < 1$ , both roots are inside the unit circle.

**(d)**  $|d_1| = 2.1843$  and  $1 + d_2 = 1.81$ . Since  $|d_1| < 1 + d_2$  and  $|d_2| < 1$ , both roots are inside the unit circle.

**4.122 (a)**  $A_3(z) = \frac{\frac{1}{12} - \frac{1}{4}z^{-1} - \frac{1}{2}z^{-2} + z^{-3}}{1 - \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{12}z^{-3}}$ . Note,  $|k_3| = \frac{1}{12} < 1$ . Using Eq. (4.177) we arrive at

$A_2(z) = \frac{-0.2098 - 0.4825z^{-1} + z^{-2}}{1 - 0.4825z^{-1} - 0.2098z^{-2}}$ . Here,  $|k_2| = 0.2098 < 1$ . Continuing this process, we get,

$A_1(z) = \frac{-0.6106 + z^{-1}}{1 - 0.6106z^{-1}}$ . Finally,  $|k_1| = 0.6106 < 1$ . Since  $|k_i| < 1$ , for  $i = 3, 2, 1$ ,  $H_a(z)$  is stable.

**(b)**  $A_3(z) = \frac{-\frac{1}{3} + \frac{1}{6}z^{-1} + \frac{13}{6}z^{-2} + z^{-3}}{1 + \frac{13}{6}z^{-1} + \frac{1}{6}z^{-2} - \frac{1}{3}z^{-3}}$ . Note,  $|k_3| = \frac{1}{3} < 1$ .  $A_2(z) = \frac{1 + 2.5z^{-1} + z^{-2}}{1 + 2.5z^{-1} + z^{-2}}$ . Since

$|k_2| = 1$ ,  $H_b(z)$  is unstable.

**(c)**  $A_4(z) = \frac{\frac{1}{36} + \frac{5}{18}z^{-1} + \frac{18.5}{18}z^{-2} + \frac{5}{3}z^{-3} + z^{-4}}{1 + \frac{5}{3}z^{-1} + \frac{18.5}{18}z^{-2} + \frac{5}{18}z^{-3} + \frac{1}{36}z^{-4}}$ . Note,  $|k_4| = \frac{1}{36} < 1$ .

$A_3(z) = \frac{0.2317 + z^{-1} + 1.6602z^{-2} + z^{-3}}{1 + 1.6602z^{-1} + z^{-2} + 0.2317z^{-3}}$ . Thus,  $|k_3| = 0.2317 < 1$ .

$A_2(z) = \frac{0.6503 + 1.5096z^{-1} + z^{-2}}{1 + 1.5096z^{-1} + 0.6503z^{-2}}$ . Here,  $|k_2| = 0.6503 < 1$ . Finally,  $A_1(z) = \frac{0.9147 + z^{-1}}{1 + 0.9147z^{-1}}$ .

Thus,  $|k_1| = 0.9147 < 1$ . Since  $|k_i| < 1$ , for  $i = 4, 3, 2, 1$ ,  $H_c(z)$  is stable.

$$(d) A_5(z) = \frac{\frac{1}{32} + \frac{5}{16}z^{-1} + \frac{5}{4}z^{-2} + \frac{5}{2}z^{-3} + \frac{5}{2}z^{-4} + z^{-5}}{1 + \frac{5}{2}z^{-1} + \frac{5}{2}z^{-2} + \frac{5}{4}z^{-3} + \frac{5}{16}z^{-4} + \frac{1}{32}z^{-5}}. \text{ This implies, } |k_5| = \frac{1}{32} < 1.$$

$$A_4(z) = \frac{0.2346 + 1.173z^{-1} + 2.4633z^{-2} + 2.4927z^{-3} + z^{-4}}{1 + 2.4927z^{-1} + 2.4633z^{-2} + 1.173z^{-3} + 0.2346z^{-4}}. \text{ Thus, } |k_4| = 0.2346 < 1.$$

$$A_3(z) = \frac{0.6225 + 1.9952z^{-1} + 2.3466z^{-2} + z^{-3}}{1 + 2.3466z^{-1} + 1.9952z^{-2} + 0.6225z^{-3}}. \text{ Implying } |k_3| = 0.6225 < 1.$$

$$A_2(z) = \frac{0.8726 + 1.8034z^{-1} + z^{-2}}{1 + 1.8034z^{-1} + 0.8726z^{-2}}. \text{ Hence, } |k_2| = 0.8726 < 1. \text{ Finally, } A_1(z) = \frac{0.9630 + z^{-1}}{1 + 0.9630z^{-1}}.$$

Thus,  $|k_1| = 0.9630 < 1$ . Since  $|k_i| < 1$ , for  $i = 5, 4, 3, 2, 1$ ,  $H_d(z)$  is stable.

$$(e) A_5(z) = \frac{\frac{1}{6} + \frac{1}{3}z^{-1} + \frac{1}{2}z^{-2} + \frac{2}{3}z^{-3} + \frac{5}{3}z^{-4} + z^{-5}}{1 + \frac{5}{6}z^{-1} + \frac{2}{3}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{6}z^{-4} + \frac{1}{6}z^{-5}}. \text{ This implies, } |k_5| = \frac{1}{6} < 1.$$

$$A_4(z) = \frac{0.2 + 0.4z^{-1} + 0.6z^{-2} + 0.8z^{-3} + z^{-4}}{1 + 0.8z^{-1} + 0.6z^{-2} + 0.4z^{-3} + 0.2z^{-4}}. \text{ Thus, } |k_4| = 0.2 < 1.$$

$$A_3(z) = \frac{0.25 + 0.5z^{-1} + 0.75z^{-2} + z^{-3}}{1 + 0.75z^{-1} + 0.5z^{-2} + 0.25z^{-3}}. \text{ Hence, } |k_3| = 0.25 < 1.$$

$$A_2(z) = \frac{0.3333 + 0.6667z^{-1} + z^{-2}}{1 + 0.6667z^{-1} + 0.3333z^{-2}}. \text{ Here, } |k_2| = 0.3333 < 1. \text{ Finally, } A_1(z) = \frac{0.5 + z^{-1}}{1 + 0.5z^{-1}}. \text{ As a}$$

result,  $|k_1| = 0.5 < 1$ . Since  $|k_i| < 1$ , for  $i = 5, 4, 3, 2, 1$ ,  $H_e(z)$  is stable.

$$4.123 (a) A_4(z) = \frac{\frac{1}{5} + \frac{2}{5}z^{-1} + \frac{3}{5}z^{-2} + \frac{4}{5}z^{-3} + z^{-4}}{1 + \frac{4}{5}z^{-1} + \frac{3}{5}z^{-2} + \frac{2}{5}z^{-3} + \frac{1}{5}z^{-4}}. \text{ Thus, } |k_4| = \frac{1}{5} < 1..$$

$$A_3(z) = \frac{0.25 + 0.5z^{-1} + 0.75z^{-2} + z^{-3}}{1 + 0.75z^{-1} + 0.5z^{-2} + 0.25z^{-3}}. \text{ Thus, } |k_3| = 0.25 < 1. \text{ Repeating the process, we}$$

$$\text{arrive at } A_2(z) = \frac{0.3333 + 0.6667z^{-1} + z^{-2}}{1 + 0.6667z^{-1} + 0.3333z^{-2}}. \text{ Thus, } |k_2| = 0.3333 < 1. \text{ Finally, we get}$$

$$A_1(z) = \frac{0.5 + z^{-1}}{1 + 0.5z^{-1}}. \text{ Thus, } |k_1| = 0.5 < 1. \text{ Since } |k_i| < 1, \text{ for } i = 5, 4, 3, 2, 1, D_a(z) \text{ has all}$$

roots inside the unit circle.

$$(b) A_3(z) = \frac{0.4 + 0.3z^{-1} + 0.2z^{-2} + z^{-3}}{1 + 0.2z^{-1} + 0.3z^{-2} + 0.4z^{-3}}. \text{ Thus, } |k_3| = 0.4 < 1. A_2(z) = \frac{0.2619 + 0.0952z^{-1} + z^{-2}}{1 + 0.0952z^{-1} + 0.2619z^{-2}}.$$

This implies,  $|k_2| = 0.2619 < 1$ . Next,  $A_1(z) = \frac{0.0755 + z^{-1}}{1 + 0.0755z^{-1}}$ . Hence,  $|k_1| = 0.0755 < 1$ . Since  $|k_i| < 1$ , for  $i = 3, 2, 1$ ,  $D_b(z)$  has all roots inside the unit circle.

**4.124**  $z = \frac{1+s}{1-s}$ . Hence,  $s = \frac{z-1}{z+1}$ . Thus, the  $k$ -th root in the  $s$ -domain is given by

$$s_k = \frac{(z_k - 1)(z_k^* + 1)}{|z_k + 1|^2} = \frac{|z_k|^2 - 1 + z_k - z_k^*}{|z_k + 1|^2} \text{ where } z_k \text{ is the } k\text{-th root in the } z\text{-domain. Hence,}$$

$$\text{Re}\{s_k\} = \frac{|z_k|^2 - 1}{|z_k + 1|^2}. \text{ Since } D(z) \text{ is a minimum phase polynomial, } |z_k| < 1. \text{ Therefore,}$$

$\text{Re}\{s_k\} < 0$ . Hence  $B(s)$  is a strictly Hurwitz polynomial.

**4.125**  $H(z) = 1 - \alpha z^{-1}$ . Hence, using Eq. (4.212) we get

$$\Phi_{yy}(z) = H(z)H(z^{-1})\Phi_{yy}(z) = (1 - \alpha z^{-1})(1 - \alpha z)\sigma_x^2. \text{ Therefore, from Eq. (4.214) we have}$$

$$P_{yy}(\omega) = |H(e^{j\omega})|^2 P_{xx}(\omega) = |H(e^{j\omega})|^2 \sigma_x^2 = (1 - \alpha e^{-j\omega})(1 - \alpha e^{j\omega})\sigma_x^2 = (1 + \alpha^2 - 2\alpha \cos\omega)\sigma_x^2.$$

Now,  $H(z)H(z^{-1}) = (1 - \alpha z^{-1})(1 - \alpha z) = -\alpha z + (1 + \alpha^2) - \alpha z^{-1}$ . Hence,

$$\phi_{yy}[n] = (-\alpha \delta[n+1] + (1 + \alpha^2)\delta[n] - \alpha \delta[n-1])\sigma_x^2.$$

As a result, average power =  $\phi_{yy}[0] = (1 + \alpha^2)\sigma_x^2$ . Note that the average power increases with increasing  $\alpha$ .

$$\mathbf{4.126} \quad H(z) = \frac{1}{1 - 0.5z^{-1}}. \quad |H(e^{j\omega})|^2 = \frac{1}{(1 - 0.5e^{-j\omega})(1 - 0.5e^{j\omega})} = \frac{1}{1.25 - \cos\omega}. \text{ Hence, rom Eq.}$$

$$(4.214) \text{ we have } P_{yy}(\omega) = |H(e^{j\omega})|^2 P_{xx}(\omega) = \frac{P_{xx}(\omega)}{(1 - 0.5e^{-j\omega})(1 - 0.5e^{j\omega})} = \frac{\sigma_x^2}{1.25 - \cos\omega}. \text{ Now,}$$

$$H(z)H(z^{-1}) = \frac{1}{1 - 0.5z^{-1}} \cdot \frac{1}{1 - 0.5z} = -\frac{z^{-1}}{(1 - 0.5z^{-1})(0.5 - z^{-1})} = -\frac{2z^{-1}}{(1 - 0.5z^{-1})(1 - 2z^{-1})}$$

$$= \frac{4/3}{1 - 0.5z^{-1}} - \frac{4/3}{1 - 2z^{-1}}. \text{ Therefore, } \phi_{yy}[n] = \frac{4\sigma_x^2}{3} \left( (0.5)^n \mu[n] + (2)^n \mu[-n-1] \right).$$

$$\mathbf{4.127} \quad (a) \quad P_{yy}(\omega) = |A(e^{j\omega})|^2 P_{xx}(\omega) = \frac{|A(e^{j\omega})|^2}{1 + d \cos(\omega)} = \frac{1}{1 + d \cos(\omega)}. \quad (16)$$

(b) No, The answer does not depend upon the choice of  $\alpha$ . However Eq. (16) holds if the filter is stable for which we require  $|d| < 1$ .

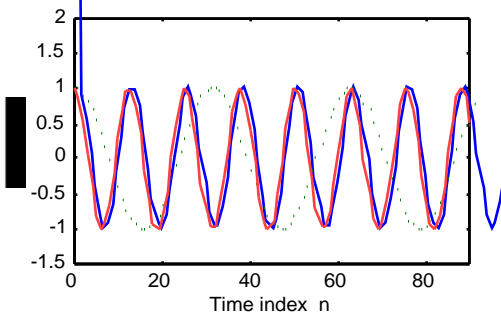
**4.128 (a)**  $\phi_{xy}[l] = E\{x[n+1]y[n]\} = E\{x[n+1]x[n]\} \otimes h[n] = \phi_{xx}[n] \otimes h[n]$ . Taking the discrete-time Fourier transform of both sides we get  $P_{xy}(\omega) = P_{xx}(\omega)H(e^{j\omega})$ . Since  $H(e^{j\omega})$  in general is not real,  $P_{xy}(\omega)$  in general is not real.

**(b)**  $\phi_{xu}[l] = E\{x[n+1]u[n]\} = E\{x[n+1]h[-n] \otimes y[n]\} = \phi_{xx}[n] \otimes h[n] \otimes h[-n]$ . Taking the discrete-time Fourier transform of both sides we get  $P_{xu}(\omega) = P_{xx}(\omega)|H(e^{j\omega})|^2$ . Thus,  $P_{xu}(\omega)$  is a real function of  $\omega$ .

**M4.1** From Table 3.2, the DTFT of  $\{n h[n]\}$  is  $j \frac{dH(e^{j\omega})}{d\omega}$ . Hence, the group delay  $t(\omega)$  using Eq. (4.203) can be computed at a set of  $N$  discrete frequency points  $\omega_k = 2\pi k/N$ ,  $k = 0, 1, \dots, N-1$ , as follows:  $\{t(\omega_k)\} = \text{Re}(\text{DFT}\{n h[n]\} / \text{DFT}\{h[n]\})$ , where all DFTs are  $N$ -points in length with  $N$  greater than or equal to length of  $\{h[n]\}$ .

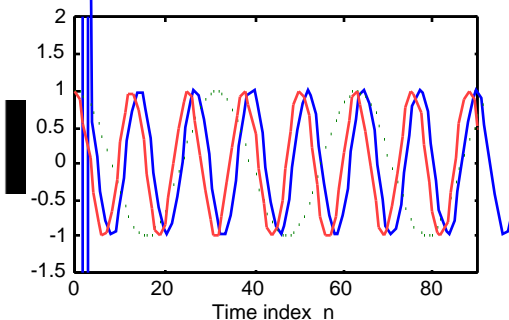
**M4.2** We modify Program 4\_2 using the following statements:

```
b = [-4.8788 9.5631 -4.8788];
x1 = cos(0.2*n); x2 = cos(0.5*n);
```

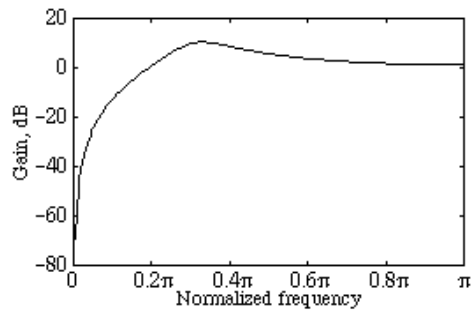


**M4.3** We modify Program 4\_2 using the following statements:

```
b = [-13.4866 45.228 -63.8089 45.228 -13.4866];
zi = [0 0 0 0];
x1 = cos(0.2*n); x2 = cos(0.5*n); x3 = cos(0.8*n);
y = filter(b, 1, x1+x2+x3, zi);
```



**M4.4**



**M4.5 (a)**

```

num1 = [0 0.75 2 2 1 0.75];
num2 = [0 -0.75 2 -2 1 -0.75];
den = [3 0 3.5 0 1];
w = 0:pi/255:pi;
h1 = freqz(num1, den, w);
h2 = freqz(num2, den, w);
plot(w/pi, abs(h1).*abs(h1)+abs(h2).*abs(h2));

```

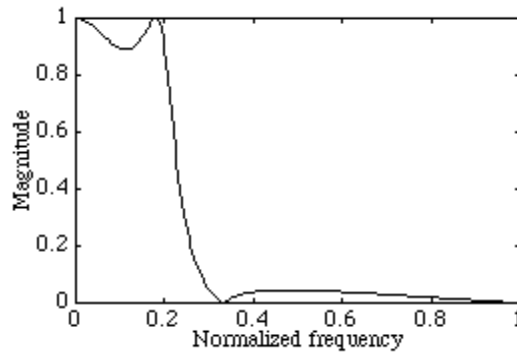
**(b)** Replace the first three lines in the above MATLAB program with the following:

```

num1 = [1 1.5 3.75 2.75 2.75 3.75 1.5 1];
num2 = [1 -1.5 3.75 -2.75 2.75 -3.75 1.5 -1];
den = [6 0 6.5 0 4.75 0 1];

```

**M4.6** The magnitude response of  $H(z)$  is as shown below from which we observe that  $H(z)$  is a lowpass filter.



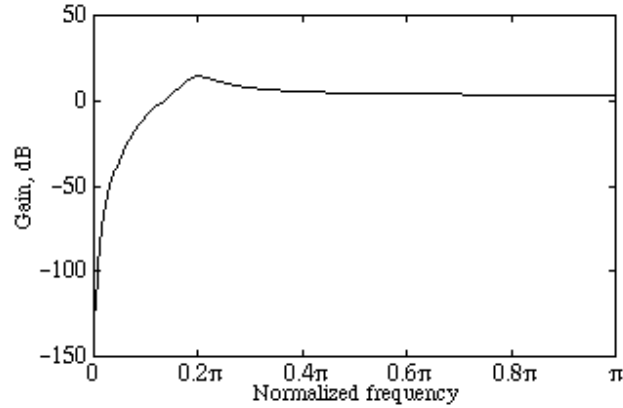
By multiplying out the factors of  $H(z)$  we get

$$H(z) = \frac{0.05634 - 0.000935z^{-1} + 0.000935z^{-2} + 0.05634z^{-3}}{1 - 2.1291z^{-1} + 1.783386z^{-2} + 0.543463z^{-3}}, \text{ The corresponding difference}$$

equation representation is therefore given by

$$y[n] = 0.05634x[n] - 0.000935x[n-1] + 0.000935x[n-2] - 0.05634x[n-3] - 2.1291y[n-1] + 1.783386y[n-2] + 0.543463y[n-3].$$

**M4.7** The magnitude response of  $H(z)$  is as shown below from which we observe that  $H(z)$  is a highpass filter.



By multiplying the factors of  $H(z)$  we get

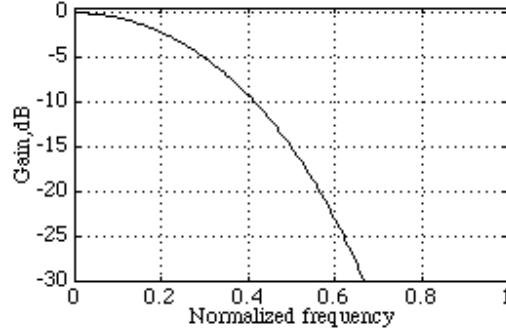
$$H(z) = \frac{1 - 4z^{-1} + 6z^{-2} - 4z^{-3} + z^{-4}}{1 - 3.0538z^{-1} + 3.8227z^{-2} - 2.2837z^{-3} + 0.5472z^{-4}}$$

The corresponding difference equation is given by

$$\begin{aligned} y[n] - 3.0538y[n-1] + 3.8227y[n-2] - 2.2837y[n-3] + 0.5472y[n-4] \\ = x[n] - 4x[n-1] + 6x[n-2] - 4x[n-3] + x[n-4]. \end{aligned}$$

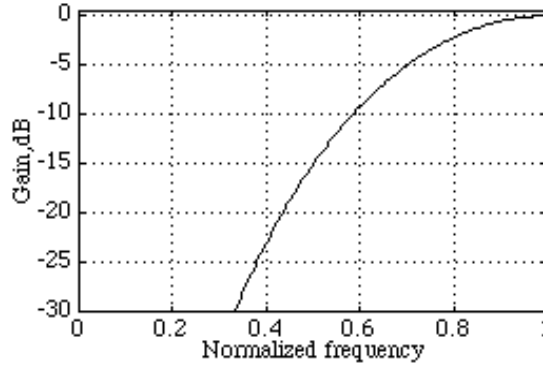
**M4.8** From Eq. (4.66), we obtain  $M = -\frac{1}{2\log_2(\cos(0.12\pi))} = 4.7599$ . We choose  $M = 5$ . A cascade

of 5 first-order lowpass FIR filter has a transfer function given by  $G(z) = \frac{1}{32}(1 + z^{-1})^5$ , whose gain response is plotted below:



**M4.9** For a cascade of  $M$  sections of the first-order highpass FIR filter of Eq. (4.67), the 3-dB cutoff frequency  $\omega_c$  is given by  $\omega_c = 2\sin^{-1}\left(2^{-1/2M}\right)$ . Hence,  $M = -\frac{1}{2\log_2(\sin(\omega_c/2))}$ .

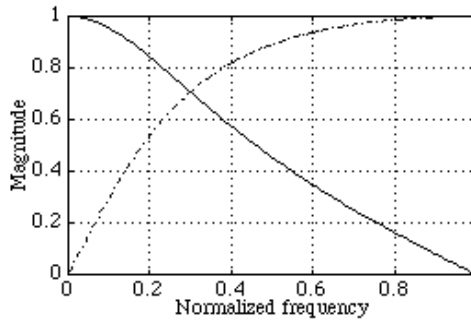
Substituting the value of  $\omega_c$  we then obtain  $M = 4.76$ . Choose  $M = 5$ . A cascade of 5 first-order highpass FIR filter has a transfer function given by  $G(z) = \frac{1}{32}(1 - z^{-1})^5$ , whose gain response is plotted below:



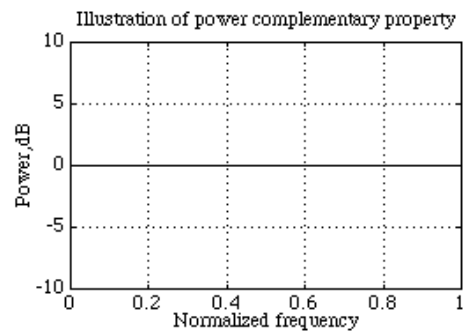
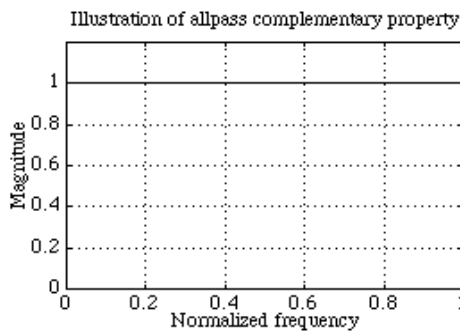
**M4.10** From Eq. (4.72b), we obtain  $\alpha = 0.32492$ . Hence, from Eq. (4.70) we get

$$H_{LP}(z) = \frac{0.33754(1+z^{-1})}{1-0.32492z^{-1}}. \text{ Likewise, from Eq. (4.73) we get } H_{HP}(z) = \frac{0.66246(1-z^{-1})}{1-0.32492z^{-1}}.$$

The magnitude responses of  $H_{LP}(z)$  and  $H_{HP}(z)$  are show below:



The magnitude response of  $H_{LP}(z) + H_{HP}(z)$  is shown below on the left-hand side, while the gain response of  $|H_{LP}(e^{j\omega})|^2 + |H_{HP}(e^{j\omega})|^2$  is shown below on the right-hand side:



**M4.11** From Eq. (4.76) we first obtain  $b = 0.309017$ . From Eq. (4.77) we arrive at two possible values of  $a$ : 0.6128 and 1.63185 leading to two possible solutions:

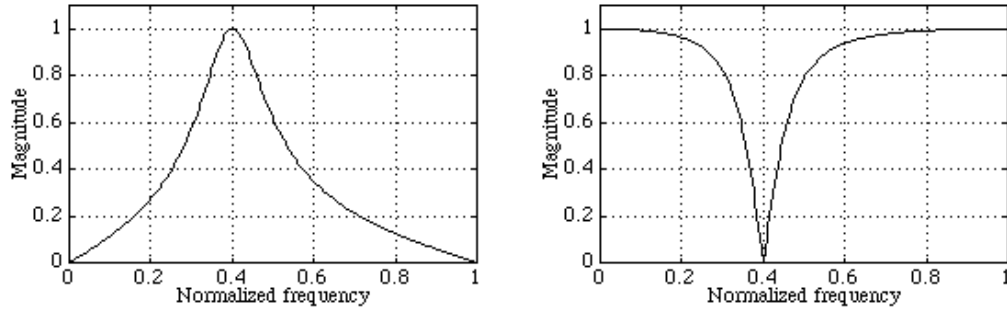
$$H'_{BP}(z) = \frac{0.1935(1-z^{-2})}{1-0.4984z^{-1}+0.6128z^{-2}}, \text{ and } H''_{BP}(z) = \frac{-0.3159(1-z^{-2})}{1-0.813287z^{-1}+1.63185z^{-2}}. \text{ It can be}$$

seen that  $H''_{BP}(z)$  is unstable. From Eq. (4.79), the bandstop transfer function corresponding to

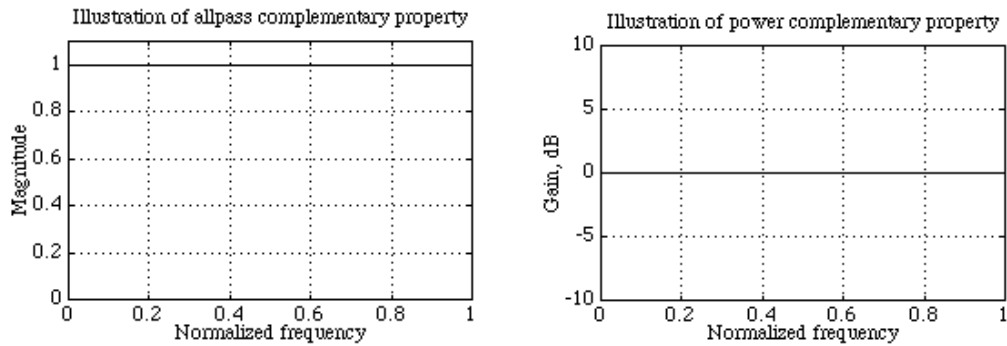


$H'_{BP}(z)$  is given by  $H'_{BS}(z) = \frac{0.8064(1 - 0.6180z^{-1} + z^{-2})}{1 - 0.498383z^{-1} + 0.6128z^{-2}}$ . A plot of the magnitude responses

of  $H'_{BP}(z)$  and  $H'_{BS}(z)$  are given below:



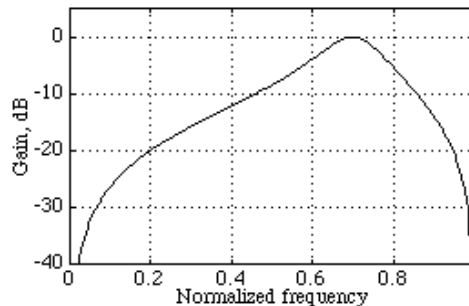
The magnitude response of  $H_{BP}(z) + H_{BS}(z)$  is shown below on the left-hand side, while the gain response of  $|H_{BP}(e^{j\omega})|^2 + |H_{BS}(e^{j\omega})|^2$  is shown below on the right-hand side:



**M4.12** From Eq. (4.76), we first obtain  $b = 0.309017$ . From Eq. (4.77) we arrive at two possible values of  $a$ : 1.376381 and 0.72654253 leading to two possible solutions:

$$H'_{BP}(z) = \frac{0.1935996(1 - z^{-2})}{1 + 0.94798z^{-1} + 0.6128z^{-2}}, \text{ and } H''_{BP}(z) = \frac{-0.3159258(1 - z^{-2})}{1 + 1.5469636z^{-1} + 1.63185168z^{-2}}.$$

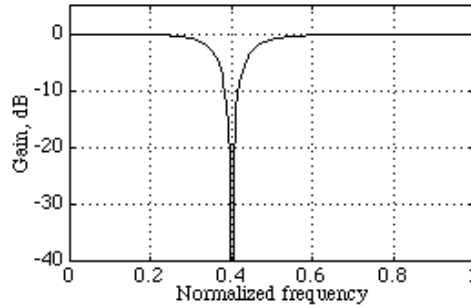
It can be seen that  $H''_{BP}(z)$  is unstable. A plot of the gain response of  $H'_{BP}(z)$  is shown below:



**M4.13** From Eq. (4.76), we first obtain  $b = 0.309016$ . From Eq. (4.77) we arrive at two possible values of  $a$ : 0.72654253 and 1.376382 leading to two possible solutions:

$$H'_{BS}(z) = \frac{0.86327126(1 + 0.618033z^{-1} + z^{-2})}{1 - 0.533531z^{-1} + 0.72654253z^{-2}}, \text{ and } H''_{BS}(z) = \frac{1.188191(1 + 0.618033z^{-1} + z^{-2})}{1 - 0.7343424z^{-1} + 1.376382z^{-2}}.$$

It can be seen that  $H''_{BS}(z)$  is unstable. A plot of the gain response of  $H'_{BS}(z)$  is shown below:

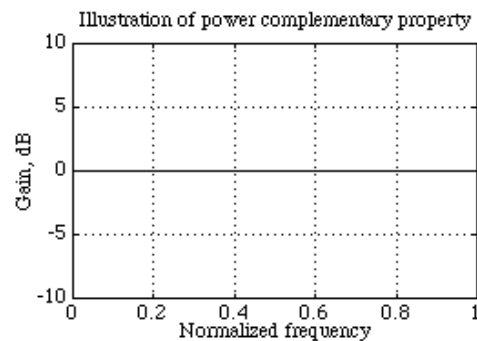
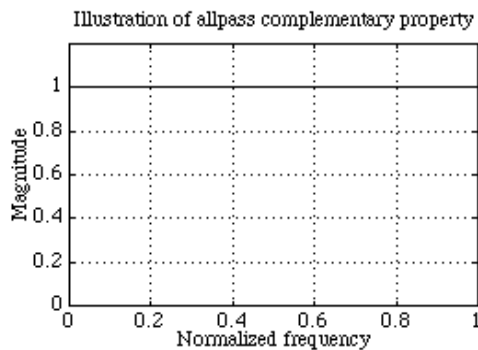


**M4.14 (a)** Using the following program we arrive at the two plots shown below:

```

b1 = [2 2]; b2 = [1 -1]; den = [3 1];
w = 0:pi/255:pi;
h1 = freqz(b1, den, w); h2 = freqz(b2, den, w);
sum = abs(h1).*abs(h1)+abs(h2).*abs(h2);
subplot(2,1,1);
plot(w/pi,abs(h1+h2));grid
axis([0 1 0 1.2]);
xlabel('Normalized frequency');ylabel('Magnitude');
title('Illustration of allpass complementary property');
subplot(2,1,2);
plot(w/pi,20*log10(sum));grid
axis([0 1 -10 10]);
xlabel('Normalized frequency');ylabel('Gain, dB');
title('Illustration of power complementary property');

```



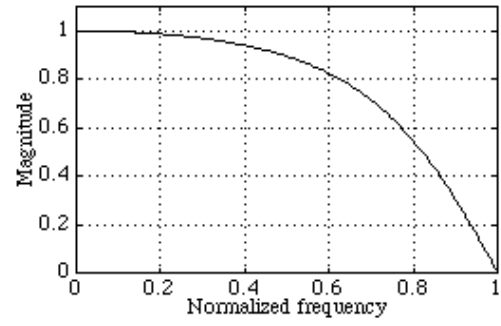
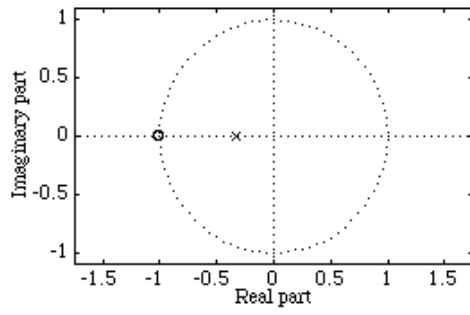
**(b)** For this part, replace the first line in the above program with the following:

```

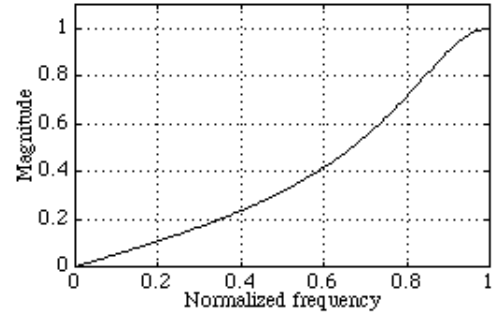
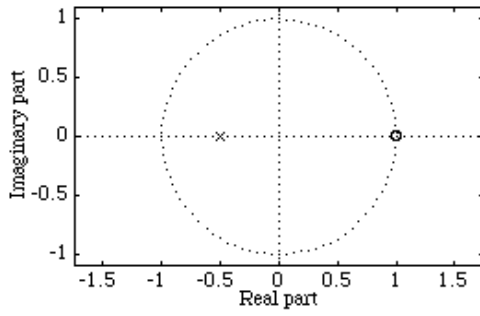
b1 = [-1 0 1]; b2 = [3 2 3]; den = [4 2 2];

```

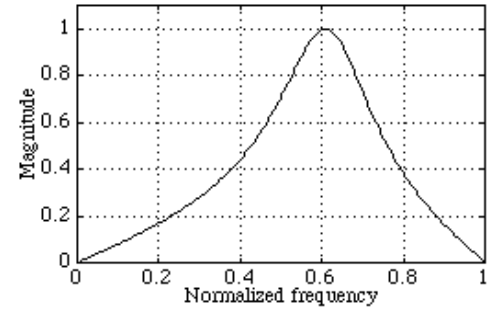
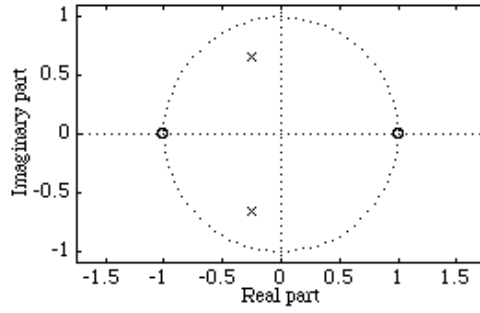
**M4.15 (a)**



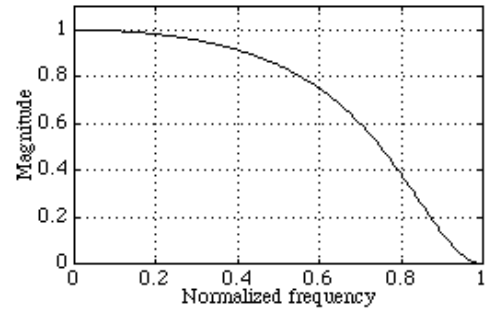
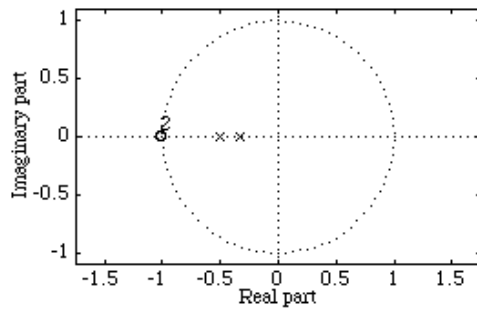
**(b)**



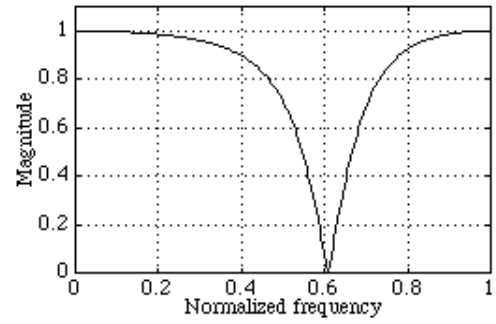
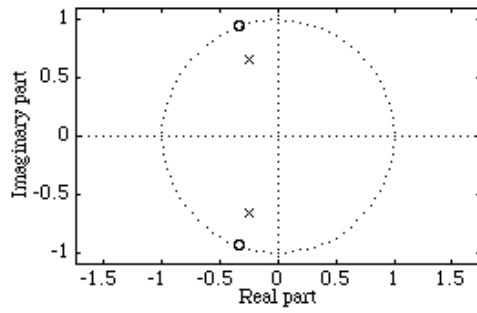
**(c)**



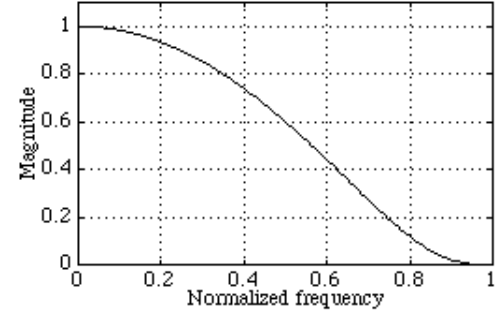
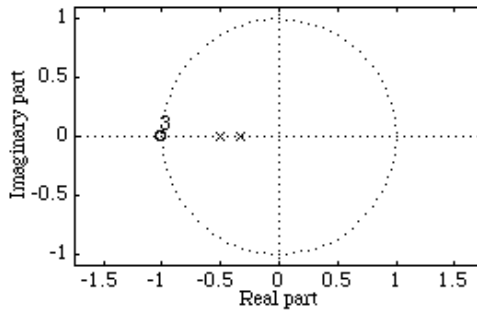
**(d)**



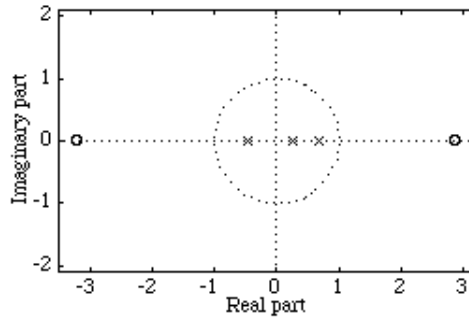
(e)



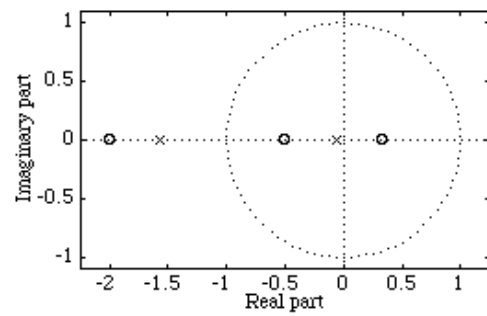
(f)



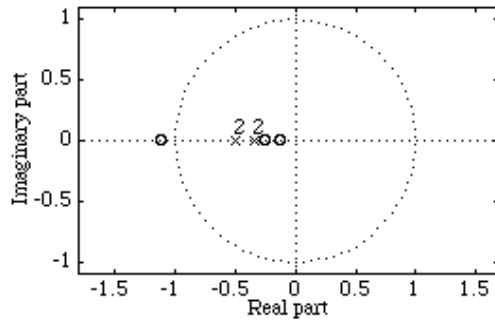
M4.16 (a)



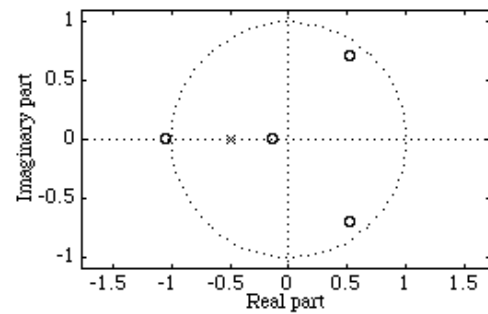
(b)

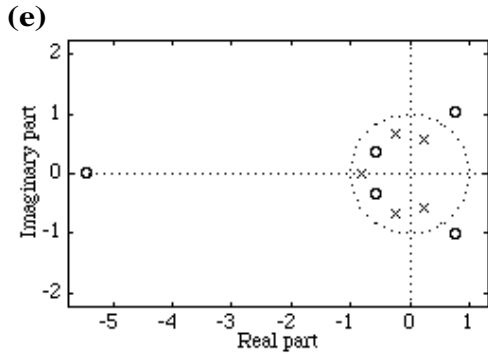


(c)



(d)





**M4.17 (a)** The stability test parameters are  
 0.0833    -0.2098    -0.6106  
 stable = 1

**(b)** The stability test parameters are  
 -0.3333    1.0000    2.0000  
 stable = 0

**(c)** The stability test parameters are  
 0.0278    0.2317    0.6503    0.9147  
 stable = 1

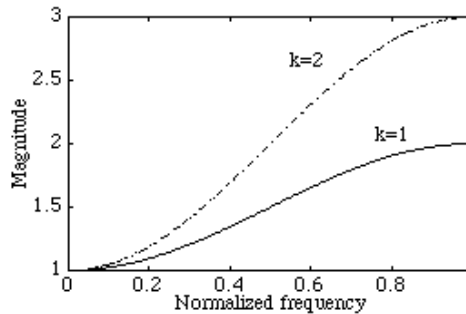
**(d)** The stability test parameters are  
 0.0312    0.2346    0.6225    0.8726    0.9630  
 stable = 1

**(e)** The stability test parameters are  
 0.1667    0.0286    0.3766    0.3208    0.5277  
 stable = 1

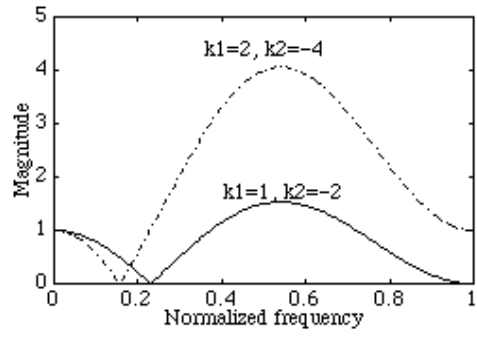
**M4.18 (a)** The stability test parameters are  
 0.2000    0.2500    0.3333    0.5000  
 stable = 1

**(b)** The stability test parameters are  
 0.4000    0.2619    0.0755  
 stable = 1

**M4.19**  $H(z) = -\frac{k}{4} + \left(1 + \frac{k}{2}\right)z^{-1} - \frac{k}{4}z^{-2}$ .



**M4.20**  $H(z) = -\frac{k_2}{4} + \frac{k_1}{4}z^{-1} + \left(1 - \frac{k_1}{2} + \frac{k_2}{2}\right)z^{-2} + \frac{k_1}{4}z^{-3} - \frac{k_2}{4}z^{-4}$ .



## Chapter 5 (2e)

**5.1**  $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ . Since  $p(t)$  is periodic function of time  $t$  with a period  $T$ , it can be

represented as a Fourier series:  $p(t) = \sum_{n=-\infty}^{\infty} c_n e^{j(2\pi n t / T)}$ , where  $c_n = \int_{-T/2}^{T/2} \delta(t) e^{-j(2\pi n t / T)} dt = \frac{1}{T}$ .

Hence  $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j(2\pi n t / T)}$ .

**5.2** Since the signal  $x_a(t)$  is being sampled at 1.5 kHz rate, there will be multiple copies of the spectrum at frequencies given by  $F_i \pm 1500$  k, where  $F_i$  is the frequency of the  $i$ -th sinusoidal component in  $x_a(t)$ . Hence,

$$\begin{array}{ll} F_1 = 250 \text{ Hz}, & F_{1m} = 250, 1250, 1750, \dots, \text{ Hz} \\ F_2 = 450 \text{ Hz}, & F_{2m} = 450, 1050, 1950, \dots, \text{ Hz} \\ F_3 = 1000 \text{ Hz}, & F_{3m} = 1000, 500, 2500, \dots, \text{ Hz} \\ F_4 = 2750 \text{ Hz}, & F_{4m} = 2750, 1250, 250, \dots, \text{ Hz} \\ F_5 = 4050 \text{ Hz}, & F_{5m} = 34050, 1050, 450, \dots, \text{ Hz} \end{array}$$

So after filtering by a lowpass filter with a cutoff at 750 Hz, the frequencies of the sinusoidal components present in  $y_a(t)$  are 250, 450, 500 Hz.

**5.3** One possible set of values are  $F_1 = 450$  Hz,  $F_2 = 625$  Hz,  $F_3 = 950$  Hz and  $F_4 = 7550$  Hz. Another possible set of values are  $F_1 = 450$  Hz,  $F_2 = 625$  Hz,  $F_3 = 950$  Hz,  $F_4 = 7375$  Hz. Hence the solution is not unique.

**5.4**  $t = nT = \frac{n}{4000}$ . Therefore,

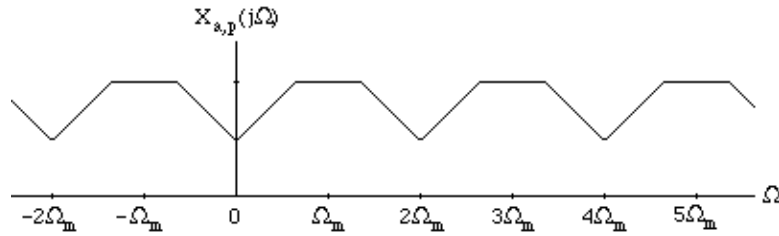
$$\begin{aligned} x[n] &= 3 \cos\left(\frac{400 \pi n}{4000}\right) + 5 \sin\left(\frac{1200 \pi n}{4000}\right) + 6 \cos\left(\frac{4400 \pi n}{4000}\right) + 2 \sin\left(\frac{5200 \pi n}{4000}\right) \\ &= 3 \cos\left(\frac{\pi n}{10}\right) + 5 \sin\left(\frac{3 \pi n}{10}\right) + 6 \cos\left(\frac{11 \pi n}{10}\right) + 2 \sin\left(\frac{13 \pi n}{10}\right) \\ &= 3 \cos\left(\frac{\pi n}{10}\right) + 5 \sin\left(\frac{3 \pi n}{10}\right) + 6 \cos\left(\frac{(20-9) \pi n}{10}\right) + 2 \sin\left(\frac{(20-7) \pi n}{10}\right) \\ &= 3 \cos\left(\frac{\pi n}{10}\right) + 5 \sin\left(\frac{3 \pi n}{10}\right) + 6 \cos\left(\frac{9 \pi n}{10}\right) - 2 \sin\left(\frac{7 \pi n}{10}\right). \end{aligned}$$

**5.5** Both the channels are sampled at 48.2 kHz. Therefore there are a total of  $2 \times 48200 = 96400$  samples/sec. Each sample is quantized using 15 bits. Hence the total bit rate of the two channels after sampling and digitization =  $15 \times 96400 = 1.446$  Mb/s.

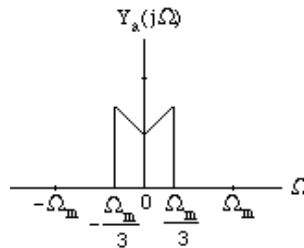
**5.6**  $h_r(t) = \frac{\sin(\Omega_c t)}{(\Omega_T t / 2)}$ . Thus,  $h_r(nT) = \frac{\sin(\Omega_c nT)}{(\Omega_T nT / 2)}$ , Since  $T = \frac{2\pi}{\Omega_T}$ , hence  $h_r(nT) = \frac{\sin\left(\frac{2\pi\Omega_c n}{\Omega_T}\right)}{\pi n}$ .

If  $\Omega_c = \Omega_T / 2$ , then,  $h_r(nT) = \frac{\sin(\pi n)}{\pi n} = \delta[n]$ .

**5.7** After sampling the spectrum of the signal is as indicated below:

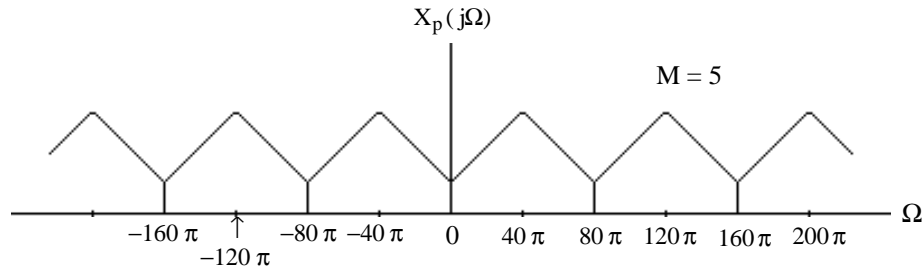


$T = \frac{2\pi}{2\Omega_m} = \frac{\pi}{\Omega_m}$ . As a result,  $\omega_c = \frac{\Omega_m\pi}{3\Omega_m} = \frac{\pi}{3}$ . Hence after the low pass filtering the spectrum of the signal will be as shown below:



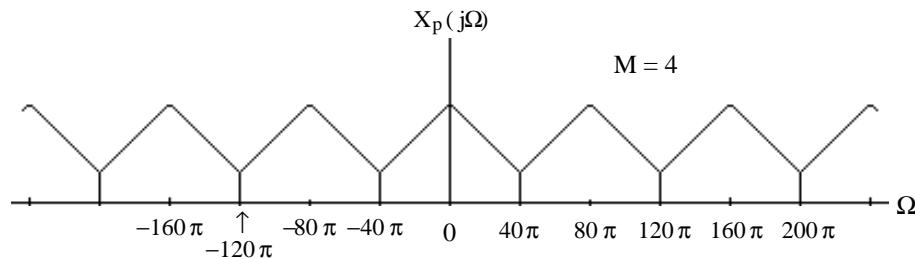
**5.8 (a)**  $\Omega_2 = 200\pi$ ,  $\Omega_1 = 160\pi$ . Thus,  $\Delta\Omega = \Omega_2 - \Omega_1 = 40\pi$ . Note  $\Delta\Omega$  is an integer multiple of  $\Omega_2$ . Hence, we choose the sampling angular frequency as

$\Omega_T = 2\Delta\Omega = 2(\Omega_2 - \Omega_1) = 80\pi = \frac{2 \times 200\pi}{M}$ , which is satisfied for  $M = 5$ . The sampling frequency is therefore  $F_T = 40$  Hz.



**(b)**  $\Omega_2 = 160\pi$ ,  $\Omega_1 = 120\pi$ . Thus,  $\Delta\Omega = \Omega_2 - \Omega_1 = 40\pi$ . Note  $\Delta\Omega$  is an integer multiple of  $\Omega_2$ . Hence, we choose the sampling angular frequency as

$\Omega_T = 2\Delta\Omega = 2(\Omega_2 - \Omega_1) = 80\pi = \frac{2 \times 160\pi}{M}$ , which is satisfied for  $M = 4$ . The sampling frequency is therefore  $F_T = 40$  Hz.

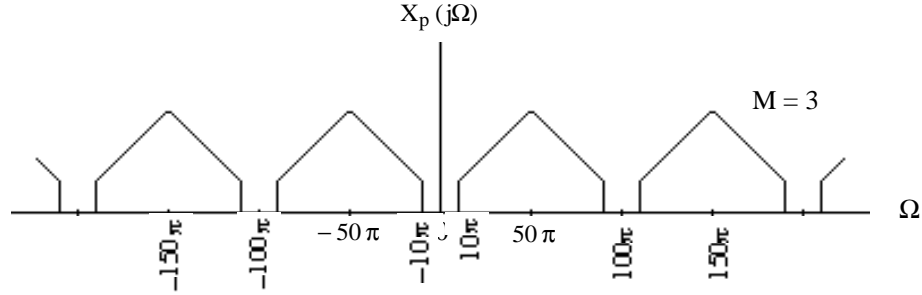




(c)  $\Omega_2 = 150\pi$ ,  $\Omega_1 = 110\pi$ . Thus,  $\Delta\Omega = \Omega_2 - \Omega_1 = 40\pi$ . Note  $\Delta\Omega$  is not an integer multiple of  $\Omega_2$ . Hence, we extend the bandwidth to the left by assuming the lowest frequency to be  $\Omega_0$  and choose the sampling angular frequency satisfying the equation

$$\Omega_T = 2\Delta\Omega = 2(\Omega_2 - \Omega_0) = \frac{2 \times 150\pi}{M}, \text{ which is satisfied for } \Omega_0 = 100\pi \text{ and } M = 3. \text{ The}$$

sampling frequency is therefore  $F_T = 50$  Hz.



5.9  $\alpha_p = -20 \log_{10}(1 - \delta_p)$  and  $\alpha_s = -20 \log_{10} \delta_s$ . Therefore,  $\delta_p = 1 - 10^{-\alpha_p/20}$  and  $\delta_s = 10^{-\alpha_s/20}$ .

(a)  $\alpha_p = 0.15$ ,  $\alpha_s = 43$ . Hence,  $\delta_p = 0.0171$  and  $\delta_s = 0.0071$

(b)  $\alpha_p = 0.04$ ,  $\alpha_s = 57$ . Hence,  $\delta_p = 0.0046$  and  $\delta_s = 0.0014$

(c)  $\alpha_p = 0.23$ ,  $\alpha_s = 39$ . Hence,  $\delta_p = 0.0261$  and  $\delta_s = 0.0112$

5.10  $H_1(s) = \frac{a}{s+a}$ . Thus,  $H_1(j\Omega) = \frac{a}{j\Omega+a}$ , and hence, i.e.  $|H_1(j\Omega)|^2 = \frac{a^2}{a^2 + \Omega^2}$ .

$|H_1(j\Omega)|^2$  is a monotonically decreasing function with  $|H_1(j0)| = 1$  and  $|H_1(j\infty)| = 0$ . Let the

3-dB cutoff frequency be given by  $\Omega_c$ . Then,  $|H_1(j\Omega_c)|^2 = \frac{a^2}{a^2 + \Omega_c^2} = \frac{1}{2}$ , and hence,  $\Omega_c = a$ .

5.11  $H_2(s) = \frac{s}{s+a}$ . Thus,  $H_2(j\Omega) = \frac{j\Omega}{j\Omega+a}$ , and hence,  $|H_2(j\Omega)|^2 = \frac{\Omega^2}{a^2 + \Omega^2}$ .

$|H_2(j\Omega)|^2$  is a monotonically increasing function with  $|H_2(j0)| = 0$  and  $|H_2(j\infty)| = 1$ . The 3-

dB cutoff frequency is given by  $\frac{\Omega_c^2}{a^2 + \Omega_c^2} = \frac{1}{2}$ , and hence,  $\Omega_c = a$ .

5.12  $H_1(s) = \frac{a}{s+a} = \frac{1}{2} \left( 1 - \frac{s-a}{s+a} \right)$ , and  $H_2(s) = \frac{s}{s+a} = \frac{1}{2} \left( 1 + \frac{s-a}{s+a} \right)$ . Thus,  $A_1(s) = 1$  and  $A_2(s) =$

$\frac{s-a}{s+a}$ . Since  $|A_1(j\Omega)| = 1$  and  $|A_2(j\Omega)| = 1 \forall \Omega$  hence  $A_1(s)$  and  $A_2(s)$  are both allpass functions.

5.13  $H_1(s) = \frac{bs}{s^2 + bs + \Omega_0^2}$ . Thus,  $H_1(j\Omega) = \frac{j b \Omega}{j b \Omega + \Omega_0^2 - \Omega^2}$ , hence  $|H_1(j\Omega)|^2 = \frac{b^2 \Omega^2}{b^2 \Omega^2 + (\Omega_0^2 - \Omega^2)^2}$ .

Now at  $\Omega = 0$ ,  $|H_1(j0)| = 0$  and at  $\Omega = \infty$ ,  $|H_1(j\infty)| = 0$  and at  $\Omega = \Omega_0$ ,  $|H_1(j\Omega_0)| = 1$ . Hence

$H_1(s)$  has a bandpass response. The 3-dB frequencies are given by  $\frac{b^2\Omega_c^2}{b^2\Omega_c^2 + (\Omega_0^2 - \Omega^2)^2} = \frac{1}{2}$ .

Thus,  $(\Omega_0^2 - \Omega_c^2)^2 = b^2\Omega_c^2$  or  $\Omega_c^4 - (b^2 + 2\Omega_0^2)\Omega_c^2 + \Omega_0^4 = 0$ . Hence if  $\Omega_1$  and  $\Omega_2$  are the roots of this equation, then so are  $-\Omega_1, -\Omega_2$ , and the product of the roots is  $\Omega_0^4$ . This implies  $\Omega_1\Omega_2 = \Omega_0^2$ . Also  $\Omega_1^2 + \Omega_2^2 = b^2 + 2\Omega_0^2$ . Hence  $(\Omega_2 - \Omega_1)^2 = b^2$  which gives the desired result  $\Omega_2 - \Omega_1 = b$ .

$$5.14 \quad H_2(s) = \frac{s^2 + \Omega_0^2}{s^2 + bs + \Omega_0^2}. \quad \text{Thus, } H_2(j\Omega) = \frac{\Omega_0^2 - \Omega^2}{\Omega_0^2 - \Omega^2 + jb\Omega}, \quad \text{hence, } |H_2(j\Omega)|^2 = \frac{(\Omega^2 - \Omega_0^2)^2}{(\Omega^2 - \Omega_0^2)^2 + b^2\Omega^2}.$$

Now,  $|H_2(j0)| = 1$ ,  $|H_2(j\infty)| = 1$  and  $|H_2(j\Omega_0)| = 1$ . Hence,  $H_2(s)$  has a bandstop response. As in the earlier problem one can show that  $\Omega_1\Omega_2 = \Omega_0^2$  and also  $\Omega_2 - \Omega_1 = b$ .

$$5.15 \quad H_1(s) = \frac{1}{2} \left( 1 - \frac{s^2 - bs + \Omega_0^2}{s^2 + bs + \Omega_0^2} \right) = \frac{1}{2} \{A_1(s) - A_2(s)\} \quad \text{and} \quad H_2(s) = \frac{1}{2} \left( 1 + \frac{s^2 - bs + \Omega_0^2}{s^2 + bs + \Omega_0^2} \right) = \frac{1}{2} \{A_1(s) + A_2(s)\}.$$

Thus,  $A_1(s) = 1$  and  $A_2(s) = \frac{s^2 - bs + \Omega_0^2}{s^2 + bs + \Omega_0^2}$ . Now

$|A_2(j\Omega)|^2 = \frac{(\Omega_0^2 - \Omega^2)^2 + b^2\Omega^2}{(\Omega_0^2 - \Omega^2)^2 + b^2\Omega^2} = 1$ , and is hence  $A_2(s)$  is an allpass function.  $A_1(s)$  is seen to be an allpass function.

$$5.16 \quad |H_a(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}. \quad \frac{d^k(1/|H_a(j\Omega)|^2)}{d\Omega^k} = 2N(2N-1)L(2N-k+1) \frac{\Omega^{2N-k}}{\Omega_c^{2N}}. \quad \text{Therefore}$$

$$\left. \frac{d^k(1/|H_a(j\Omega)|^2)}{d\Omega^k} \right|_{\Omega=0} = 0 \quad \text{for } k = 1, 2, \dots, 2N-1. \quad \text{or, equivalently, } \left. \frac{d^k(|H_a(j\Omega)|^2)}{d\Omega^k} \right|_{\Omega=0} = 0 \quad \text{for } k = 1, 2, \dots, 2N-1.$$

$$5.17 \quad 10 \log_{10} \left( \frac{1}{1+\epsilon^2} \right) = -0.5, \quad \text{which yields } \epsilon = 0.3493. \quad 10 \log_{10} \left( \frac{1}{A^2} \right) = -30, \quad \text{which yields}$$

$$A^2 = 1000. \quad \text{Now, } \frac{1}{k} = \frac{\Omega_s}{\Omega_p} = \frac{8}{2.1} = 3.8095238 \quad \text{and} \quad \frac{1}{k_1} = \frac{\sqrt{A^2 - 1}}{\epsilon} = \frac{\sqrt{999}}{0.3493} = 90.486576. \quad \text{Then,}$$

$$\text{from Eq. (5.33) we get } N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = \frac{90.4866}{3.8095} = 3.3684. \quad \text{Hence choose } N = 4 \text{ as the order.}$$

5.18 The poles are given by  $p_1 = \Omega_c e^{j\frac{\pi(N+2l-1)}{2N}}, l = 1, 2, \dots, N$ . Here,  $N = 5$  and  $\Omega_c = 1$ . Hence

$$p_1 = e^{j6\pi/10} = -0.3090 + j0.9511, \quad p_2 = e^{j8\pi/10} = -0.8090 + j0.5878, \quad p_3 = e^{j10\pi/10} = -1, \\ p_4 = e^{j12\pi/10} = -0.8090 - j0.5878 = p_2^*, \quad \text{and} \quad p_5 = e^{j14\pi/10} = -0.3090 - j0.9511 = p_1^*.$$

**5.19** From Eq. (5.39)  $T_N(x) = 2xT_{N-1}(x) - T_{N-2}(x)$  where  $T_N(x)$  is defined in Eq. (5.38).

Case 1:  $|x| \leq 1$ . Making use of Eq. (5.38) in Eq. (5.39) we get

$$\begin{aligned} T_N(x) &= 2x \cos\left((N-1) \cdot \cos^{-1}(x)\right) - \cos\left((N-2) \cdot \cos^{-1}(x)\right) \\ &= 2x \cos(N \cos^{-1} x - \cos^{-1} x) - \cos(N \cos^{-1} x - 2 \cos^{-1} x) \\ &= 2x \left[ \cos(N \cos^{-1} x) \cos(\cos^{-1} x) + \sin(N \cos^{-1} x) \sin(\cos^{-1} x) \right] \\ &\quad - \left[ \cos(N \cos^{-1} x) \cos(2 \cos^{-1} x) + \sin(N \cos^{-1} x) \sin(2 \cos^{-1} x) \right] \\ &= 2x \cos(N \cos^{-1} x) \cdot \cos(\cos^{-1} x) - \cos(N \cos^{-1} x) \cdot \cos(2 \cos^{-1} x) \\ &= 2x^2 \cos(N \cos^{-1} x) - \cos(N \cos^{-1} x) \cdot \left[ 2 \cos^2(\cos^{-1} x) - 1 \right] \\ &= \cos(N \cos^{-1} x) \left[ 2x^2 - 2x^2 + 1 \right] = \cos(N \cos^{-1} x). \end{aligned}$$

Case 2:  $|x| > 1$ . Making use of Eq. (5.38) in Eq. (5.39) we get

$$\begin{aligned} T_N(x) &= 2x \cosh\left((N-1) \cdot \cosh^{-1}(x)\right) - \cosh\left((N-2) \cdot \cosh^{-1}(x)\right). \text{ Using the trigonometric} \\ &\text{identities } \cosh(A-B) = \cosh(A)\cosh(B) - \sinh(A)\sinh(B), \sinh(2A) = 2\sinh(A)\cosh(A), \text{ and} \\ &\cosh(2A) = 2\cosh^2(A) - 1, \text{ and following a similar algebra as in Case 1, we can show} \\ T_N(x) &= 2x \cosh\left((N-1) \cdot \cosh^{-1}(x)\right) - \cosh\left((N-2) \cdot \cosh^{-1}(x)\right) = \cosh(N \cosh^{-1} x). \end{aligned}$$

**5.20**  $10 \log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -0.5$ , which yields  $\varepsilon = 0.3493$ .  $10 \log_{10}\left(\frac{1}{A^2}\right) = -60$ , which yields  $A^2 =$

$$1000. \text{ Now, } \frac{1}{k} = \frac{\Omega_s}{\Omega_p} = \frac{8}{2.1} = 3.8095238 \text{ and } \frac{1}{k_1} = \frac{\sqrt{A^2-1}}{\varepsilon} = \frac{\sqrt{999}}{0.3493} = 90.486576. \text{ Then,}$$

from Eq. (5.41) we get  $N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = \frac{\cosh^{-1}(90.486576)}{\cosh^{-1}(3.8095238)} = \frac{5.1983}{2.013} = 2.5824$ . We thus choose  $N = 3$  as the order.

**5.21** From Problem 5.18 solution, we observe  $\frac{1}{k} = 3.8095238$  or  $k = 0.2625$ , and  $\frac{1}{k_1} = 90.4865769$

or  $k_1 = 0.011051362$ . Substituting the value of  $k$  in Eq. (5.52a) we get  $k' = 0.964932$ . Then, from Eq. (5.52b) we arrive at  $\rho_0 = 0.017534$ . Substituting the value of  $\rho_0$  in Eq. (5.52c) we get  $\rho = 0.017534$ . Finally from Eq. (5.51) we arrive at  $N = 2.9139$ . We choose  $N = 3$ .

**5.22** From Eq. (5.53)  $B_N(s) = (2N-1)B_{N-1}(s) + s^2 B_{N-2}(s)$ , where  $B_1(s) = s+1$  and  $B_2(s) = s^2+3s+3$ . Thus,  $B_3(s) = 5B_2(s) + s^2 B_1(s) = 5(s^2+3s+3) + s^2(s+1) = s^3+6s^2+15s+15$ , and  $B_4(s) = 7B_3(s) + s^2 B_2(s) = 7(s^3+6s^2+15s+15) + s^2(s^2+3s+3) = s^4+10s^3+45s^2+105s+105$ .

Thus,  $B_5(s) = 9B_4(s) + s^2 B_3(s) = 9(s^4+10s^3+45s^2+105s+105) + s^2(s^3+6s^2+15s+15) = s^5+15s^4+105s^3+420s^2+945s+945$ , and

$$B_6(s) = 11B_5(s) + s^2 B_4(s) = 11(s^5 + 15s^4 + 105s^3 + 420s^2 + 945s + 945) + s^2(s^4 + 10s^3 + 45s^2 + 105s + 105) = s^6 + 21s^5 + 210s^4 + 1260s^3 + 4725s^2 + 10395s + 10395.$$

**5.23** We use the lowpass-to-highpass transformation given in Eq. (5.59) where now

$\Omega_p = 2\pi(0.2) = 0.4\pi$  and  $\hat{\Omega}_p = 2\pi(2) = 4\pi$ . Therefore, from Eq. (5.59) we get the desired

transformation as  $s \rightarrow \frac{\Omega_p \hat{\Omega}_p}{s} = \frac{1.6\pi^2}{s} = \frac{15.791367}{s}$ . Therefore,

$$H_{HP}(s) = H_{LP}(s) \Big|_{s \rightarrow \frac{15.791367}{s}} = \frac{4.52}{\left(\frac{15.791367}{s}\right)^2 + 3\left(\frac{15.791367}{s}\right) + 4.52} = \frac{4.52 s^2}{4.52 s^2 + 47.3741s + 2.49367}.$$

**5.24** We use the lowpass-to-bandpass transformation given in Eq. (5.61) where now

$\Omega_p = 2\pi(0.16) = 0.32\pi$ ,  $\hat{\Omega}_o = 2\pi(3) = 6\pi$ , and  $\hat{\Omega}_{p2} - \hat{\Omega}_{p1} = 2\pi(0.5) = \pi$ . Substituting these

values in Eq. (5.59) we get the desired transformation  $s \rightarrow 0.32\pi \left( \frac{s^2 + 36\pi^2}{\pi s} \right) = \frac{s^2 + 36\pi^2}{3.125s}$ .

$$\begin{aligned} \text{Therefore } H_{BP}(s) &= H_{LP}(s) \Big|_{s \rightarrow \frac{s^2 + 36\pi^2}{3.125s}} = \frac{0.056 \left[ \left( \frac{s^2 + 36\pi^2}{3.125s} \right)^2 + 17.95 \right]}{\left( \frac{s^2 + 36\pi^2}{3.125s} \right)^2 + 1.06 \left( \frac{s^2 + 36\pi^2}{3.125s} \right) + 1.13} \\ &= \frac{0.056(s^4 + 49.61s^2 + 70695.62)}{s^4 + 3.3125s^3 + 721.64667s^2 + 1176.95s + 126242.182}. \end{aligned}$$

**5.25**  $\hat{\Omega}_p = 2\pi(5) \times 10^6$ ,  $\hat{\Omega}_s = 2\pi(0.5) \times 10^6$ ,  $\hat{\alpha}_p = 0.3$  dB and  $\hat{\alpha}_s = 45$  dB. From Figure 5.14 we

observe  $10 \log_{10} \left( \frac{1}{1 + \epsilon^2} \right) = -0.3$ , hence,  $\epsilon^2 = 10^{0.03} - 1 = 0.0715193$ , or  $\epsilon = 0.26743$ . From

Figure 5.14 we also note that  $10 \log_{10} \left( \frac{1}{A^2} \right) = -45$ , and hence  $A^2 = 10^{4.5} = 31622.7766$ . Using

these values in Eq. (5.30) we get  $k_1 = \frac{\epsilon}{\sqrt{A^2 - 1}} = 0.001503898$ .

To develop the bandedges of the lowpass prototype, we set  $\Omega_p = 1$  and obtain using Eq. (5.59)

$\Omega_s = \frac{\hat{\Omega}_p}{\hat{\Omega}_s} = \frac{5}{0.5} = 10$ . Next, using Eq. (5.29) we obtain  $k = \frac{\Omega_p}{\Omega_s} = \frac{1}{10} = 0.1$ . Substituting the

values of  $k$  and  $k_1$  in Eq. (5.33) we get  $N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = 2.82278$ . We choose  $N = 3$  as the order of the lowpass filter which is also the order of the highpass filter.

To verify using MATLAB er use the statement

```
[N, Wn] = buttord(1, 10, 0.3, 45, 's')
```

which yields  $N = 3$  and  $Wn = 1.77828878$ .

**5.26**  $\hat{F}_{p1} = 20 \times 10^3$ ,  $\hat{F}_{p2} = 45 \times 10^3$ ,  $\hat{F}_{s1} = 10 \times 10^3$ ,  $\hat{F}_{s2} = 60 \times 10^3$ ,  $\alpha_p = 0.5$  dB and  $\alpha_s = 40$  dB.

We observe  $\hat{F}_{p1}\hat{F}_{p2} = 20 \times 45 \times 10^6 = 9 \times 10^8$ , and  $\hat{F}_{s1}\hat{F}_{s2} = 10 \times 60 \times 10^6 = 6 \times 10^8$ . Since  $\hat{F}_{s1}\hat{F}_{s2} \neq \hat{F}_{p1}\hat{F}_{p2}$ , we adjust the the stopband edge on the left to  $\hat{F}_{s1} = 15 \times 10^3$  in which case  $\hat{F}_{s1}\hat{F}_{s2} = \hat{F}_{p1}\hat{F}_{p2} = F_o^2 = 9 \times 10^8$ . The angular center frequency of the desired bandpass filter is therefore  $\Omega_o = 2\pi F_o = 2\pi \times 30 \times 10^3$ . The passband bandwidth is  $B_w = \hat{\Omega}_{p2} - \hat{\Omega}_{p1} = 2\pi \times 25 \times 10^3$ .

To determine the banedges of the prototype lowpass filter we set  $\Omega_p = 1$  and then using Eq.

$$(5.62) \text{ we obtain } \Omega_s = \Omega_p \frac{\hat{\Omega}_o^2 - \hat{\Omega}_{s1}^2}{\hat{\Omega}_{s1} B_w} = \frac{30^2 - 15^2}{15 \times 25} = 1.8. \text{ Thus, } k = \frac{\Omega_p}{\Omega_s} = \frac{1}{1.8} = 0.5555555556.$$

Now,  $10 \log_{10} \left( \frac{1}{1 + \varepsilon^2} \right) = -0.5$ , and hence,  $\varepsilon^2 = 10^{0.05} - 1 = 0.1220184543$ , or  $\varepsilon = 0.34931140019$ .

In addition  $10 \log_{10} \left( \frac{1}{A^2} \right) = -40$  or  $A^2 = 10^4 = 10000$ . Using these values in Eq. (5.30) we get

$$k_1 = \frac{\varepsilon}{\sqrt{A^2 - 1}} = 0.00349328867069. \text{ From Eq. (5.52a) we get}$$

$$k' = \sqrt{1 - k^2} = 0.831479419283, \text{ and then from Eq. (5.52b) we get}$$

$$\rho_o = \frac{1 - \sqrt{k'}}{2(1 + \sqrt{k'})} = 0.02305223718137. \text{ Substituting the value of } \rho_o \text{ in Eq. (5.52c) we then get}$$

$\rho = 0.02305225$ . Finally, substituting the values of  $\rho$  and  $k_1$  in Eq. (5.51) we arrive at

$$N = \frac{2 \log_{10}(4/k_1)}{\log_{10}(1/\rho)} = 3.7795. \text{ We choose } N = 4 \text{ as the order for the prototype lowpass filter.}$$

The order of the desired bandpass filter is therefore 8.

Using the statement `[N,Wn]=ellipord(1,1.8,0.5,40,'s')` we get  $N = 4$  and  $Wn = 1$ .

**5.27**  $\hat{F}_{p1} = 10 \times 10^6$ ,  $\hat{F}_{p2} = 70 \times 10^6$ ,  $\hat{F}_{s1} = 20 \times 10^6$ , and  $\hat{F}_{s2} = 45 \times 10^6$ . We observe  $\hat{F}_{p1}\hat{F}_{p2} = 700 \times 10^{12}$ , and  $\hat{F}_{s1}\hat{F}_{s2} = 900 \times 10^{12}$ . Since  $\hat{F}_{p1}\hat{F}_{p2} \neq \hat{F}_{s1}\hat{F}_{s2}$ . we adjust the right stopband edge to  $\hat{F}_{s2} = 35 \times 10^6$ , in which case  $\hat{F}_{p1}\hat{F}_{p2} = \hat{F}_{s1}\hat{F}_{s2} = \hat{F}_o^2 = 700 \times 10^{12}$ . The stopband bandwidth is  $B_w = \hat{\Omega}_{s2} - \hat{\Omega}_{s1} = 2\pi \times 15 \times 10^6$ .

Now,  $10 \log_{10} \left( \frac{1}{1 + \epsilon^2} \right) = -0.5$ , and hence,  $\epsilon^2 = 10^{0.05} - 1 = 0.1220184543$ , or  $\epsilon = 0.34931140019$ .

In addition  $10 \log_{10} \left( \frac{1}{A^2} \right) = -30$  or  $A^2 = 10^3 = 1000$ . Using these values in Eq. (5.30) we get

$$k_1 = \frac{\epsilon}{\sqrt{A^2 - 1}} = 0.01105172361656.$$

To determine the bandedges of the prototype lowpass filter we set  $\Omega_s = 1$  and then using Eq.

$$(5.65) \text{ we obtain } \Omega_p = \Omega_s \frac{\hat{\Omega}_{sl} B_w}{\hat{\Omega}_o^2 - \hat{\Omega}_{sl}^2} = 0.3157894. \text{ Therefore, } k = \frac{\Omega_p}{\Omega_s} = 0.3157894.$$

Substituting the values of  $k$  and  $k_1$  in Eq. (5.41) we get  $N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = 2.856$ . We choose  $N = 3$  as the order for the prototype lowpass filter.

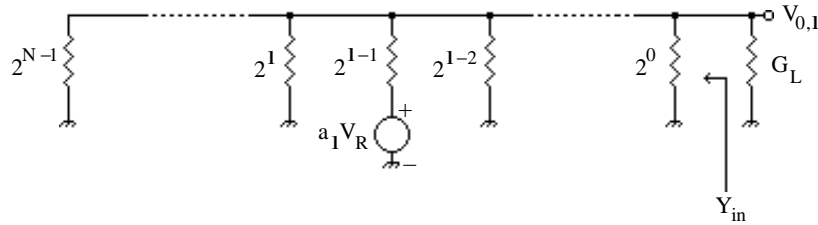
The order of the desired bandstop filter is therefore 6.

Using the statement `[N, Wn] = cheblord(0.3157894, 1, 0.5, 30, 's')` we get  $N = 3$  and  $Wn = 0.3157894$ .

**5.28** From Eq. (5.59), the difference in dB in the attenuation levels at  $\Omega_p$  and  $\Omega_o$  is given by

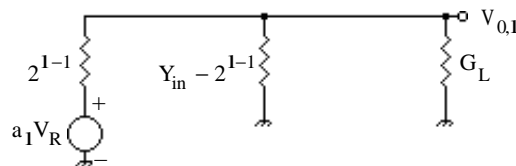
$20N \log_{10} \left( \frac{\Omega_o}{\Omega_p} \right)$ . Hence for  $\Omega_o = 2\Omega_p$ , attenuation difference in dB is equal to  $20N \log_{10} 2 = 6.0206 N$ . Likewise, for  $\Omega_o = 3\Omega_p$ , attenuation difference in dB is equal to  $20N \log_{10} 3 = 9.5424 N$ . Finally, for  $\Omega_o = 4\Omega_p$ , attenuation difference in dB is equal to  $20N \log_{10} 4 = 12.0412 N$ .

**5.29** The equivalent representation of the D/A converter of Figure 5.39 reduces to the circuit shown below if the  $l$ -th is ON and the remaining bits are OFF, i.e.  $a_l = 1$ , and  $a_k = 0$ ,  $k \neq l$ .



In the above circuit,  $Y_{in}$  is the total conductance seen by the load conductance  $G_L$  which is

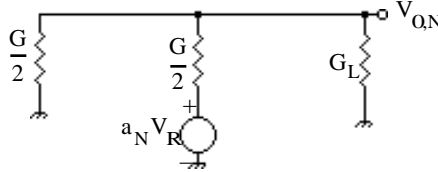
given by  $Y_{in} = \sum_{i=0}^{N-1} 2^i = 2^N - 1$ . The above circuit can be redrawn as indicated below:



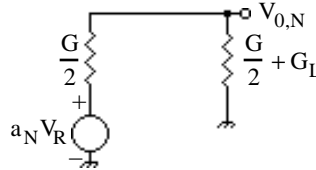
Using the voltage divider relation we then get  $V_{0,1} = \frac{2^{1-1}}{Y_i + G_L} a_1 V_R$ . Using the superposition theorem, the general expression for the output voltage  $V_0$  is thus given by

$$V_0 = \sum_{i=1}^N \frac{2^{i-1}}{Y_i + G_L} a_i V_R = \sum_{i=1}^N 2^{i-1} a_i \left( \frac{R_L}{1 + (2^N - 1)R_L} \right) V_R.$$

**5.30** The equivalent representation of the D/A converter of Figure 5.40 reduces to the circuit shown below if the N-th bit is ON and the remaining bits are OFF, i.e.  $a_N = 1$ , and  $a_k = 0$ ,  $k \neq N$ .

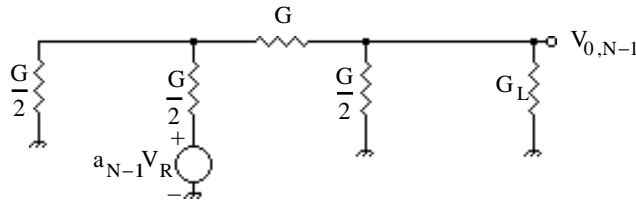


which simplifies to the one shown below

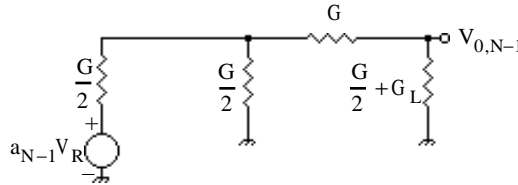


Using the voltage-divider relation we then get  $V_{0,N} = \frac{\frac{G}{2}}{\frac{G}{2} + G_L + \frac{G}{2}} a_N V_R = \frac{R_L}{2(R + R_L)} a_N V_R$ .

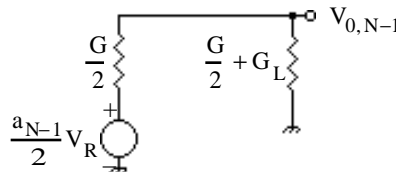
The equivalent representation of the D/A converter of Figure 5.40 reduces to the circuit shown below if the (N-1)-th bit is ON and the remaining bits are OFF, i.e.  $a_{N-1} = 1$ , and  $a_k = 0$ ,  $k \neq N$ .



The above circuit simplifies to the one shown below:



Its Thevenin equivalent circuit is indicated below:



from which we readily obtain

$$V_{0,N-1} = \frac{\frac{G}{2}}{G+G_L} \frac{a_{N-1}}{2} V_R = \frac{R_L}{2(R_L+R)} \frac{a_{N-1}}{2} V_R.$$

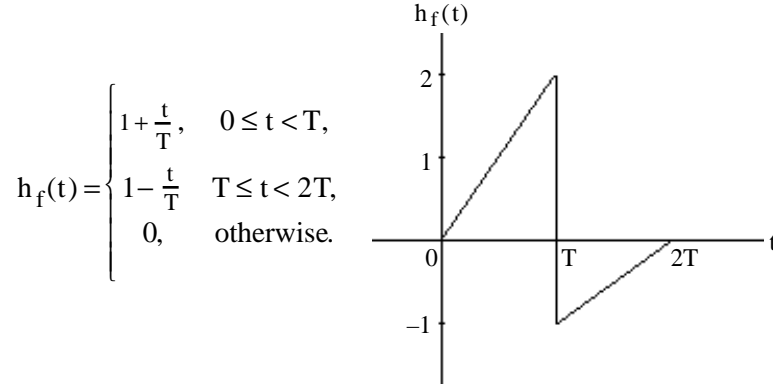
Following the same procedure we can show that if the 1 -th bit is ON and the remaining bits are OFF, i.e.  $a_1 = 1$ , and  $a_k = 0$ ,  $k \neq 1$ , then

$$V_{0,1} = \frac{R_L}{2(R_L+R)} \frac{a_1}{2^{N-1}} V_R.$$

Hence, in general we have

$$V_0 = \sum_{l=1}^N \frac{R_L}{2(R_L+R)} \frac{a_l}{2^{N-1}} V_R.$$

**5.31** From the input-output relation of the first-order hold we get the expression for the impulse response as  $h_f(t) = \delta(nT) + \frac{\delta(nT) - \delta(nT - T)}{T}(t - nT)$ ,  $nT \leq t < (n+1)T$ . In the range  $0 \leq t < T$ , the impulse response is given by  $h_f(t) = \delta(0) + \frac{\delta(0) - \delta(-T)}{T}t = 1 + \frac{t}{T}$ . Likewise, in the range  $T \leq t < 2T$ , the impulse response is given by  $h_f(t) = \delta(T) + \frac{\delta(T) - \delta(T - T)}{T}(t - T) = 1 - \frac{t}{T}$ . Outside these two ranges,  $h_f(t) = 0$ . Hence we have



Using the unit step function  $\mu(t)$  we can write

$$\begin{aligned} h_f(t) &= \left(1 + \frac{t}{T}\right) [\mu(t) - \mu(t - T)] + \left(1 - \frac{t}{T}\right) [\mu(t - T) - \mu(t - 2T)] \\ &= \mu(t) + \frac{t}{T} \mu(t) - \frac{2(t - T)}{T} \mu(t - T) + 2\mu(t - T) - \mu(t - 2T) + \left(\frac{t - 2T}{T}\right) \mu(t - 2T) + 2\mu(t - 2T). \end{aligned}$$

Taking the Laplace transform of the above equation we arrive at the transfer function

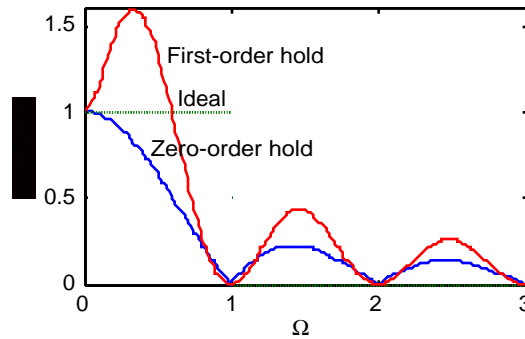
$$H_f(s) = \frac{1}{s} + \frac{1}{Ts^2} - \frac{2}{T} \cdot \frac{e^{-sT}}{s^2} + 2 \frac{e^{-sT}}{s} - \frac{e^{-2sT}}{s} + \frac{1}{T} \cdot \frac{e^{-2sT}}{s^2} + 2 \frac{e^{-2sT}}{s} = \left(\frac{1+sT}{T}\right) \left(\frac{1-e^{-sT}}{s}\right)^2.$$

Hence, the frequency response is given by

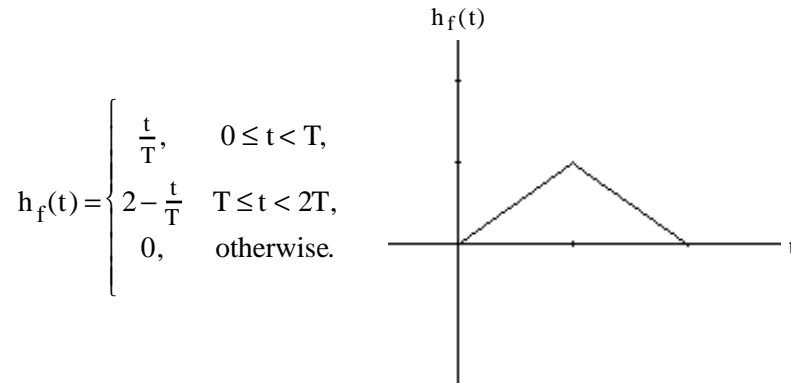
$$H_f(j\Omega) = \left(\frac{1+j\Omega T}{T}\right) \left(\frac{1-e^{-j\Omega T}}{j\Omega}\right)^2 = T \sqrt{1+\Omega^2 T^2} \left(\frac{2\sin(\Omega T/2)}{\Omega T}\right)^2 e^{j\Omega T} e^{\tan^{-1} \Omega T}.$$



A plot of the magnitude responses of the ideal filter, zero-order hold and the first-order hold is shown below:



**5.32** From the input-output relation of the linear interpolator we get the expression for the impulse response as  $h_f(t) = \delta(nT - T) + \frac{\delta(nT) - \delta(nT - T)}{T}(t - NT)$ ,  $nT \leq t < (n+1)T$ . In the range  $0 \leq t < T$ , the impulse response is given by  $h_f(t) = \delta(-T) + \frac{\delta(0) - \delta(-T)}{T}t$ , and in the range  $T \leq t < 2T$ , the impulse response is given by  $h_f(t) = \delta(0) + \frac{\delta(T) - \delta(0)}{T}(t - T)$ , Outside these two ranges,  $h_f(t) = 0$ . Hence we have



Using the unit step function  $\mu(t)$  we can write

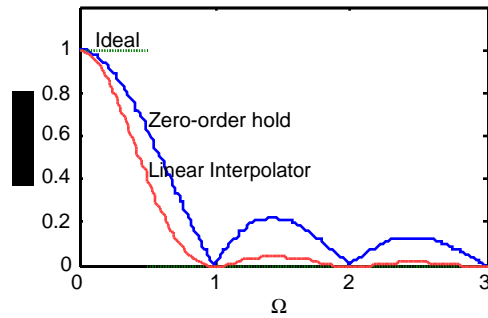
$$\begin{aligned} h_f(t) &= \frac{t}{T} [\mu(t) - \mu(t - T)] + \left(2 - \frac{t}{T}\right) [\mu(t - T) - \mu(t - 2T)] \\ &= \frac{t}{T} \mu(t) - \frac{2(t - T)}{T} \mu(t - T) + \frac{(t - 2T)}{T} \mu(t - 2T). \end{aligned}$$

Taking the Laplace transform of the above equation we arrive at the transfer function

$$H_f(s) = \frac{1}{s^2 T} - \frac{2e^{-sT}}{s^2 T} + \frac{e^{-2sT}}{s^2 T} = T \left( \frac{1 - e^{-sT}}{sT} \right)^2. \text{ Hence, the frequency response is given by}$$

$$H_f(j\Omega) = T \left( \frac{1 - e^{-j\Omega T}}{j\Omega T} \right)^2 = T \left( \frac{\sin(\Omega T / 2)}{\Omega T / 2} \right)^2 e^{-j\Omega T}.$$

A plot of the magnitude responses of the ideal filter, zero-order hold and the first-order hold is shown below:



**M5.1** Using the MATLAB statement

```
[N,Wn] = buttord(2*pi*2100, 2*pi*8000, 0.5, 30, 's')
```

we arrive at  $N = 4$  and the 3-dB passband edge frequency  $W_n = 21199.429284$  rad/sec.

**M5.2** Using the MATLAB statement

```
[N,Wn] = cheblord(2*pi*2100, 2*pi*8000, 0.5, 30, 's')
```

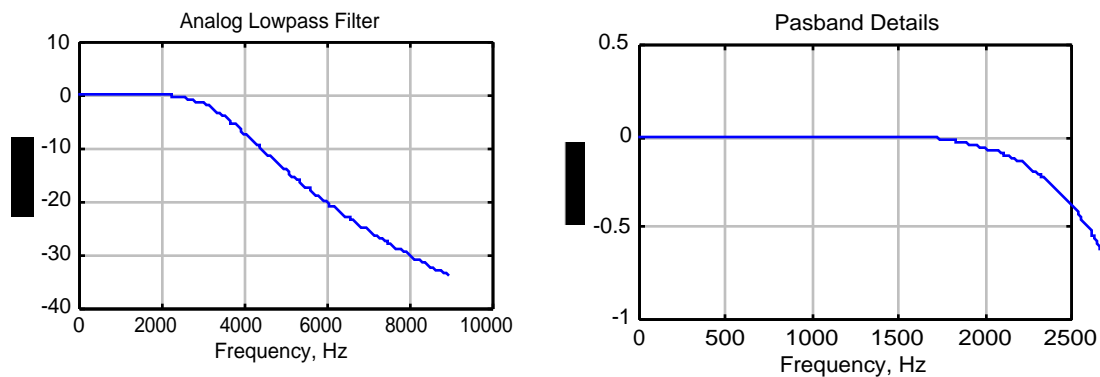
we arrive at  $N = 3$  and the passband edge frequency  $W_n = 13194.689145$  rad/sec.

**M5.3** Using the MATLAB statement

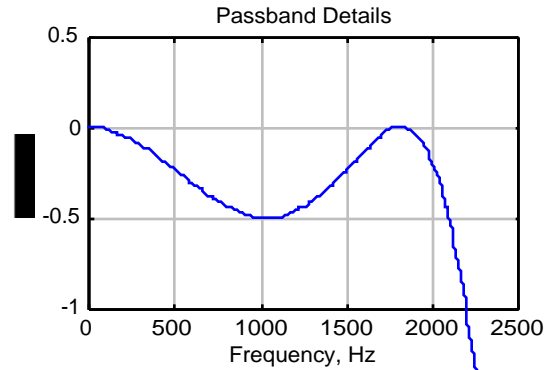
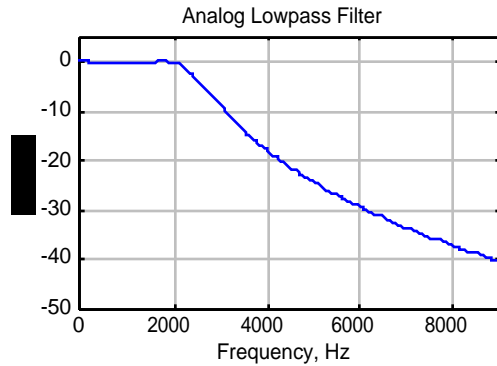
```
[N,Wn] = ellipord(2*pi*2100, 2*pi*8000, 0.5, 30, 's')
```

we arrive at  $N = 3$  and the passband edge frequency  $W_n = 1.3194689145$  rad/sec.

**M5.4** We use  $N = 4$  and  $W_n = 21199.429284$  computed in Exercise M5.1 in Program M5\_2 and use  $\omega = 0:2\pi:2\pi*9000;$  to evaluate the frequency points. The gain plot developed by running Program 5\_2 is as shown below:



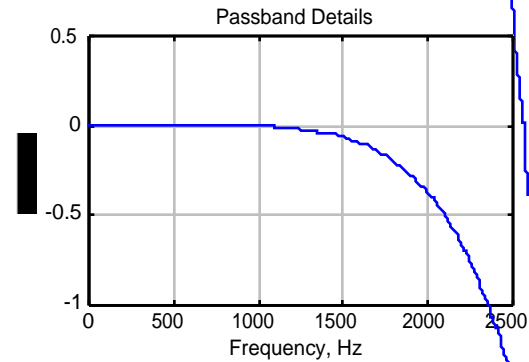
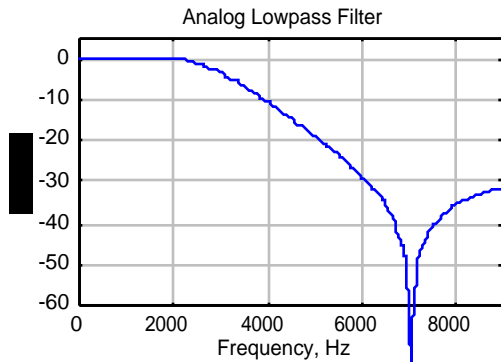
**M5.5** We use  $N = 3$  computed in Exercise M5.2 and  $F_p = 2\pi*2100$  and  $R_p = 0.5$  in Program M5\_3 and use  $\omega = 0:2\pi:2\pi*8000;$  to evaluate the frequency points. The gain plot developed by running Program 5\_3 is as shown below:



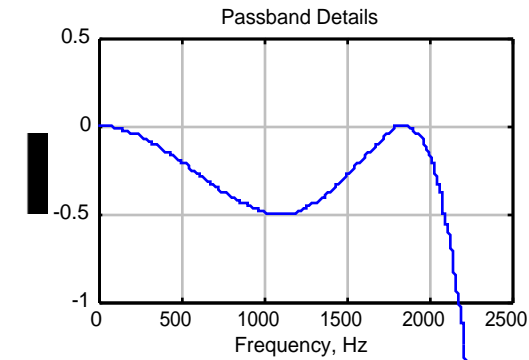
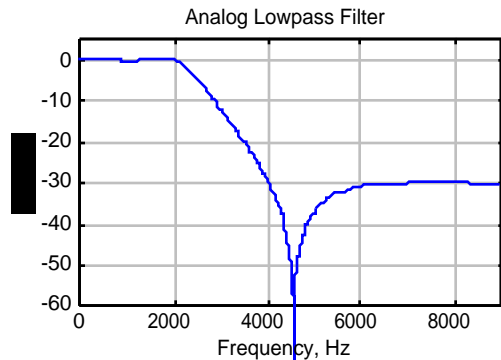
**M5.6** We replace the statement

```
Rp = input('Passband edge frequency in Hz = '); with
Rs = input('Stopband edge frequency in Hz = '); and replace
[num,den] = cheby1(N,Rp,Fp,'s'); with
[num,den] = cheby2(N,Rs,Fs,'s'); to modify Program 5_3 and use N = 3
computed in Exercise M5.2 and Rs = 30 in the modified Program M5_3 and use w =
0:2*pi:2*pi*8000; to evaluate the frequency points.
```

The gain plot developed by running the modified Program 5\_3 is as shown below:



**M5.7** We use  $N = 3$  and  $\omega_n = 1.3194689145$  computed in Exercise M5.3 in Program M5\_3 and use  $w = 0:2\pi:2\pi*9000$ ; to evaluate the frequency points. The gain plot developed by running Program 5\_3 is as shown below:



**M5.8** The MATLAB program used is as given below:

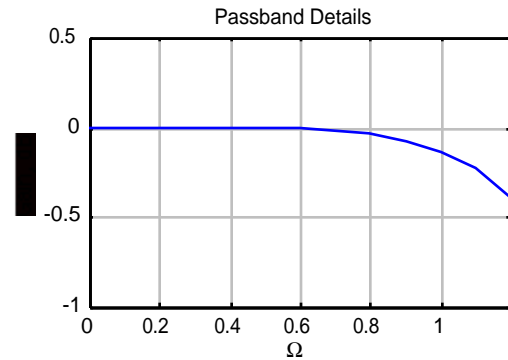
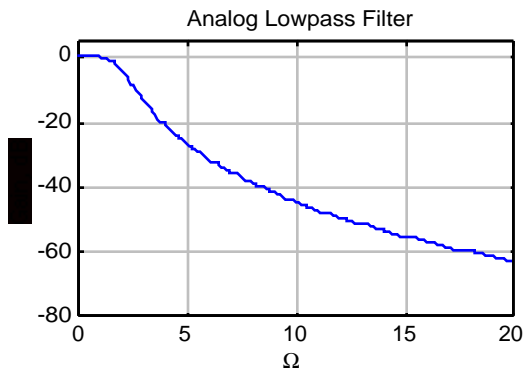
```
[N,Wn] = buttord(1,10,0.3,45,'s')
```

```

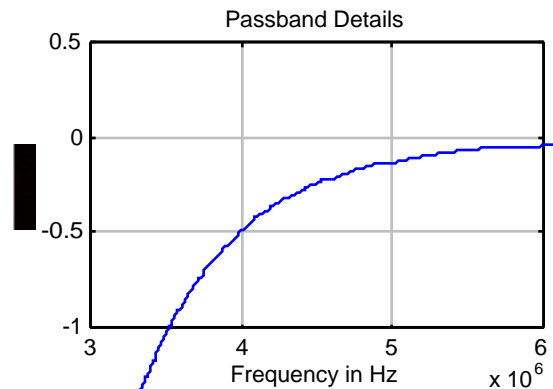
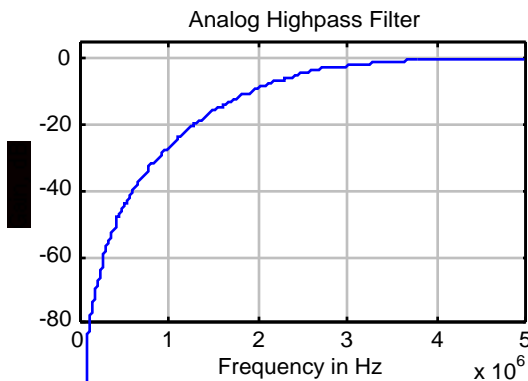
[B,A] = butter(N,Wn,'s')
[num,den] = lp2hp(B,A,2*pi*5e6)
figure(1)
omega = [0: 0.1: 20];
h = freqs(B,A,omega);
gain = 20*log10(abs(h));
plot(omega,gain);grid;axis([0 20 -80 5]);
xlabel('\Omega'); ylabel('Gain, dB')
title('Analog Lowpass Filter')
figure(2)
omega = [0: 10000: 10e6*pi];
h = freqs(num,den,omega);
gain = 20*log10(abs(h));
plot(omega/(2*pi),gain);grid;axis([0 5e6 -80 5]);
xlabel('Frequency in Hz'); ylabel('Gain, dB')
title('Analog Highpass Filter')

```

$$H_{LP}(s) = \frac{5.6235}{s^3 + 3.5566s^2 + 6.3246s + 5.6235}$$



$$H_{HP}(s) = \frac{s^3 + 4.4345 \times 10^{-9} s^2 + 9.0568 \times 10^{-2} s + 4.2847 \times 10^{-11}}{s^3 + 3.5333 \times 10^7 s^2 + 6.242 \times 10^{14} s + 5.5137 \times 10^{21}}$$



**M5.9** The MATLAB program used is as given below:

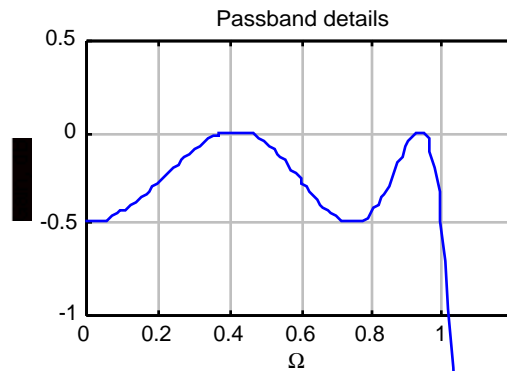
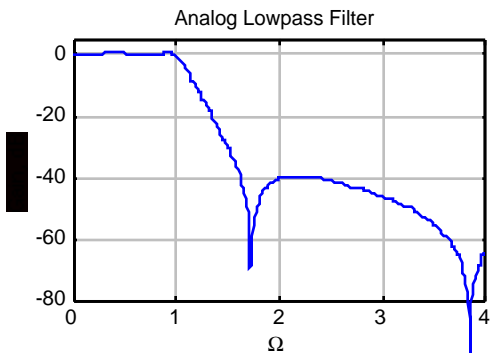
```
[N,Wn] = ellipord(1,1.8,0.5,40,'s')
```

```

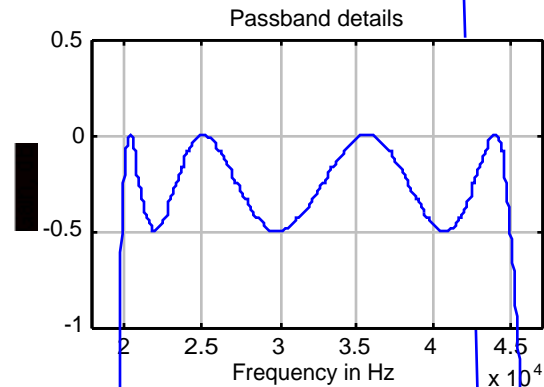
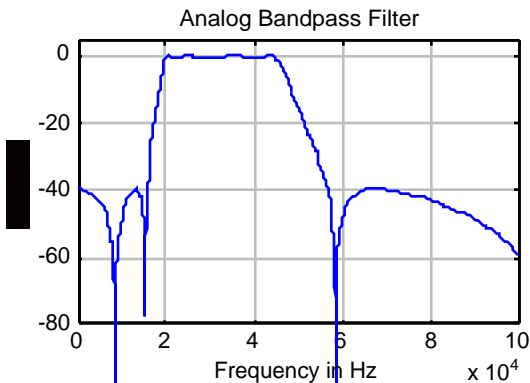
[B,A] = ellip(N,0.5,40,Wn,'s')
[num,den] = lp2bp(B,A,2*pi*30e3,2*pi*25e3)
figure(1)
omega = [0: 0.01: 4];
h = freqs(B,A,omega);
gain = 20*log10(abs(h));
plot(omega,gain);grid;axis([0 4 -80 5]);
xlabel('\Omega'); ylabel('Gain, dB')
title('Analog Lowpass Filter')
figure(2)
omega = [0: 200: 200e3*pi];
h = freqs(num,den,omega);
gain = 20*log10(abs(h));
plot(omega/(2*pi),gain);grid;axis([0 100e3 -80 5]);
xlabel('Frequency in Hz'); ylabel('Gain, dB')
title('Analog Bandpass Filter')

```

$$H_{LP}(s) = \frac{0.01s^4 + 0.17921s^2 + 0.44866}{s^4 + 1.1789s^3 + 1.7558s^2 + 1.0845s + 0.47524}$$



$$H_{BP}(s) = \frac{1.0002 \times 10^{-2} s^8 - 2.3081 \times 10^{-12} s^7 + 5.8434 \times 10^9 s^6 - 6.4605 \times 10^{-1} s^5 + 6.6313 \times 10^{20} s^4 - 3.9221 \times 10^{10} s^3 + 7.3769 \times 10^{30} s^2 - 3.1971 \times 10^{20} s + 1.5941 \times 10^{40}}{s^8 + 1.8519 \times 10^5 s^7 + 1.8544 \times 10^{11} s^6 + 2.3943 \times 10^{16} s^5 + 1.0942 \times 10^{22} s^4 + 8.507 \times 10^{26} s^3 + 2.3411 \times 10^{32} s^2 + 8.3065 \times 10^{36} s + 1.5937 \times 10^{42}}$$



**M5.10** The MATLAB program used is as given below:

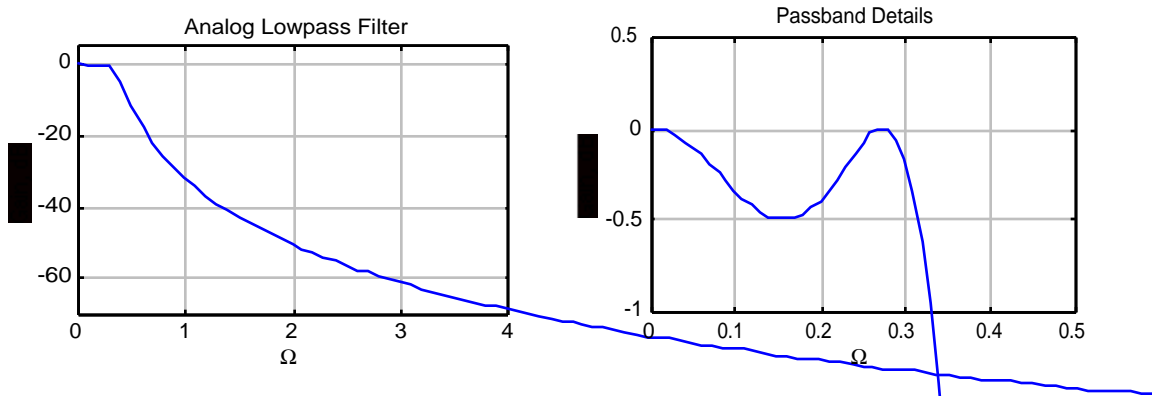
```
[N,Wn] = cheblord(0.3157894,1,0.5,30,'s')
```

```

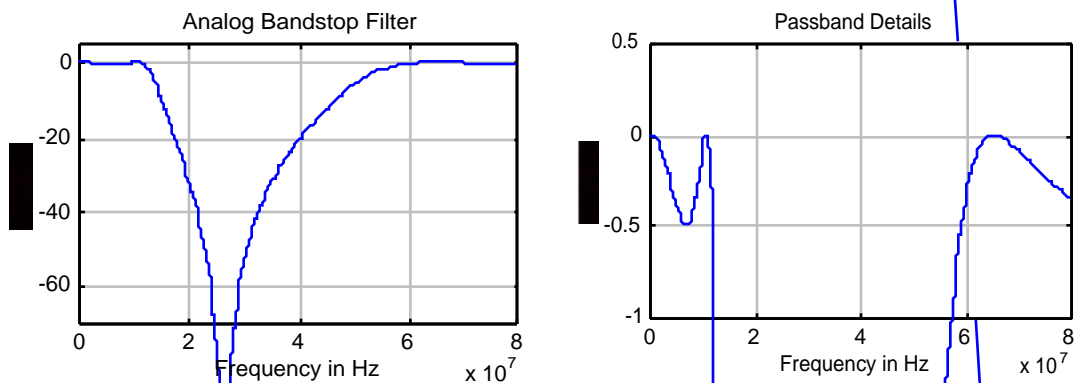
[B,A] = cheby1(N,0.5, Wn,'s')
[num,den] = lp2bs(B,A,2*pi*sqrt(700)*10^6,2*pi*15e6)
figure(1)
omega = [0: 0.01: 10];
h = freqs(B,A,omega);
gain = 20*log10(abs(h));
plot (omega,gain);grid;axis([0 4 -70 5]);
xlabel('\Omega'); ylabel('Gain, dB')
title('Analog Lowpass Filter')
figure(2)
omega = [0: 100000: 160e6*pi];
h = freqs(num,den,omega);
gain = 20*log10(abs(h));
plot (omega/(2*pi),gain);grid;axis([0 80e6 -70 5]);
xlabel('Frequency in Hz'); ylabel('Gain, dB')
title('Analog Bandstop Filter')

```

$$H_{LP}(s) = \frac{0.022538}{s^3 + 0.39566s^2 - 0.15306s + 0.022538}$$



$$H_{BS}(s) = \frac{s^6 - 2.6591 \times 10^{-8} s^5 + 8.2905 \times 10^{16} s^4 - 1.4698 \times 10^9 s^3 + 2.2911 \times 10^{33} s^2 - 2.031 \times 10^{25} s + 2.1104 \times 10^{49}}{s^6 + 6.4007 \times 10^8 s^5 + 2.3884 \times 10^{17} s^4 + 7.2521 \times 10^{25} s^3 + 6.6003 \times 10^{33} s^2 + 4.8881 \times 10^{41} s + 2.1104 \times 10^{49}}$$

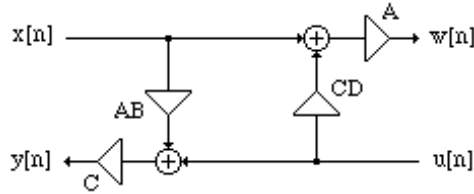


**M5.11** The MATLAB program to generate the plots of Figure 5.48 is given below:

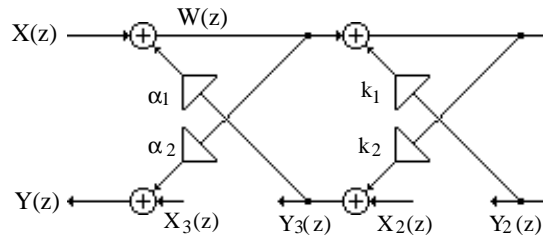
```
% Droop Compensation
w = 0:pi/100:pi;
h1 = freqz([-1/16 9/8 -1/16], 1, w);
h2 = freqz(9, [8 1], w);
w1 = 0;
for n = 1:101
h3(n) = sin(w1/2)/(w1/2);
w1 = w1 + pi/100;
end
m1 = 20*log10(abs(h1));
m2 = 20*log10(abs(h2));
m3 = 20*log10(abs(h3));
plot(w/pi, m3, 'r-', w/pi, m1+m3, 'b--', w/pi, m2+m3, 'r-');
grid
xlabel('Normalized frequency'); ylabel('Gain, dB');
text(0.43, -0.58, 'DAC', 'sc'); text(0.85, -0.58, 'IIR', 'sc');
text(0.85, -1.67, 'FIR', 'sc');
```

## Chapter 6 (2e)

**6.1** From Figure P6.1, we get  $w[n] = A(x[n] + CDu[n])$ , and  $y[n] = C(ABx[n] + u[n])$ . These two equations can be rewritten as  $w[n] = Ax[n] + ACDu[n]$ , and  $y[n] = CABx[n] + Cu[n]$ . The corresponding realization shown below has no delay-free loop:



**6.2 (a)**

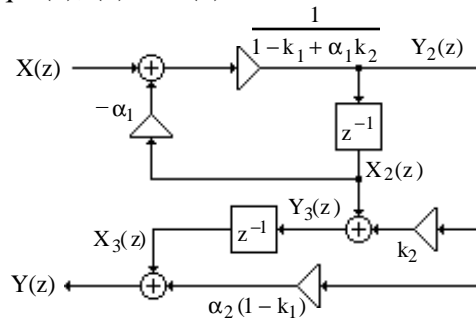


Analyzing the above figure we get (1):  $W(z) = X(z) - \alpha_1 Y_3(z)$ , (2):  $Y_2(z) = W(z) + k_1 Y_2(z)$ , (3):  $Y_3(z) = k_2 Y_2(z) + X_2(z)$ , and (4):  $Y(z) = \alpha_2 W(z) + X_3(z)$ .

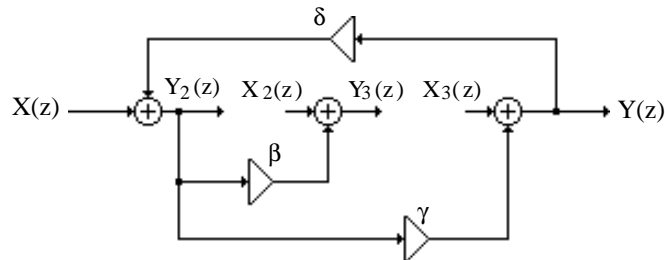
From Eq. (2) we get (5):  $W(z) = (1 - k_1) Y_2(z)$ . Substituting Eqs. (3) and (5) in Eq. (1) we get  $(1 - k_2) Y_2(z) = X(z) - \alpha_1 (k_2 Y_2(z) + X_2(z))$  or (6):  $Y_2(z) = \frac{1}{1 - k_1 + \alpha_1 k_2} (X(z) - \alpha_1 X_2(z))$ .

Substituting Eq. (5) in Eq. (4) we get (7):  $Y(z) = \alpha_2 (1 - k_1) Y_2(z) + X_3(z)$ .

A realization based on Eqs. (6), (7) and (3) shown below has no delay-free loops.



**(b)**



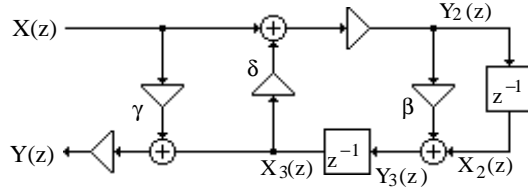


Analyzing the above figure we get (1):  $Y_2(z) = X(z) + \delta Y(z)$ , (2):  $Y_3(z) = \beta Y_2(z) + X_2(z)$ , and (3):  $Y(z) = X_3(z) + \gamma Y_2(z)$ . From Eqs. (1) and (3) we get

$Y_2(z) = X_1(z) + \delta(X(z) + \gamma Y_2(z))$ , which yields (4):  $Y_2(z) = \frac{1}{1 - \delta\gamma}(\delta X(z) + X_3(z))$ . Next, substituting Eq. (4) in Eq. (3) we get

$$(5): Y(z) = X_3(z) + \gamma Y_2(z) = X_3(z) + \frac{\gamma}{1 - \delta\gamma}(X(z) + \delta X_3(z)) = \frac{1}{1 - \delta\gamma}(\gamma X(z) + X_3(z)).$$

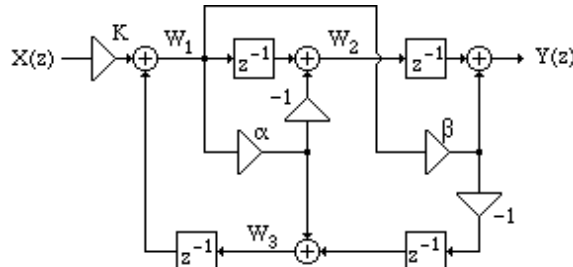
A realization based on Eqs. (2), (4) and (5) shown below has no delay-free loops.



**6.3**  $H(z) = \frac{G_1(z)}{1 - G_1(z)G_2(z)} = \frac{1}{1 - \frac{K}{1 - 2z^{-1}}} = \frac{1}{(1 - K) - 2z^{-1}}$ . The pole is at  $z = \frac{2}{1 - K}$ . Therefore, the

system is stable if  $\left| \frac{2}{1 - K} \right| < 1$ , i.e. if  $-1 > K > 3$ .

**6.4**



From the above figure, we get  $W_1 = KX + z^{-1}W_3$ ,  $W_2 = (z^{-1} - \alpha)W_1$ ,

$W_3 = \alpha W_1 - \beta z^{-1}W_1 = (\alpha - \beta z^{-1})W_1$ , and  $Y = z^{-1}W_2 + \beta W_1$ . Substituting the third equation in the first we get  $W_1 = KX + z^{-1}(\alpha - \beta z^{-1})W_1$ , or  $[1 - \alpha z^{-1} + \beta z^{-2}]W_1 = KX$ . Next, substituting the second equation in the last one we get  $Y = [z^{-1}(z^{-1} - \alpha) + \beta]W_1$ . From the last two equations

we finally arrive at  $H(z) = \frac{Y}{X} = K \left( \frac{\beta - \alpha z^{-1} + z^{-2}}{1 - \alpha z^{-1} + \beta z^{-2}} \right)$ .

(a) Since the structure employs 4 unit delays to implement a second-order transfer function, it is noncanonic.

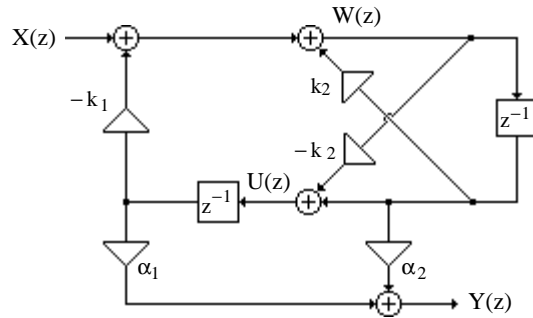
(b) and (c) We next form  $H(z)H(z^{-1}) = K^2 \left( \frac{\beta - \alpha z^{-1} + z^{-2}}{1 - \alpha z^{-1} + \beta z^{-2}} \right) \left( \frac{\beta - \alpha z + z^2}{1 - \alpha z + \beta z^2} \right)$

$$= K^2 \left( \frac{\beta - \alpha z^{-1} + z^{-2}}{1 - \alpha z^{-1} + \beta z^{-2}} \right) \left( \frac{\beta z^{-2} - \alpha z^{-1} + 1}{z^{-2} - \alpha z^{-1} + \beta z} \right) = K^2. \text{ Therefore, } |H(e^{j\omega})| = K, \text{ for all}$$

values of  $\omega$ . Hence  $|H(e^{j\omega})| = 1$  if  $K = 1$ .

(d) Note  $H(z)$  is an allpass transfer function with a constant magnitude at all values of  $\omega$ .

### 6.5



Analyzing the figure we get (1):  $W(z) = X(z) - k_1 z^{-1} U(z) + k_2 z^{-1} W(z)$ ,

(2):  $U(z) = -k_2 W(z) + z^{-1} W(z) = (-k_2 + z^{-1}) W(z)$ , and (3):  $Y(z) = \alpha_1 z^{-1} U(z) + \alpha_2 z^{-1} W(z)$ .

Substituting Eq. (2) in Eq. (1) we get  $W(z) = X(z) - k_1 z^{-1} (-k_2 + z^{-1}) W(z) + k_2 z^{-1} W(z)$ , or

$$\left[ 1 - (1 + k_1) k_2 z^{-1} + k_1 z^{-2} \right] W(z) = X(z), \text{ or (4): } W(z) = \frac{1}{1 - (1 + k_1) k_2 z^{-1} + k_1 z^{-2}} X(z).$$

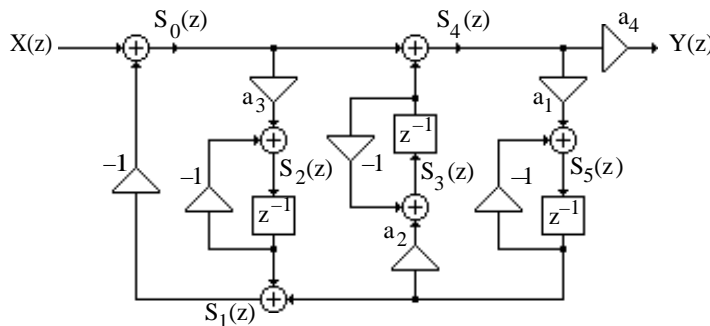
Substituting Eq. (4) in Eq. (2) we get (5):  $U(z) = \frac{-k_2 + z^{-1}}{1 - (1 + k_1) k_2 z^{-1} + k_1 z^{-2}} X(z)$ . Finally,

substituting Eqs. (4) and (5) in Eq. (3) we get  $Y(z) = \frac{(\alpha_2 - \alpha_1 k_2) z^{-1} + \alpha_1 z^{-2}}{1 - (1 + k_1) k_2 z^{-1} + k_1 z^{-2}} X(z)$ , or

$$H(z) = \frac{(\alpha_2 - \alpha_1 k_2) z^{-1} + \alpha_1 z^{-2}}{1 - (1 + k_1) k_2 z^{-1} + k_1 z^{-2}}. \text{ For stability we must have } |k_1| < 1, \text{ and}$$

$$|k_2(1 + k_1)| < 1 + k_1, \text{ or } |k_2| < 1.$$

### 6.6

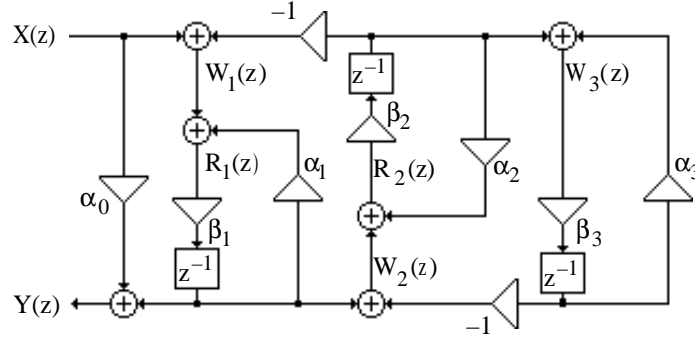


Analysis yields  $S_0(z) = X(z) - S_1(z)$ ,  $S_1(z) = z^{-1} S_2(z) + z^{-1} S_5(z)$ ,  $S_2(z) = a_3 S_0(z) - z^{-1} S_2(z)$ ,

$S_3(z) = a_3z^{-1}S_5(z) - z^{-1}S_3(z)$ ,  $S_4(z) = S_0(z) + z^{-1}S_3(z)$ ,  $S_5(z) = a_1S_4(z) - z^{-1}S_5(z)$ , and  $Y(z) = a_4S_4(z)$ . Eliminating  $S_0(z)$ ,  $S_1(z)$ ,  $S_2(z)$ ,  $S_3(z)$ ,  $S_4(z)$  and  $S_5(z)$  from these equations we get after some algebra

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_4(1+z^{-1})^3}{1 + (3+a_1+a_3)z^{-1} + (3+2a_1+2a_3-a_1a_2)z^{-2} + (1+a_1+a_3-a_1a_2-a_1a_2a_3)z^{-3}}.$$

## 6.7



Analysis yields  $Y(z) = \alpha_0 X(z) + \beta_1 z^{-1} R_1(z)$ ,  $W_1(z) = X(z) - \beta_2 z^{-1} R_2(z)$ ,  
 $R_1(z) = W_1(z) + \alpha_1 \beta_1 z^{-1} R_1(z)$ ,  $W_2(z) = \beta_1 z^{-1} R_1(z) - \beta_3 z^{-1} W_3(z)$ ,  
 $R_2(z) = W_2(z) + \alpha_2 \beta_2 z^{-1} R_2(z)$ ,  $W_3(z) = \beta_2 z^{-1} R_2(z) + \alpha_3 \beta_3 z^{-1} W_3(z)$ .

From the third equation we get  $W_1(z) = (1 - \alpha_1 \beta_1 z^{-1}) R_1(z)$ . From the sixth equation we get

$$W_3(z) = \frac{\beta_2 z^{-1} R_2(z)}{1 - \alpha_3 \beta_3 z^{-1}}. \text{ From the fifth equation we get } R_2(z) = \frac{W_2(z)}{1 - \alpha_2 \beta_2 z^{-1}}.$$

Rewriting the fourth equation we get  $W_2(z) - \beta_3 z^{-1} W_3(z) = \beta_1 z^{-1} R_1(z)$ , in which we substitute the expressions for  $W_3(z)$  and  $W_1(z)$  resulting in

$$R_2(z) \left\{ (1 - \alpha_2 \beta_2 z^{-1}) + \frac{\beta_2 \beta_3 z^{-1}}{(1 - \alpha_3 \beta_3 z^{-1})} \right\} = \beta_1 z^{-1} R_1(z), \text{ or}$$

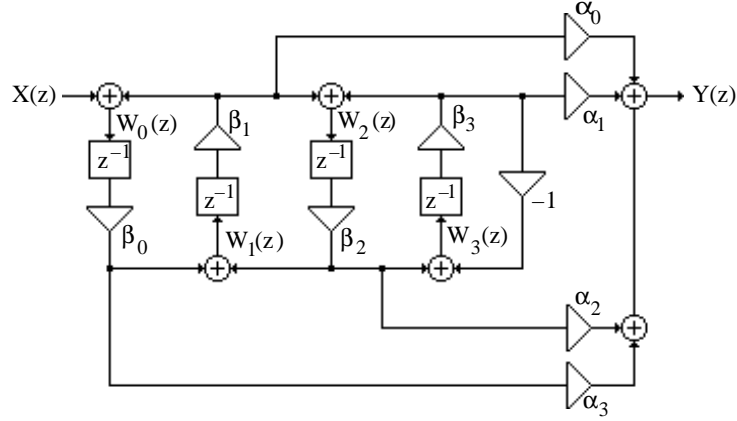
$$R_2(z) = \frac{\beta_1 z^{-1} (1 - \alpha_3 \beta_3 z^{-1})}{\beta_2 \beta_3 z^{-1} + (1 - \alpha_2 \beta_2 z^{-1})(1 - \alpha_3 \beta_3 z^{-1})} R_1(z).$$

Combining  $W_1(z) = X(z) - \beta_2 z^{-1} R_2(z)$ ,  $W_1(z) = (1 - \alpha_1 \beta_1 z^{-1}) R_1(z)$ . and making use of the expression for  $R_2(z)$  we arrive at

$$R_1(z) = \frac{\beta_2 \beta_3 z^{-2} + (1 - \alpha_2 \beta_2 z^{-1})(1 - \alpha_3 \beta_3 z^{-1})}{\beta_2 \beta_3 z^{-2} (1 - \alpha_1 \beta_1 z^{-1}) + (1 - \alpha_1 \beta_1 z^{-1})(1 - \alpha_2 \beta_2 z^{-1})(1 - \alpha_3 \beta_3 z^{-1}) + \beta_1 \beta_2 z^{-2} (1 - \alpha_3 \beta_3 z^{-1})} X(z).$$

Substituting the above in  $Y(z) = \alpha_0 X(z) + \beta_1 z^{-1} R_1(z)$ , we finally get

$$H(z) = \frac{Y(z)}{X(z)} = \alpha_0 + \frac{\beta_1 \beta_2 \beta_3 z^{-2} + \beta_1 z^{-1} (1 - \alpha_2 \beta_2 z^{-1})(1 - \alpha_3 \beta_3 z^{-1})}{(1 - \alpha_1 \beta_1 z^{-1})(1 - \alpha_2 \beta_2 z^{-1})(1 - \alpha_3 \beta_3 z^{-1}) + \beta_2 \beta_3 z^{-2} (1 - \alpha_1 \beta_1 z^{-1}) + \beta_1 \beta_2 z^{-2} (1 - \alpha_3 \beta_3 z^{-1})}.$$



Analysis yields  $W_0(z) = X(z) + \beta_1 z^{-1} W_1(z)$ ,  $W_1(z) = \beta_0 z^{-1} W_0(z) + \beta_2 z^{-1} W_2(z)$ ,  
 $W_2(z) = \beta_1 z^{-1} W_1(z) + \beta_3 z^{-1} W_3(z)$ ,  $W_3(z) = \beta_2 z^{-1} W_2(z) - \beta_3 z^{-1} W_3(z)$ .

From these equations we get  $W_3(z) = \frac{\beta_2 z^{-1}}{1 + \beta_3 z^{-1}} W_2(z)$ , and

$$\left( \frac{1 + \beta_3 z^{-1} - \beta_2 \beta_3 z^{-2}}{1 + \beta_3 z^{-1}} \right) W_2(z) = \beta_1 z^{-1} W_1(z), \quad \text{Or, } W_2(z) = \left( \frac{\beta_1 z^{-1} (1 + \beta_3 z^{-1})}{1 + \beta_3 z^{-1} - \beta_2 \beta_3 z^{-2}} \right) W_1(z),$$

$$\text{In addition, } W_1(z) = \left( \frac{\beta_0 z^{-1} (1 + \beta_3 z^{-1} - \beta_2 \beta_3 z^{-2})}{1 + \beta_3 z^{-1} - (\beta_2 \beta_3 + \beta_1 \beta_2) z^{-2} - \beta_1 \beta_2 \beta_3 z^{-3}} \right) W_0(z).$$

$$\text{Now, } W_0(z) \left( 1 - \frac{\beta_0 \beta_1 z^{-2} (1 + \beta_3 z^{-1} - \beta_2 \beta_3 z^{-2})}{1 + \beta_3 z^{-1} - \beta_2 (\beta_1 + \beta_3) z^{-2} - \beta_1 \beta_2 \beta_3 z^{-3}} \right) = X(z), \quad \text{Hence,}$$

$$W_0(z) = \left( \frac{1 + \beta_3 z^{-1} - \beta_2 (\beta_1 + \beta_3) z^{-2} - \beta_1 \beta_2 \beta_3 z^{-3}}{1 + \beta_3 z^{-1} - (\beta_1 \beta_2 + \beta_2 \beta_3 + \beta_0 \beta_1) z^{-2} - \beta_1 \beta_3 (\beta_2 + \beta_0) z^{-3} + \beta_0 \beta_1 \beta_2 \beta_3 z^{-4}} \right) X(z),$$

$$W_1(z) = \left( \frac{\beta_0 z^{-1} (1 + \beta_3 z^{-1} - \beta_2 \beta_3 z^{-2})}{1 + \beta_3 z^{-1} - (\beta_1 \beta_2 + \beta_2 \beta_3 + \beta_0 \beta_1) z^{-2} - \beta_1 \beta_3 (\beta_2 + \beta_0) z^{-3} + \beta_0 \beta_1 \beta_2 \beta_3 z^{-4}} \right) X(z),$$

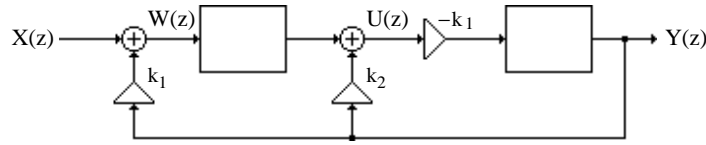
$$W_2(z) = \left( \frac{\beta_1 \beta_0 z^{-2} (1 + \beta_3 z^{-1})}{1 + \beta_3 z^{-1} - (\beta_1 \beta_2 + \beta_2 \beta_3 + \beta_0 \beta_1) z^{-2} - \beta_1 \beta_3 (\beta_2 + \beta_0) z^{-3} + \beta_0 \beta_1 \beta_2 \beta_3 z^{-4}} \right) X(z),$$

$$W_3(z) = \left( \frac{\beta_0 \beta_1 \beta_2 z^{-3}}{1 + \beta_3 z^{-1} - (\beta_1 \beta_2 + \beta_2 \beta_3 + \beta_0 \beta_1) z^{-2} - \beta_1 \beta_3 (\beta_2 + \beta_0) z^{-3} + \beta_0 \beta_1 \beta_2 \beta_3 z^{-4}} \right) X(z).$$

Finally,  $Y(z) = \alpha_3 \beta_0 z^{-1} W_0(z) + \alpha_0 \beta_1 z^{-1} W_1(z) + \alpha_2 \beta_2 z^{-1} W_2(z) + \alpha_1 \beta_3 z^{-1} W_3(z)$ . Substituting the expressions for  $W_0(z)$ ,  $W_1(z)$ ,  $W_2(z)$  and  $W_3(z)$  in the above we get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\alpha_3 \beta_0 z^{-1} + (\alpha_3 \beta_3 \beta_0 + \alpha_0 \beta_1 \beta_0) z^{-2} + (\alpha_0 \beta_0 \beta_1 \beta_3 + \alpha_2 \beta_1 \beta_2 \beta_0 - \alpha_3 \beta_2 \beta_1 \beta_0 - \alpha_3 \beta_0 \beta_2 \beta_3) z^{-3} + (-\alpha_3 + \alpha_2 - \alpha_0 + \alpha_1) \beta_0 \beta_1 \beta_2 \beta_3 z^{-3}}{1 + \beta_3 z^{-1} - (\beta_1 \beta_2 + \beta_2 \beta_3 + \beta_0 \beta_1) z^{-2} - \beta_1 \beta_3 (\beta_2 + \beta_0) z^{-3} + \beta_0 \beta_1 \beta_2 \beta_3 z^{-4}}$$

6.9



Analysis yields (1):  $W(z) = X(z) + k_1 Y(z)$ , (2):  $U(z) = \frac{1}{1-z^{-1}} W(z) + k_2 Y(z)$ , and

(3):  $Y(z) = \frac{-k_1}{1-z^{-1}} U(z)$ . Substituting Eq. (2) in Eq. (3) we get

(4):  $Y(z) = \frac{-k_1}{1-z^{-1}} \left( \frac{1}{1-z^{-1}} W(z) + k_2 Y(z) \right) = -\frac{k_1}{(1-z^{-1})^2} W(z) - \frac{k_1 k_2}{1-z^{-1}} Y(z)$ . Substituting

Eq. (1) in Eq. (4) we then get  $Y(z) = -\frac{k_1}{(1-z^{-1})^2} [X(z) + k_1 Y(z)] - \frac{k_1 k_2}{1-z^{-1}} Y(z)$

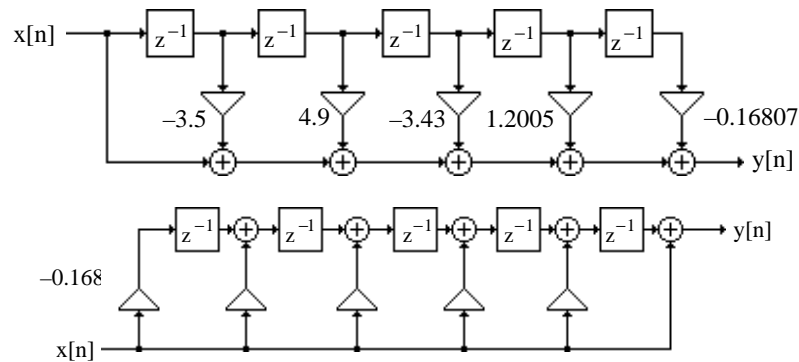
$= -\frac{k_1}{(1-z^{-1})^2} X(z) - \frac{k_1}{1-z^{-1}} \left[ \frac{k_1 + k_2 - k_2 z^{-1}}{1-z^{-1}} \right] Y(z)$ , or

$\left[ 1 + \frac{k_1(k_1 + k_2 - k_2 z^{-1})}{(1-z^{-1})^2} \right] Y(z) = -\frac{k_1}{(1-z^{-1})^2} X(z)$ . Hence,

$$H(z) = \frac{Y(z)}{X(z)} = -\frac{k_1}{[1 + k_1(k_1 + k_2)] - (2 + k_1 k_2)z^{-1} + z^{-2}}$$

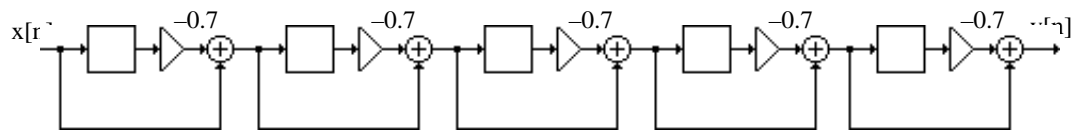
6.10 (a) A direct form realization of

$H(z) = 1 - 3.5z^{-1} + 4.9z^{-2} - 3.43z^{-3} + 1.2005z^{-4} - 0.16807z^{-5}$  and its transposed structure are shown below:

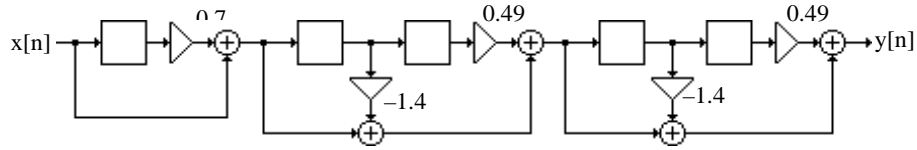


(b) A cascade realization of

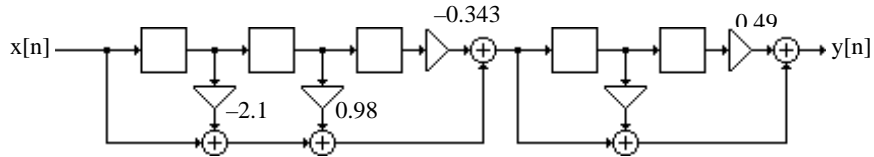
$H(z) = (1 - 0.7z^{-1})(1 - 0.7z^{-1})(1 - 0.7z^{-1})(1 - 0.7z^{-1})(1 - 0.7z^{-1})$  is shown below:



(c) A cascade realization of  $H(z) = (1 - 0.7z^{-1})(1 - 1.4z^{-1} + 0.49z^{-2})(1 - 1.4z^{-1} + 0.49z^{-2})$  is shown below:



(d) A cascade realization of  $H(z) = (1 - 2.1z^{-1} + 1.47z^{-2} - 0.343z^{-3})(1 - 1.4z^{-1} + 0.49z^{-2})$  is shown below:



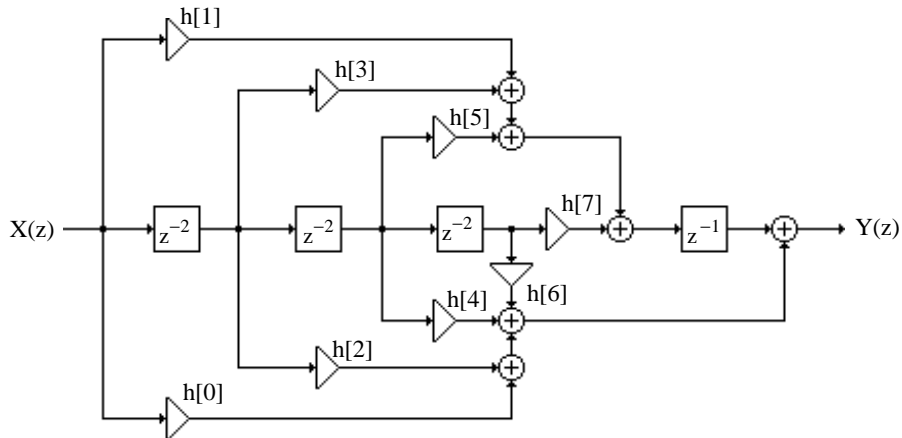
**6.11 (a)**  $H(z) = (h[0] + h[3]z^{-3} + h[6]z^{-6}) + z^{-1}(h[1] + h[4]z^{-3} + h[7]z^{-6}) + z^{-2}(h[2] + h[5]z^{-3})$ .  
Hence,  $E_0(z) = h[0] + h[3]z^{-3} + h[6]z^{-6}$ ,  $E_1(z) = h[1] + h[4]z^{-3} + h[7]z^{-6}$ ,  $E_2(z) = h[2] + h[5]z^{-3}$ .

(b)

Figure is same as in Figure 6.9 with  $h[8] = 0$ .

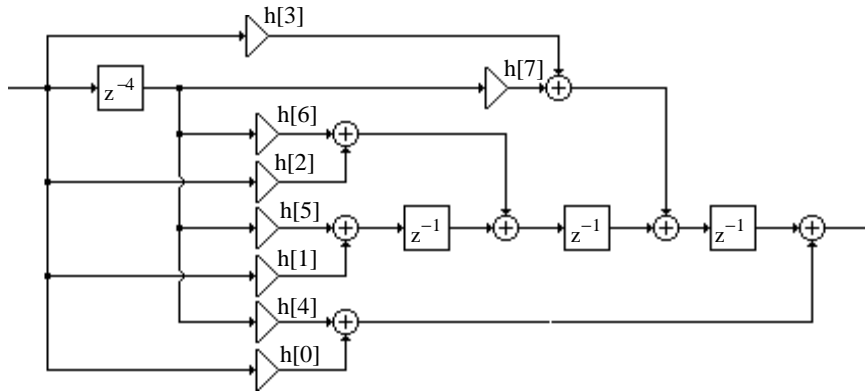
**6.12 (a)**  $H(z) = (h[0] + h[2]z^{-2} + h[4]z^{-4} + h[6]z^{-6}) + z^{-1}(h[1] + h[3]z^{-2} + h[5]z^{-4} + h[7]z^{-6})$ .  
Hence,  $E_0(z) = h[0] + h[2]z^{-2} + h[4]z^{-4} + h[6]z^{-6}$ ,  $E_1(z) = h[1] + h[3]z^{-2} + h[5]z^{-4} + h[7]z^{-6}$ .

(b)

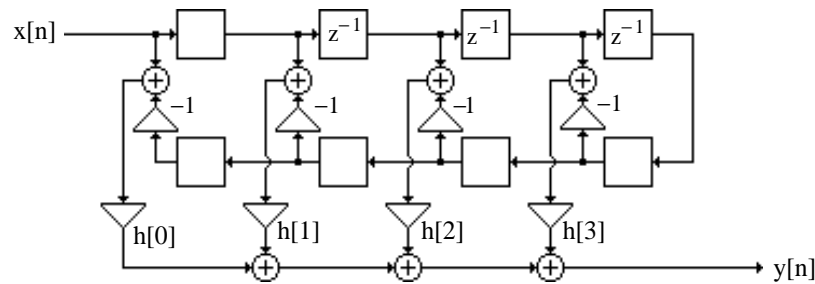


**6.13 (a)**  $H(z) = (h[0] + h[4]z^{-4}) + z^{-1}(h[1] + h[5]z^{-4}) + z^{-2}(h[2] + h[6]z^{-4}) + z^{-3}(h[3] + h[7]z^{-4})$ .  
Hence,  $E_0(z) = h[0] + h[4]z^{-4}$ ,  $E_1(z) = h[1] + h[5]z^{-4}$ ,  
 $E_2(z) = h[2] + h[6]z^{-4}$ ,  $E_3(z) = h[3] + h[7]z^{-4}$ .

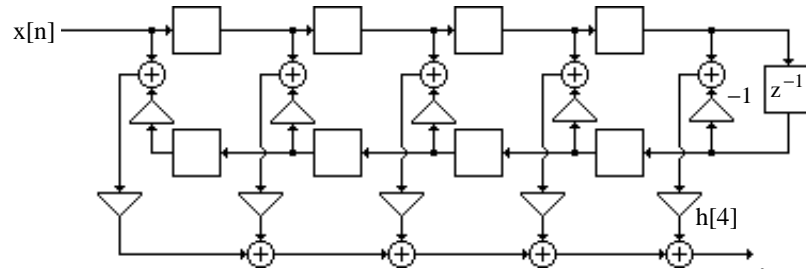
(b)



6.14



6.15



6.16 Consider first the case of  $N = 4$ .

$$\begin{aligned} H(z) &= b_0 + b_1 z^{-1} \left( 1 + b_2 z^{-1} \left( 1 + b_3 z^{-1} \left( 1 + b_4 z^{-1} \right) \right) \right) = b_0 + b_1 z^{-1} \left( 1 + b_2 z^{-1} \left( 1 + b_3 z^{-1} + b_3 b_4 z^{-2} \right) \right) \\ &= b_0 + b_1 z^{-1} \left( 1 + b_2 z^{-1} + b_2 b_3 z^{-2} + b_2 b_3 b_4 z^{-3} \right) \\ &= b_0 + b_1 z^{-1} + b_1 b_2 z^{-2} + b_1 b_2 b_3 z^{-3} + b_1 b_2 b_3 b_4 z^{-4} = h[0] + h[1] z^{-1} + h[2] z^{-2} + h[3] z^{-3} + h[4] z^{-4}. \end{aligned}$$

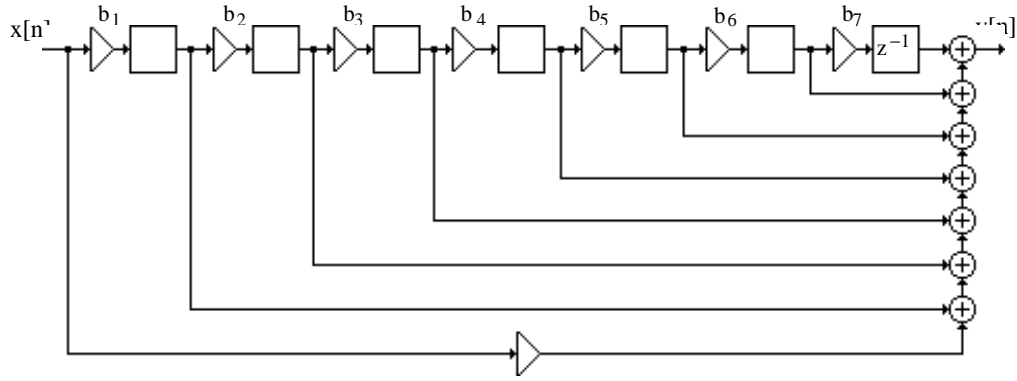
Comparing like powers of  $z^{-1}$  on both sides, we obtain  $h[0] = b_0$ ,  $h[1] = b_1$ ,  $h[2] = b_1 b_2$ ,

$h[3] = b_1 b_2 b_3$ ,  $h[4] = b_1 b_2 b_3 b_4$ . Solving these equations we get  $b_0 = h[0]$ ,  $b_1 = h[1]$ ,  $b_2 = \frac{h[2]}{h[1]}$ ,

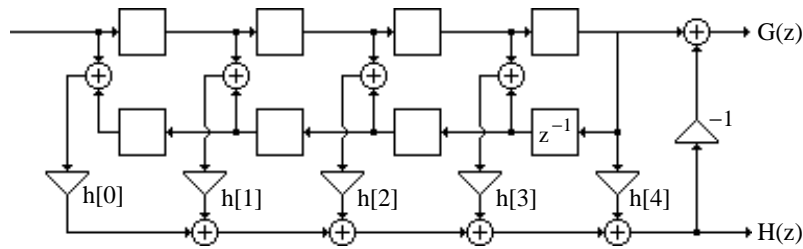
$b_3 = \frac{h[3]}{h[2]}$ , and  $b_4 = \frac{h[4]}{h[3]}$ . In the general case, we have

$$b_0 = h[0], \quad b_1 = h[1], \quad \text{and} \quad b_k = \frac{h[k]}{h[k-1]}, \quad 2 \leq k \leq N.$$

A nested realization of  $H(z)$  for  $N = 7$  is shown below:



**6.17**  $G(z) = z^{-N/2} - H(z)$ . A canonic realization of both  $G(z)$  and  $H(z)$  is as indicated on next page for  $N = 8$ .



**6.18** Without any loss of generality, assume  $M = 5$  which means  $N = 11$ . In this case the transfer function is given by  $H(z) =$

$$z^{-5} \left[ h[5] + h[4](z + z^{-1}) + h[3](z^2 + z^{-2}) + h[2](z^3 + z^{-3}) + h[1](z^4 + z^{-4}) + h[0](z^5 + z^{-5}) \right].$$

Now, the recursion relation for the Chebyshev polynomial is given by

$$T_r(x) = 2xT_{r-1}(x) - T_{r-2}(x), \text{ for } r \geq 2 \text{ with } T_0(x) = 1 \text{ and } T_1(x) = x. \text{ Hence,}$$

$$T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1, \quad T_3(x) = 2xT_2(x) - T_1(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x,$$

$$T_4(x) = 2xT_3(x) - T_2(x) = 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1,$$

$$T_5(x) = 2xT_4(x) - T_3(x) = 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x) = 16x^5 - 20x^3 + 5x.$$

We can thus rewrite the expression inside the square brackets given above as

$$\begin{aligned} & h[5] + 2h[4]T_1\left(\frac{z+z^{-1}}{2}\right) + 2h[3]T_2\left(\frac{z+z^{-1}}{2}\right) + 2h[2]T_3\left(\frac{z+z^{-1}}{2}\right) \\ & \quad + 2h[1]T_4\left(\frac{z+z^{-1}}{2}\right) + h[0]T_5\left(\frac{z+z^{-1}}{2}\right) \\ & = h[5] + 2h[4]\left(\frac{z+z^{-1}}{2}\right) + 2h[3]\left[2\left(\frac{z+z^{-1}}{2}\right)^2 - 1\right] + 2h[2]\left[4\left(\frac{z+z^{-1}}{2}\right)^3 - 3\left(\frac{z+z^{-1}}{2}\right)\right] \\ & \quad + 2h[1]\left[8\left(\frac{z+z^{-1}}{2}\right)^4 - 8\left(\frac{z+z^{-1}}{2}\right)^2 + 1\right] + 2h[0]\left[16\left(\frac{z+z^{-1}}{2}\right)^5 - 20\left(\frac{z+z^{-1}}{2}\right)^3 + 5\left(\frac{z+z^{-1}}{2}\right)\right] \end{aligned}$$



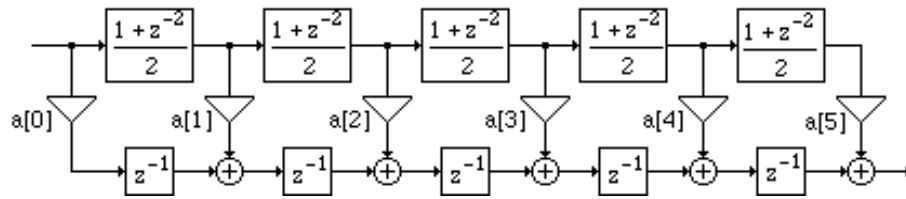
$$= \sum_{n=0}^5 a[n] \left( \frac{z+z^{-1}}{2} \right)^n, \text{ where } a[0]=h[5]-2h[3]+2h[1], \quad a[1]=2h[4]-6h[2]+10h[0],$$

$$a[2]=4h[3]-16h[1], \quad a[3]=8h[2]-40h[1], \quad a[4]=16h[1], \text{ and } a[5]=32h[0].$$

A realization of  $H(z) = z^{-5} \left[ \sum_{n=0}^5 a[n] \left( \frac{z+z^{-1}}{2} \right)^n \right] = a[0]z^{-5} + a[1]z^{-4} \left( \frac{1+z^{-2}}{2} \right)$

$$+ a[2]z^{-3} \left( \frac{1+z^{-2}}{2} \right)^2 + a[3]z^{-2} \left( \frac{1+z^{-2}}{2} \right)^3 + a[4]z^{-1} \left( \frac{1+z^{-2}}{2} \right)^4 + a[5] \left( \frac{1+z^{-2}}{2} \right)^5$$

is thus as shown below:



**6.19** Consider  $H(z) = \frac{P(z)}{D(z)} = \frac{P_1(z)}{D_1(z)} \cdot \frac{P_2(z)}{D_2(z)} \cdot \frac{P_3(z)}{D_3(z)}$ . Assume all zeros of  $P(z)$  and  $D(z)$  are complex.

Note that the numerator of the first stage can be one of the 3 factors,  $P_1(z)$ ,  $P_2(z)$ , and  $P_3(z)$ , the numerator of the second stage can be one of the remaining 2 factors, and the numerator of the third stage is the remaining factor. Likewise, the denominator of the first stage can be one of the 3 factors,  $D_1(z)$ ,  $D_2(z)$ , and  $D_3(z)$ , the denominator of the second stage can be one of the remaining 2 factors, and the denominator of the third stage is the remaining factor. Hence, there are  $(3!)^2 = 36$  different types of cascade realizations.

If the zeros of  $P(z)$  and  $D(z)$  are all real, then  $P(z)$  has 6 real zeros and  $D(z)$  has 6 real zeros. In this case then there are  $(6!)^2 = 518400$  different types of cascade realizations.

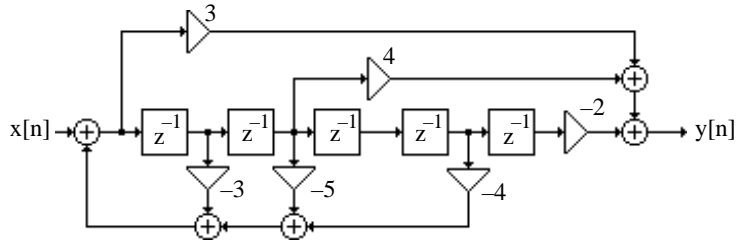
**6.20**  $H(z) = \prod_{i=1}^K \frac{P_i(z)}{D_i(z)}$ . Here the numerator of the first stage can be chosen in  $\binom{K}{1}$  ways, the

numerator of the second stage can be chosen in  $\binom{K-1}{1}$  ways, and until there is only one possible choice for the numerator of the  $K$ -th stage. Likewise, the denominator of the first stage can be chosen in  $\binom{K}{1}$  ways, the denominator of the second stage can be chosen in

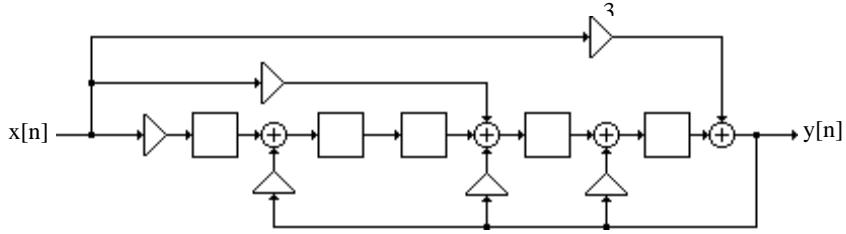
$\binom{K-1}{1}$  ways, and until there is only one possible choice for the denominator of the  $K$ -th stage. Hence the total number of possible cascade realizations are equal to

$$\binom{K}{1}^2 \binom{K-1}{1}^2 \binom{K-2}{1}^2 \cdots \binom{2}{1}^2 \binom{1}{1}^2 = (K!)^2$$

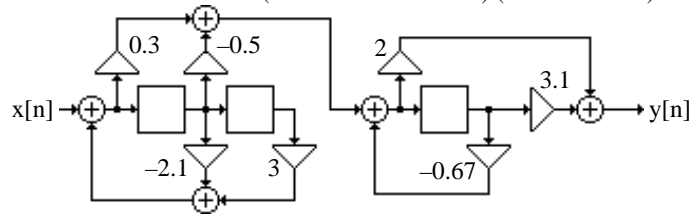
**6.21** A canonic direct form II realization of  $H(z) = \frac{3+4z^{-1}-2z^{-2}}{1+3z^{-1}+5z^{-2}+4z^{-4}}$  is shown on next page:



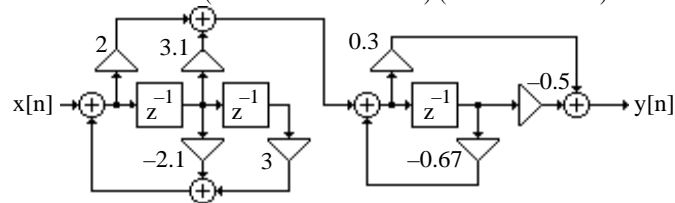
A canonic direct form  $II_1$  realization of  $H(z)$  is as indicated below:



**6.22 (a)** A cascade realization of  $H_1(z) = \left( \frac{0.3 - 0.5z^{-1}}{1 + 2.1z^{-1} - 3z^{-2}} \right) \left( \frac{2 + 3.1z^{-1}}{1 + 0.67z^{-1}} \right)$  is shown below:

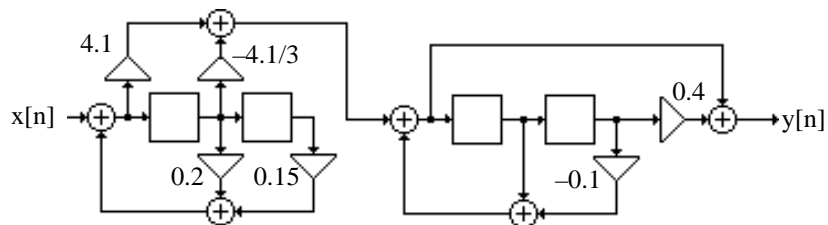


A cascade realization of  $H_1(z) = \left( \frac{2 + 3.1z^{-1}}{1 + 2.1z^{-1} - 3z^{-2}} \right) \left( \frac{0.3 - 0.5z^{-1}}{1 + 0.67z^{-1}} \right)$  is shown below:

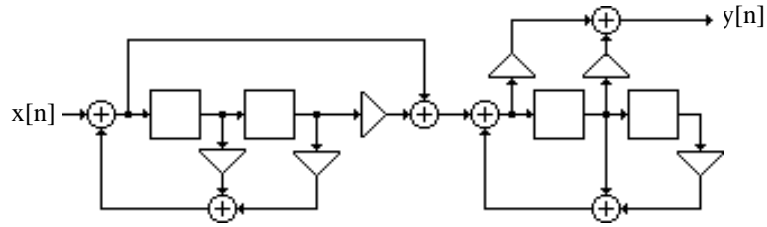


**(b)** A cascade realization of  $H_2(z) = \left( \frac{4.1z(z - \frac{1}{3})}{(z + 0.3)(z - 0.5)} \right) \left( \frac{z^2 + 0.4}{z^2 - z - 0.1} \right)$

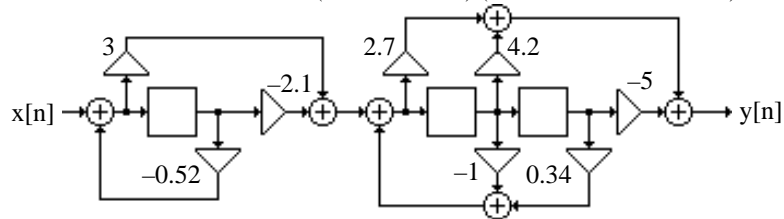
$= \left( \frac{4.1 - 1.3667z^{-1}}{1 - 0.2z^{-1} - 0.15z^{-2}} \right) \left( \frac{1 + 0.4z^{-2}}{1 - z^{-1} + 0.1z^{-2}} \right)$  is shown below:



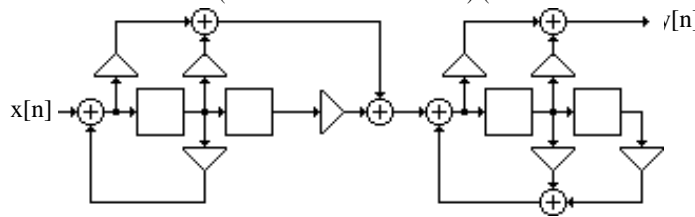
A cascade realization of  $H_2(z) = \left( \frac{1+0.4z^{-2}}{1-0.2z^{-1}-0.15z^{-2}} \right) \left( \frac{4.1-1.3667z^{-1}}{1-z^{-1}+0.1z^{-2}} \right)$  is shown below:



(c) A cascade realization of  $H_3(z) = \left( \frac{3-2.1z^{-1}}{1+0.52z^{-1}} \right) \left( \frac{2.7+4.2z^{-1}-5z^{-2}}{1+z^{-1}-0.34z^{-2}} \right)$  is shown below:



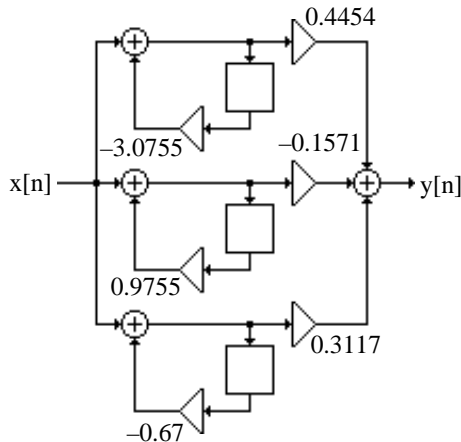
A cascade realization of  $H_3(z) = \left( \frac{2.7+4.2z^{-1}-5z^{-2}}{1+0.52z^{-1}} \right) \left( \frac{3-2.1z^{-1}}{1+z^{-1}-0.34z^{-2}} \right)$  is shown below:



**6.23 (a)** A partial-fraction expansion of  $H_1(z)$  in  $z^{-1}$  obtained using the M-file `residuez` gives

$$H_1(z) = \frac{0.6 - 0.07z^{-1} - 1.55z^{-2}}{1 + 2.77z^{-1} - 1.593z^{-2} - 2.01z^{-3}} = \frac{0.4454}{1 + 3.0755z^{-1}} - \frac{0.1571}{1 - 0.9755z^{-1}} + \frac{0.3117}{1 + 0.67z^{-1}}$$

realization yields the Parallel Form I structure shown below:

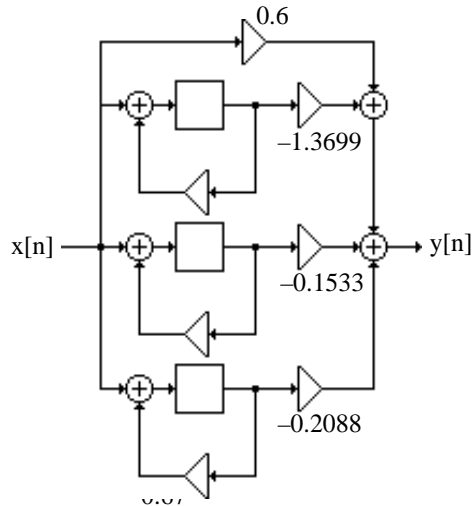


A partial-fraction expansion of  $H_1(z)$  in  $z$  obtained using the M-file `residue` gives

$$H_1(z) = \frac{0.6z^3 - 0.07z^2 - 1.55z}{z^3 + 2.77z^2 - 1.593z - 2.01} = 0.6 + \frac{-1.3699}{z + 3.0755} + \frac{0.1533}{z - 0.9755} + \frac{-2088}{z + 0.67}$$

$$= 0.6 + \frac{-1.3699z^{-1}}{1 + 3.0755z^{-1}} + \frac{0.1533z^{-1}}{1 - 0.9755z^{-1}} + \frac{-2088z^{-1}}{1 + 0.67z^{-1}}, \text{ whose realization yields the Parallel Form II}$$

structure shown below:

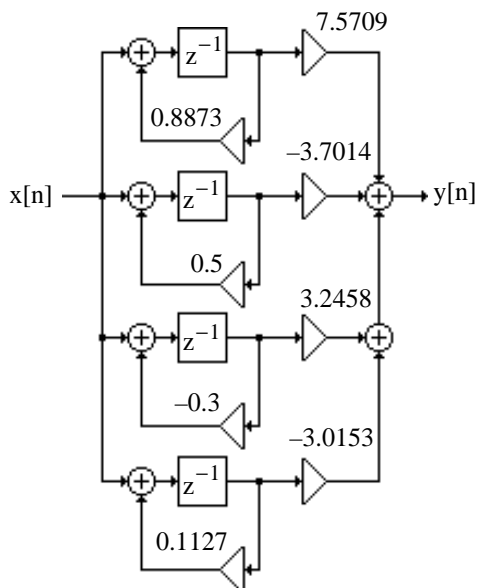


(b) A partial-fraction expansion of  $H_2(z)$  in  $z^{-1}$  obtained using the M-file `residuez` gives

$$H_2(z) = \frac{4.1z^4 - 1.3667z^3 + 1.64z^2 - 0.5467z}{z^4 - 1.2z^3 + 0.15z^2 + 0.13z - 0.015}$$

$$= \frac{7.5709}{1 - 0.8873z^{-1}} + \frac{-3.7014}{1 - 0.5z^{-1}} + \frac{3.2458}{1 + 0.3z^{-1}} + \frac{-3.0153}{1 - 0.1127z^{-1}}$$

whose realization yields the Parallel Form I structure shown below:

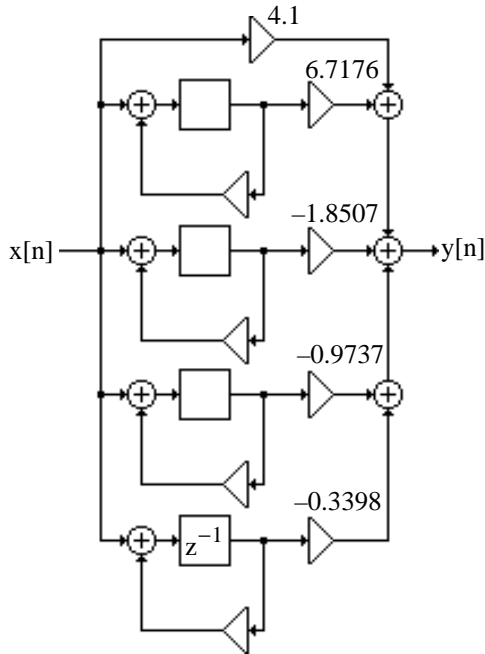


A partial-fraction expansion of  $H_2(z)$  in  $z$  obtained using the M-file `residue` gives

$$H_1(z) = 4.1 + \frac{6.7176}{z - 0.8873} + \frac{-1.8507}{z - 0.5} + \frac{-0.9737}{z + 0.3} + \frac{-0.3398}{z - 0.1127}$$

$$= 4.1 + \frac{6.7176z^{-1}}{1 - 0.8873z^{-1}} + \frac{-1.8507z^{-1}}{1 - 0.5z^{-1}} + \frac{-0.9737z^{-1}}{1 + 0.3z^{-1}} + \frac{-0.3398z^{-1}}{1 - 0.1127z^{-1}}$$

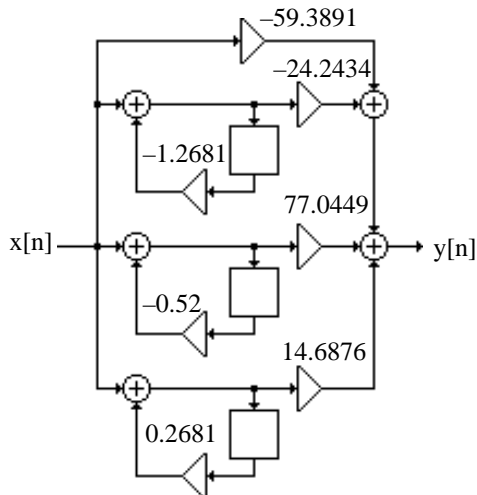
whose realization yields the Parallel Form II structure shown below:



(c) A partial-fraction expansion of  $H_3(z)$  in  $z^{-1}$  obtained using the M-file `residuez` gives

$$H_3(z) = \frac{8.1 + 6.93z^{-1} - 23.82z^{-2} + 10.5z^{-3}}{1 + 1.52z^{-1} + 0.18z^{-2} - 0.1768z^{-3}} = -59.3891 + \frac{-24.2434}{1 + 1.2681z^{-1}} + \frac{77.0449}{1 + 0.52z^{-1}} + \frac{14.6876}{1 - 0.2681z^{-1}}$$

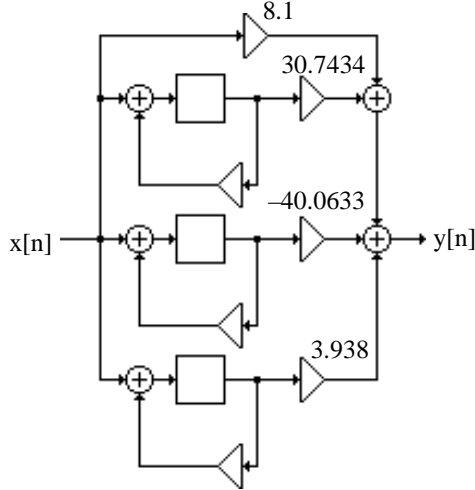
whose realization yields the Parallel Form I structure shown below:



A partial-fraction expansion of  $H_3(z)$  in  $z$  obtained using the M-file `residue` gives

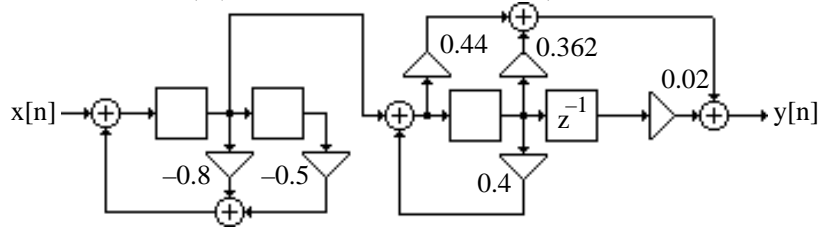
$$H_3(z) = 8.1 + \frac{30.7434}{z+1.2681} + \frac{-40.0633}{z+0.52} + \frac{3.938}{z-0.2681} = 8.1 + \frac{30.7434z^{-1}}{1+1.2681z^{-1}} + \frac{-40.0633z^{-1}}{1+0.52z^{-1}} + \frac{3.938z^{-1}}{1-0.2681z^{-1}}$$

whose realization yields the Parallel Form II structure shown below:



**6.24** A cascade realization based on the factorization

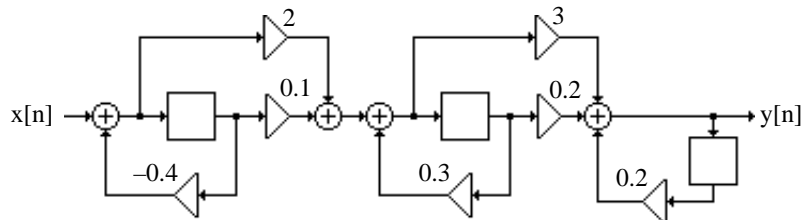
$$H(z) = \left( \frac{z^{-1}}{1+0.8z^{-1}+0.5z^{-2}} \right) \left( \frac{0.44+0.362z^{-1}+0.02z^{-2}}{1-0.4z^{-1}} \right) \text{ is indicated below:}$$



**6.25 (a)**  $H(z) = \frac{2+0.1z^{-1}}{1+0.4z^{-1}} \cdot \frac{3+0.2z^{-1}}{1-0.3z^{-1}} \cdot \frac{1}{1-0.2z^{-1}} = \frac{6+0.7z^{-1}+0.02z^{-2}}{1-0.1z^{-1}-0.14z^{-2}+0.024z^{-3}}$

**(b)**  $y[n] = 6x[n] + 0.7x[n-1] + 0.02x[n-2] + 0.1y[n-1] + 0.14y[n-2] - 0.024y[n-3]$ .

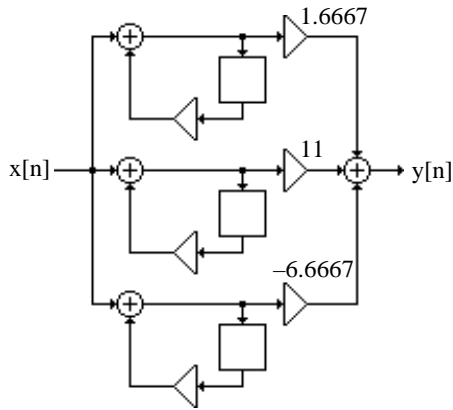
**(c)** A cascade realization of  $H(z)$  is shown below:



**(d)** A parallel form I realization of  $H(z)$  is obtained by making a partial-fraction expansion in

using M-File `residuez`  $H(z) = \frac{1.6667}{1+0.4z^{-1}} + \frac{11}{1-0.3z^{-1}} - \frac{6.6667}{1-0.2z^{-1}}$ ,

whose realization is shown below:



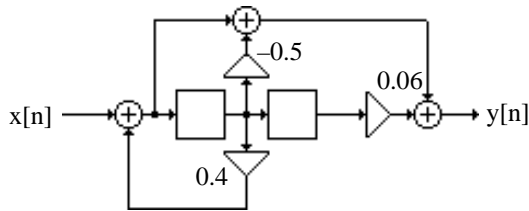
(e)  $h[n] = 1.6667(-0.4)^n \mu[n] + 11(0.3)^n \mu[n] - 6.6667(0.2)^n \mu[n]$ .

6.26 (a)  $Y(z) = \frac{1}{1-0.4z^{-1}} - \frac{0.3z^{-1}}{1-0.4z^{-1}} = \frac{1-0.3z^{-1}}{1-0.4z^{-1}}$ .  $X(z) = \frac{1}{1-0.2z^{-1}}$ . Thus,

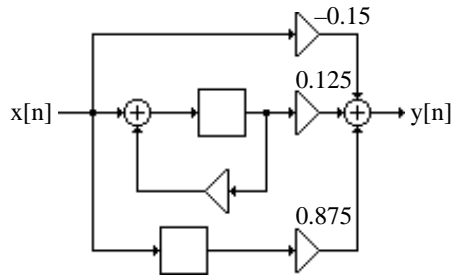
$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1-0.3z^{-1})(1-0.2z^{-1})}{1-0.4z^{-1}} = \frac{1-0.5z^{-1}+0.06z^{-2}}{1-0.4z^{-1}}$$

(b)  $y[n] = x[n] - 0.5x[n-1] + 0.06x[n-2] + 0.4y[n-1]$ .

(c)



(d) A partial-fraction expansion of  $H(z)$  in  $z^{-1}$  obtained using M-file residuez is given by  $H(z) = 0.875z^{-1} - 0.15 + \frac{0.125}{1-0.4z^{-1}}$  whose realization yields the Parallel Form I structure as indicated below:



(e) The inverse z-transform of  $H(z)$  yields  $h[n] = 0.875\delta[n-1] - 0.15\delta[n] + 0.125(0.4)^n \mu[n]$ .

(f)  $X(z) = \frac{1}{1-0.3z^{-1}} - \frac{0.4z^{-1}}{1-0.3z^{-1}} = \frac{1-0.4z^{-1}}{1-0.3z^{-1}}$ . Therefore

$$Y(z) = H(z)X(z) = \left( \frac{(1-0.3z^{-1})(1-0.2z^{-1})}{1-0.4z^{-1}} \right) \left( \frac{1-0.4z^{-1}}{1-0.3z^{-1}} \right) = 1-0.2z^{-1},$$

whose inverse z-transform yields  $y[n] = \delta[n] - 0.2\delta[n-1]$ .

**6.27** Figure P6.11 can be seen to be a Parallel Form II structure. A partial-fraction expansion of

$$H(z) = \frac{3z^2 + 18.5z + 17.5}{(z+0.5)(z+2)}$$

in  $z$  obtained using the M-file residue is given by

$$H(z) = 3 + \frac{5}{z+2} + \frac{6}{z+0.5}.$$

Comparing the coefficients of the expansion with the corresponding

multiplier coefficients in Figure P6.11 we conclude that the multiplier coefficient 2 should be replaced with 6 and the multiplier coefficient 0.5 should be replaced with  $-0.6$ .

**6.28** Figure P6.11 can be seen to be a Parallel Form I structure. A partial-fraction expansion of

$$H(z) = \frac{3z(5z-1)}{(z+0.5)(4z+1)} = \frac{3.75-1.5z^{-1}}{1+0.75z^{-1}+0.125z^{-2}}$$

in  $z^{-1}$  obtained using the M-file residuez is

$$\text{given by } H(z) = \frac{13.5}{1+0.5z^{-1}} - \frac{9.5}{1+0.25z^{-1}}.$$

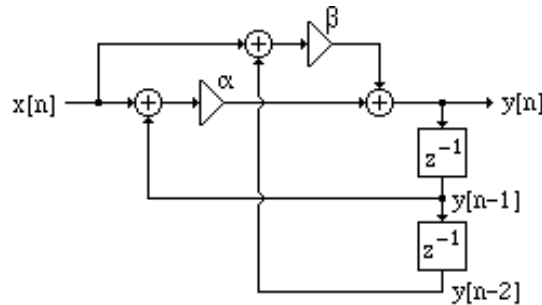
Comparing the coefficients of the expansion with the

corresponding multiplier coefficients in Figure P6.11 we conclude that the multiplier coefficient  $A = 13.5$  and the multiplier coefficient  $B = -0.25$ .

**6.29** The difference equation corresponding to the transfer function  $H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \alpha + \beta}{1 + \alpha z^{-1} + \beta z^{-2}}$

is given by  $y[n] + \alpha y[n-1] + \beta y[n-2] = (1 + \alpha + \beta)x[n]$ , which can be rewritten in the form

$y[n] = x[n] + \alpha(x[n] - y[n-1]) + \beta(x[n] - y[n-2])$ . A realization of  $H(z)$  based on this equation is thus as shown below:



**6.30 (a)** The difference equation corresponding to the transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1 + \alpha_1 + \alpha_2)(1 + 2z^{-1} + z^{-2})}{1 - \alpha_1 z^{-1} + \alpha_2 z^{-2}}$$

is given by

$y[n] + \alpha_1 y[n-1] - \alpha_2 y[n-2] = (1 + \alpha_1 + \alpha_2)(x[n] + 2x[n-1] + x[n-2])$ , which can be rewritten as

$$y[n] = (x[n] + 2x[n-1] + x[n-2]) + \alpha_1 (y[n-1] - x[n] - 2x[n-1] - x[n-2])$$

$$- \alpha_2 (y[n-2] - x[n] - 2x[n-1] - x[n-2]).$$

Denoting  $w[n] = x[n] + 2x[n-1] + x[n-2]$ ,

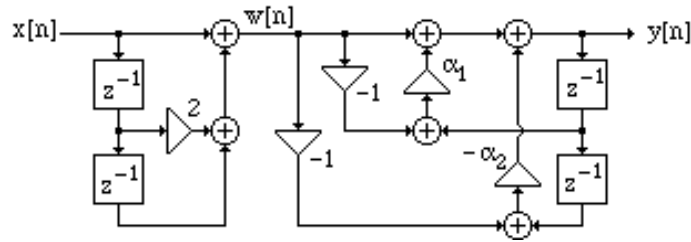
the difference equation representation becomes

$$y[n] = w[n] + \alpha_1 (y[n-1] - w[n]) - \alpha_2 (y[n-2] - w[n]).$$

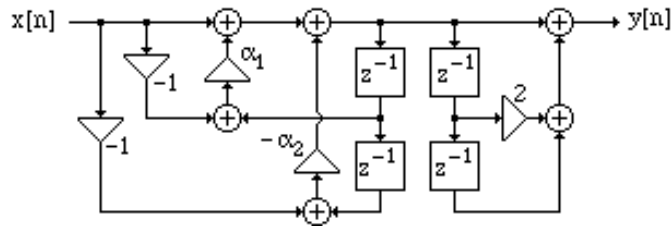
A realization of  $H(z)$  based on the last two



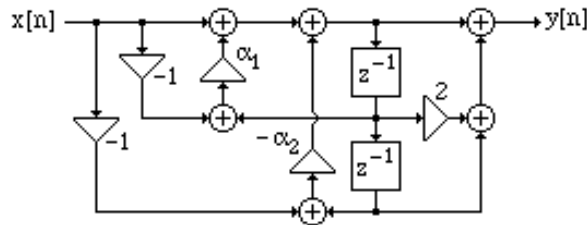
equations is as indicated below where the first stage realizes  $w[n]$  while the second stage realizes  $y[n]$ .



An interchange of the two stages leads to an equivalent realization shown below:



Finally, by delay sharing the above structure reduces to a canonic realization as shown below:



(b) The difference equation corresponding to the transfer function

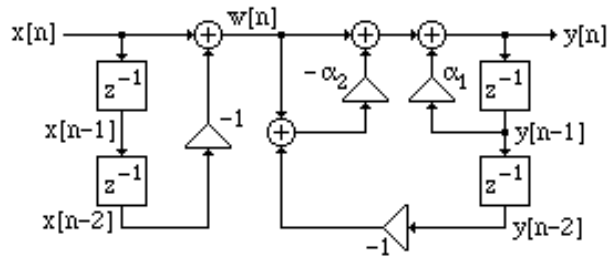
$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1 - \alpha_2)(1 - z^{-2})}{1 - \alpha_1 z^{-1} + \alpha_2 z^{-2}} \text{ is given by}$$

$$y[n] - \alpha_1 y[n-1] + \alpha_2 y[n-2] = (1 - \alpha_2)(x[n] - x[n-2]) \text{ which can be rewritten as}$$

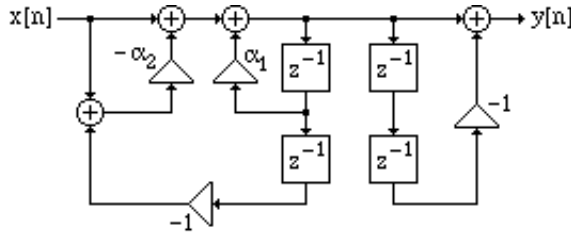
$$y[n] = \alpha_1 y[n-1] - \alpha_2 y[n-2] + x[n] - \alpha_2 x[n] - x[n-2] - \alpha_2 x[n-2]$$

$$= \alpha_1 y[n-1] - \alpha_2 (x[n] - x[n-2] + y[n-2]) + (x[n] - x[n-2]). \text{ Denoting } w[n] = x[n] - x[n-2], \text{ we}$$

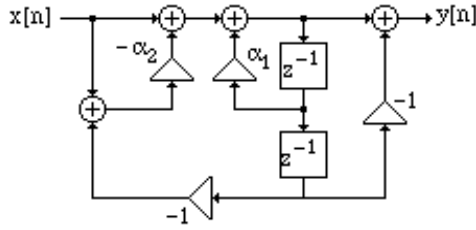
can rewrite the last equation as  $y[n] = \alpha_1 y[n-1] - \alpha_2 (w[n] + y[n-2]) + w[n]$ . A realization of  $H(z)$  based on the last two equations is as shown below:



An interchange of the two stages leads to an equivalent realization shown below:



Finally, by delay sharing the above structure reduces to a canonic realization as shown below:



**6.31 (a)** From the structure of Figure P.6/14 it follows that  $H_N(z) = \frac{Y_1}{X_1} = \frac{C + DH_{N-1}(z)}{A + BH_{N-1}(z)}$ , from

which we get  $H_{N-1}(z) = \frac{C - AH_N(z)}{BH_N(z) - D}$ . Substituting the expression for  $H_N(z)$  we then arrive

$$\text{at } H_{N-1}(z) = \frac{C \left( 1 + \sum_{i=1}^N d_i z^{-i} \right) - A \left( \sum_{i=0}^N p_i z^{-i} \right)}{B \left( \sum_{i=0}^N p_i z^{-i} \right) - D \left( 1 + \sum_{i=1}^N d_i z^{-i} \right)}$$

$$= \frac{(C - Ap_0) + (Cd_1 - Ap_1)z^{-1} + \mathbf{L} + (Cd_{N-1} - Ap_{N-1})z^{-N+1} + (Cd_N - Ap_N)z^{-N}}{(Bp_0 - D) + (Bp_1 - Dd_1)z^{-1} + \mathbf{L} + (Bp_{N-1} - Dd_{N-1})z^{-N+1} + (Bp_N - Dd_N)z^{-N}}$$

Substituting the values  $A = 1$ ,  $B = d_N z^{-1}$ ,  $C = p_0$ , and  $D = p_N z^{-1}$ , we get  $H_{N-1}(z)$

$$\begin{aligned} &= \frac{(p_0 - p_0) + (p_0 d_1 - p_1)z^{-1} + \mathbf{L} + (p_0 d_{N-1} - p_{N-1})z^{-N+1} + (p_0 d_N - p_N)z^{-N}}{(d_N p_0 - p_N)z^{-1} + (d_N p_1 - p_N d_1)z^{-2} + \mathbf{L} + (d_N p_{N-1} - p_N d_{N-1})z^{-N} + (d_N p_N - p_N d_N)z^{-N-1}} \\ &= \frac{(p_0 d_1 - p_1) + (p_0 d_2 - p_2)z^{-1} + \mathbf{L} + (p_0 d_{N-1} - p_{N-1})z^{-N+2} + (p_0 d_N - p_N)z^{-N+1}}{(p_0 d_N - p_N) + (p_1 d_N - p_N d_1)z^{-1} + \mathbf{L} + (p_{N-1} d_N - p_N d_{N-1})z^{-N+1}} \\ &= \frac{p_0' + p_1' z^{-1} + \mathbf{L} + p_{N-2}' z^{-N+2} + p_{N-1}' z^{-N+1}}{1 + d_1' z^{-1} + \mathbf{L} + d_{N-1}' z^{-N+1}} \quad \text{where} \end{aligned}$$

$$p_k' = \frac{p_k d_{k+1} - p_{k+1}}{p_0 d_N - p_N}, \quad k = 0, 1, \mathbf{K}, N-1, \quad \text{and} \quad d_k' = \frac{p_k d_N - p_N d_k}{p_0 d_N - p_N}, \quad k = 1, 2, \mathbf{K}, N-1.$$

**(b)** From the chain parameters, we obtain for the first stage  $t_{11} = \frac{C}{A} = p_0$ ,

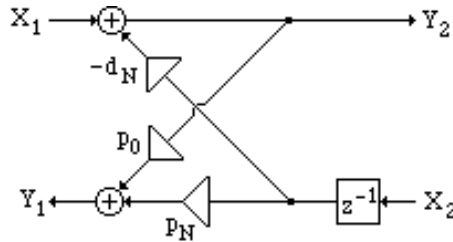
$$t_{12} = \frac{AD - BC}{A} = (p_N - p_0 d_N)z^{-1}, \quad t_{21} = \frac{1}{A} = 1, \quad \text{and} \quad t_{22} = -\frac{B}{A} = -d_N z^{-1}.$$

The corresponding input-output relations are then given by

$$Y_1 = p_0 X_1 + (p_N - p_0 d_N) z^{-1} = p_0 (X_1 - d_N z^{-1} X_2) + p_N z^{-1} X_2,$$

$Y_2 = X_1 - d_N z^{-1} X_2$ . Substituting the second equation into the first we rewrite it as

$Y_1 = p_0 Y_2 + p_N z^{-1} X_2$ . A realization of the two-pair based on the last two equations is therefore as indicated below:

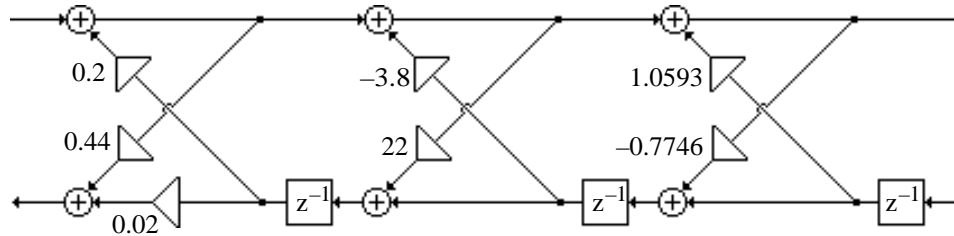


Except for the first stage, all other stages require 2 multipliers. Hence the total number of multipliers needed to implement an  $N$ -th order transfer function  $H_N(z)$  is  $2N+1$ . The total number of two-input adders required is  $2N$  while the overall realization is canonic requiring  $N$  delays.

**6.32** From  $H_3(z) = \frac{0.44z^{-1} + 0.362z^{-2} + 0.02z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$ , using Eq. (6.135) we arrive at

$$H_2(z) = \frac{22 + 18.1z^{-1} + z^{-2}}{1 + 4.8z^{-1} + 8.8z^{-2}}. \text{ Repeating the procedure we obtain } H_1(z) = \frac{1.0593 + z^{-1}}{1 + 0.7746z^{-1}}.$$

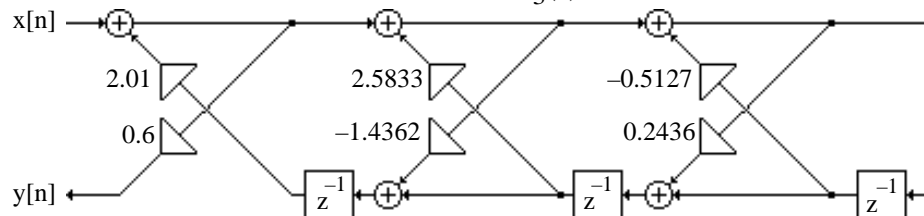
From  $H_3(z)$ ,  $H_2(z)$  and  $H_1(z)$  we then arrive at the cascaded lattice realization of  $H_3(z)$  as shown below:



**6.33 (a)** From  $H_3(z) = \frac{0.6 - 0.07z^{-1} - 1.55z^{-2}}{1 + 2.77z^{-1} - 1.593z^{-2} - 2.01z^{-3}}$  (Note the change in the labelling of the

transfer function) using Eq. (6.135) we arrive at  $H_2(z) = \frac{-1.4362 - 0.4927z^{-1} + z^{-2}}{1 - 0.1167z^{-1} - 2.5833z^{-2}}$ .

Repeating the procedure we obtain  $H_1(z) = \frac{0.2436 + z^{-1}}{1 + 0.5127z^{-1}}$ . From  $H_3(z)$ ,  $H_2(z)$  and  $H_1(z)$  we then arrive at the cascaded lattice realization of  $H_3(z)$  as shown below:

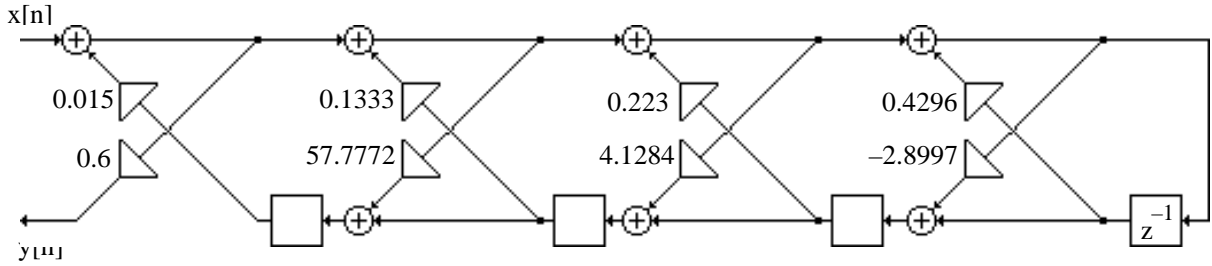


(b) From  $H_4(z) = \frac{4.1 - 1.3667z^{-1} + 1.64z^{-2} - 0.5467z^{-3}}{1 - 1.2z^{-1} + 0.15z^{-2} + 0.13z^{-3} - 0.015z^{-4}}$  (Note the change in the labelling of the transfer function) using Eq. (6.135) we arrive at

$$H_3(z) = \frac{57.7772 + 16.6667z^{-1} - 17.5561z^{-2} + z^{-3}}{1 - 0.3333z^{-1} + 0.4z^{-2} - 0.1333z^{-3}}. \text{ Repeating the procedure we obtain}$$

$$H_2(z) = \frac{4.1284 - 4.6735z^{-1} + z^{-2}}{1 + 0.2170z^{-1} - 0.2230z^{-2}}, \text{ and } H_2(z) = \frac{-2.8997 + z^{-1}}{1 - 0.4296z^{-1}}. \text{ From } H_4(z), H_3(z), H_2(z)$$

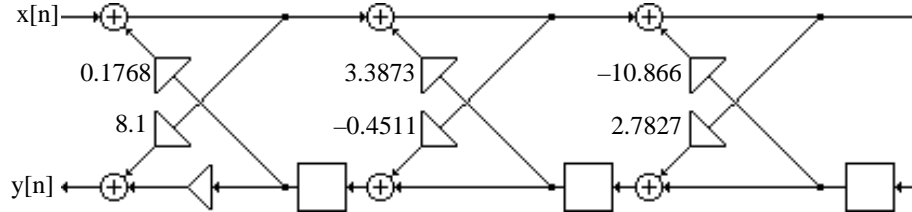
and  $H_1(z)$  we then arrive at the cascaded lattice realization of  $H_3(z)$  as shown below:



(c) From  $H_3(z) = \frac{8.1 + 6.93z^{-1} - 23.82z^{-2} + 10.5z^{-3}}{1 + 1.52z^{-1} + 0.18z^{-2} - 0.1768z^{-3}}$  using Eq. (6.135) we arrive at

$$H_2(z) = \frac{-0.4511 - 2.1185z^{-1} + z^{-2}}{1 + 1.4403z^{-1} - 3.3873z^{-2}}. \text{ Repeating the procedure we obtain } H_2(z) = \frac{2.7827 + z^{-1}}{1 + 10.866z^{-1}}.$$

From  $H_3(z)$ ,  $H_2(z)$  and  $H_1(z)$  we then arrive at the cascaded lattice realization of  $H_3(z)$  as shown below:



### 6.34 When $H_N(z)$ is an allpass transfer function of the form

$$H_N(z) = A_N(z) = \frac{d_N + d_{N-1}z^{-1} + \dots + z^{-N}}{1 + d_1z^{-1} + \dots + d_Nz^{-N}}, \text{ then from Eq. (6.135a), the numerator coefficients}$$

of  $H_{N-1}(z)$  are given by  $p_k' = \frac{p_0 d_{k+1} - p_{k+1}}{p_0 d_N - p_N} = \frac{d_N d_{k+1} - d_{N-k-1}}{d_N^2 - 1}$ , and from Eq. (6.135b) the

denominator coefficients of  $H_{N-1}(z)$  are given by

$$d'_{N-k-1} = \frac{p_{N-k-1} d_N - d_{N-k-1}}{p_0 d_N - p_N} = \frac{d_{k+1} d_N - d_{N-k-1}}{d_N^2 - 1} = p_k', \text{ implying } H_{N-1}(z) \text{ is an allpass}$$

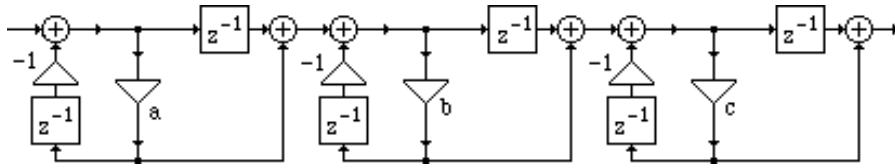
transfer function of order  $N-1$ . Since here  $p_N = 1$  and  $p_0 = d_N$ , the lattice structure of Problem 6.31 then reduces to the lattice structure employed in the Gray-Markel realization procedure.

**6.35 (a)** Consider the realization of Type 1B allpass transfer function. From its transfer parameters given in Eq. (6.62b) we arrive at  $Y_1 = z^{-1}X_1 + (1+z^{-1})X_2 = z^{-1}(X_1 + X_2) + X_2$ , and  $Y_2 = (1-z^{-1})X_1 - z^{-1}X_2 = X_1 - z^{-1}(X_1 + X_2)$ . A realization of the two-pair based on these two equations is as shown below which leads to the structure of Figure 6.36(b).

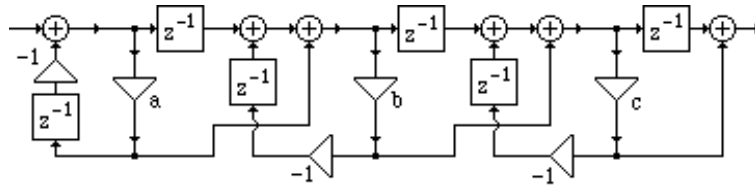
**(b)** From the transfer parameters of Type 1A<sub>t</sub> allpass given in Eq. (6.62c) we obtain  $Y_1 = z^{-1}X_1 + X_2$ , and  $Y_2 = (1-z^{-2})X_1 - z^{-1}X_2 = X_1 - z^{-1}(z^{-1}X_1 + X_2) = X_1 - z^{-1}Y_1$ . A realization of the two-pair based on these two equations is as shown below which leads to the structure of Figure 6.36(c).

**(c)** From the transfer parameters of Type 1B<sub>t</sub> allpass given in Eq. (6.62d) we obtain  $Y_1 = z^{-1}X_1 + (1-z^{-1})X_2 = z^{-1}(X_1 - X_2) + X_2$ , and  $Y_2 = (1+z^{-1})X_1 - z^{-1}X_2 = X_1 + z^{-1}(X_1 - X_2)$ . A realization of the two-pair based on these two equations is as shown below which leads to the structure of Figure 6.36(d).

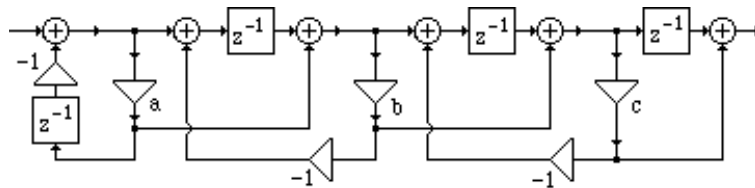
**6.36 (a)** A cascade connection of three Type 1A first-order allpass networks is shown below which is seen to require 6 delays:



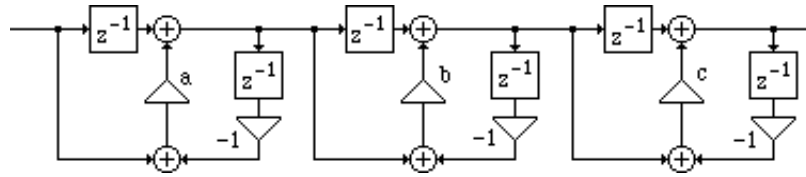
Simple block-diagram manipulations results in the structure shown below:



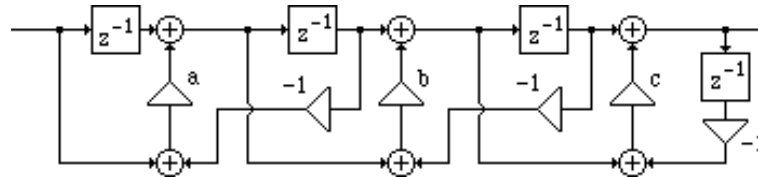
By delay-sharing between adjacent allpass sections we arrive at the following equivalent realization requiring now 4 delays.



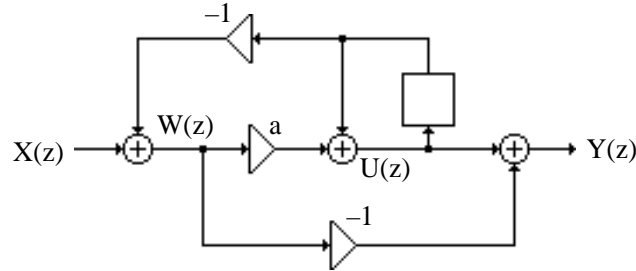
**(b)** A cascade connection of three Type 1A<sub>t</sub> first-order allpass networks is shown below which is seen to require 6 delays:



By delay-sharing between adjacent allpass sections we arrive at the following equivalent realization requiring now 4 delays.



**6.37** Analyzing the structure of Figure P6.14 we obtain



(1):  $W(z) = X(z) - z^{-1}U(z)$ , (2)  $U(z) = aW(z) + z^{-1}U(z)$ , and (3):  $Y(z) = -W(z) + U(z)$ .

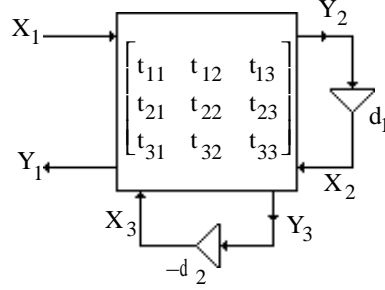
From Eq. (2) we obtain (4):  $U(z) = \frac{a}{1 - z^{-1}} W(z)$ . Substituting Eq. (4) in Eq. (3) we get

(5):  $Y(z) = -\left(\frac{1 - a - z^{-1}}{1 - z^{-1}}\right)W(z)$ . Substituting Eq. (4) in Eq. (1) we get

(6):  $X(z) = \left(\frac{1 - z^{-1} + az^{-1}}{1 - z^{-1}}\right)W(z)$ . From Eqs. (5) and (6) we finally get

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-(1-a) + z^{-1}}{1 - (1-a)z^{-1}}.$$

**6.38** We realize  $A_2(z) = \frac{d_1 d_2 + d_1 z^{-1} + z^{-2}}{1 + d_1 z^{-1} + d_1 d_2 z^{-2}}$  in the form of a constrained three-pair as indicated below:



From the above figure, we have  $\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ , and  $X_2 = d_1 Y_2$ ,  $X_3 = -d_2 Y_3$ .

From these equations, after some algebra we get  $A_2(z) = \frac{Y_1}{X_1} = \frac{N(z)}{D(z)}$ , where

$$D(z) = 1 - d_1 t_{22} + d_2 t_{33} + d_1 d_2 (t_{23} t_{32} - t_{22} t_{33}), \text{ and}$$

$$N(z) = t_{11} - d_1 (t_{11} t_{22} - t_{12} t_{21}) + d_2 (t_{11} t_{33} - t_{13} t_{31}) + d_1 d_2 \{ t_{21} (t_{12} t_{33} - t_{13} t_{32}) + t_{31} (t_{22} t_{13} - t_{12} t_{23}) + t_{11} (t_{23} t_{32} - t_{22} t_{33}) \}.$$

Comparing the denominator of the desired allpass transfer function with  $D(z)$  we get

$t_{22} = z^{-1}$ ,  $t_{33} = 0$ ,  $t_{23} t_{32} = z^{-2}$ . Next, comparing the numerator of the desired allpass transfer function with  $N(z)$  we get  $t_{11} = z^{-1}$ ,  $t_{12} t_{21} = z^{-1}(z^{-2} - 1)$ ,  $t_{13} t_{31} = 0$ , and  $t_{32}(t_{11} t_{23} - t_{21} t_{13}) + t_{31}(t_{22} t_{13} - t_{12} t_{23}) = 1$ . Substituting the appropriate transfer parameters from the previous equations into the last equation we simplify it to  $t_{13} t_{21} t_{32} + t_{31} t_{12} t_{23} = z^{-4} - 1$ . Since  $t_{13} t_{31} = 0$ , either  $t_{13} = 0$ , or  $t_{31} = 0$ . (Both cannot be simultaneously equal to zero, as this will violate the condition  $t_{13} t_{21} t_{32} + t_{31} t_{12} t_{23} = z^{-4} - 1$ .)

Consider the case  $t_{13} = 0$ . Then the equation above reduces to  $t_{31} t_{12} t_{23} = z^{-4} - 1$ . From this equation and  $t_{23} t_{32} = z^{-2}$ , it follows that

$$t_{32} = z^{-2}, t_{23} = 1, t_{31} t_{12} = z^{-4} - 1 = (z^{-1} - 1)(z^{-1} + 1)(z^{-2} + 1).$$

There are four possible realizable sets of values of  $t_{21}$  and  $t_{31}$  satisfying the last equation and  $t_{12} t_{21} = z^{-1}(z^{-2} - 1)$ . These lead to four different realizable transfer matrices for the three-port:

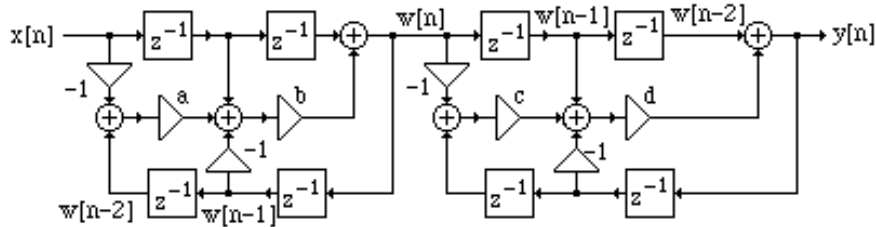
$$\begin{aligned} \text{Type 2A: } & \begin{bmatrix} z^{-2} & z^{-2} - 1 & 0 \\ z^{-1} & z^{-1} & 1 \\ z^{-2} + 1 & z^{-2} & 0 \end{bmatrix}, & \text{Type 2B: } & \begin{bmatrix} z^{-2} & z^{-1} + 1 & 0 \\ z^{-1}(z^{-1} - 1) & z^{-1} & 1 \\ (z^{-2} + 1)(z^{-1} - 1) & z^{-2} & 0 \end{bmatrix}, \\ \text{Type 2C: } & \begin{bmatrix} z^{-2} & z^{-1} - 1 & 0 \\ z^{-1}(z^{-1} + 1) & z^{-1} & 1 \\ (z^{-2} + 1)(z^{-1} + 1) & z^{-2} & 0 \end{bmatrix}, & \text{Type 2D: } & \begin{bmatrix} z^{-2} & 1 & 0 \\ z^{-1}(z^{-2} - 1) & z^{-1} & 1 \\ z^{-4} - 1 & z^{-2} & 0 \end{bmatrix}. \end{aligned}$$

A realization of each Type 2 allpass structures is obtained by implementing its respective transfer matrix, and then constraining the  $Y_2$  and  $X_2$  variables through the multiplier  $d_1$  and

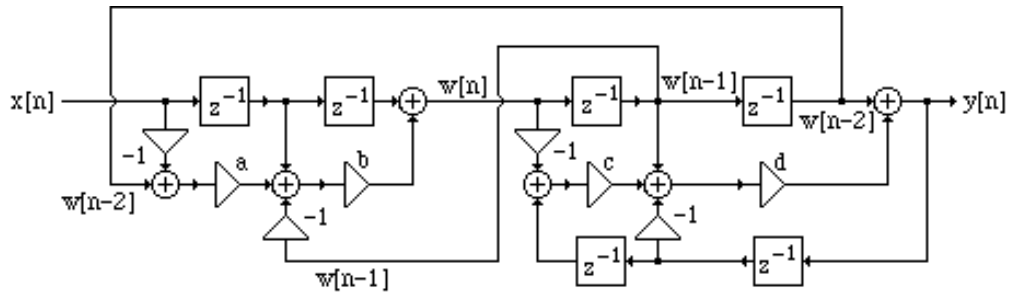
constraining the  $Y_3$  and  $X_3$  variables through the multiplier  $-d_2$  resulting in the four structures shown in Figure 6.38 of text.

It can be easily shown that the allpass structures obtained for the case  $t_{31} = 0$  are precisely the transpose of the structures of Figure 6.38.

**6.39** A cascade connection of two Type 2D second-order allpass networks is shown below which is seen to require 8 delays:



By delay-sharing between adjacent allpass sections we arrive at the following equivalent realization requiring now 6 delays.



The minimum number of multipliers needed to implement a cascade of  $M$  Type 2D second-order allpass sections is thus  $4 + 2(M-1) = 2(M + 1)$ .

**6.40** We realize  $A_2(z) = \frac{d_2 + d_1 z^{-1} + z^{-2}}{1 + d_1 z^{-1} + d_2 z^{-2}}$  in the form of a constrained three-pair as indicated in

the figure in the solution of Problem 6.46. Comparing the numerator and the denominator of the Type 3 allpass transfer function with  $N(z)$  and  $D(z)$  given in the solution of Problem 6.46 we arrive at  $t_{11} = z^{-2}$ ,  $t_{22} = z^{-1}$ ,  $t_{33} = z^{-2}$ ,  $t_{23}t_{32} = z^{-2}$ ,  $t_{23}t_{32} = z^{-1}(z^{-2} - 1)$ ,  $t_{13}t_{31} = z^{-4} - 1$ ,  $t_{13}t_{21}t_{32} + t_{13}t_{21}t_{32} = z^{-3}(z^{-2} - 1) + z^{-1}(z^{-4} - 1)$ . To solve the last four equations, we preselect  $t_{23}$  and  $t_{32}$  satisfying  $t_{23}t_{32} = z^{-2}$ , and then determine realizable values for  $t_{12}$ ,  $t_{21}$ ,  $t_{13}$ , and  $t_{31}$ ,



Choice #1:  $t_{23} = z^{-1}$ ,  $t_{32} = z^{-2}$ . This leads to four possible realizable sets of values of  $t_{12}$ ,  $t_{21}$ ,  $t_{13}$ , and  $t_{31}$ , satisfying the constraint equations given earlier and resulting in the transfer matrices given below:

$$\begin{aligned} \text{Type 3A: } & \begin{bmatrix} z^{-2} & z^{-2}-1 & z^{-2}-1 \\ z^{-1} & z^{-1} & z^{-1} \\ z^{-2}+1 & z^{-2} & z^{-2} \end{bmatrix}, \text{ Type 3B: } \begin{bmatrix} z^{-2} & z^{-1}+1 & z^{-1}+1 \\ z^{-1}(z^{-1}-1) & z^{-1} & z^{-1} \\ (z^{-1}-1)(z^{-2}+1) & z^{-2} & z^{-2} \end{bmatrix}, \\ \text{Type 3C: } & \begin{bmatrix} z^{-2} & z^{-1}-1 & z^{-1}-1 \\ z^{-1}(z^{-1}+1) & z^{-1} & z^{-1} \\ (z^{-1}+1)(z^{-2}+1) & z^{-2} & z^{-2} \end{bmatrix}, \text{ Type 3D: } \begin{bmatrix} z^{-2} & 1 & 1 \\ z^{-1}(z^{-2}-1) & z^{-1} & z^{-1} \\ z^{-4}-1 & z^{-2} & z^{-2} \end{bmatrix}. \end{aligned}$$

A realization of each Type 3 allpass structures is obtained by implementing its respective transfer matrix, and then constraining the  $Y_2$  and  $X_2$  variables through the multiplier  $d_1$  and constraining the  $Y_3$  and  $X_3$  variables through the multiplier  $-d_2$ . Realizations of Types 3A, 3C and 3D allpass are shown in Figure 6.38 of text. The realization of Type 3B allpass is shown below:

Choice #2:  $t_{23} = 1$ ,  $t_{32} = z^{-3}$ . This leads to four possible realizable sets of values of  $t_{12}$ ,  $t_{21}$ ,  $t_{13}$ , and  $t_{31}$ , satisfying the constraint equations given earlier and resulting in the transfer matrices given below:

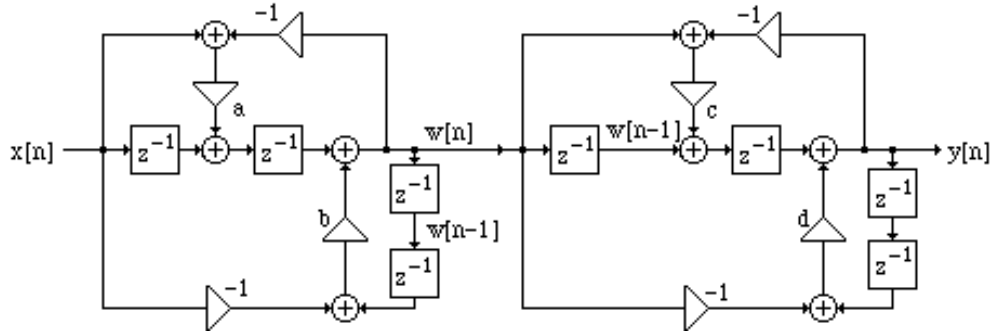
$$\begin{aligned} \text{Type 3E: } & \begin{bmatrix} z^{-2} & z^{-1}(z^{-2}-1) & z^{-2}-1 \\ z^{-1} & z^{-1} & 1 \\ z^{-2}+1 & z^{-3} & z^{-2} \end{bmatrix}, \text{ Type 3F: } \begin{bmatrix} z^{-2} & z^{-1}(z^{-1}+1) & z^{-1}+1 \\ z^{-1}-1 & z^{-1} & 1 \\ (z^{-1}-1)(z^{-2}+1) & z^{-3} & z^{-2} \end{bmatrix}, \\ \text{Type 3G: } & \begin{bmatrix} z^{-2} & z^{-1}(z^{-1}-1) & z^{-1}-1 \\ z^{-1}+1 & z^{-1} & 1 \\ (z^{-1}+1)(z^{-2}+1) & z^{-3} & z^{-2} \end{bmatrix}, \text{ Type 3H: } \begin{bmatrix} z^{-2} & z^{-1}(z^{-1}-1) & z^{-1}-1 \\ z^{-2}-1 & z^{-1} & 1 \\ z^{-4}-1 & z^{-3} & z^{-2} \end{bmatrix}. \end{aligned}$$

Realization of Types 3H is shown in Figure 6.38 of text. The realizations of Types 3E, 3F and 3G allpass are shown below:

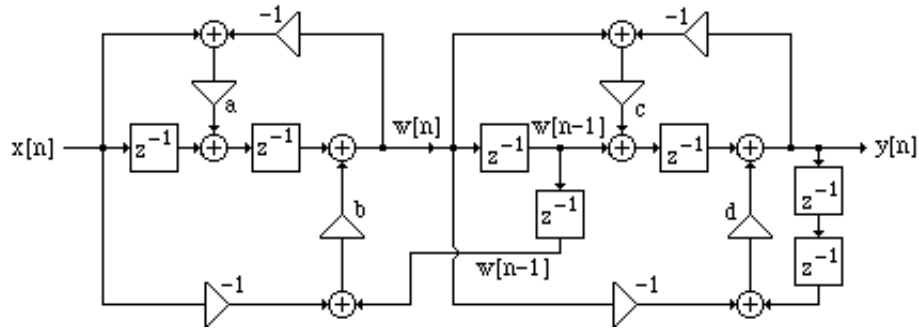
Choice #3:  $t_{23} = z^{-2}$ ,  $t_{32} = z^{-1}$ . The structures in this case are the transpose of the Types 3A, 3B, 3C and 3D allpass networks of Choice #1 given above.

Choice #4:  $t_{23} = z^{-3}$ ,  $t_{32} = 1$ . The structures in this case are the transpose of the Types 3E, 3F, 3G and 3H allpass networks of Choice #2 given above.

**6.41** A cascade connection of two Type 3H second-order allpass networks is shown below which is seen to require 8 delays:

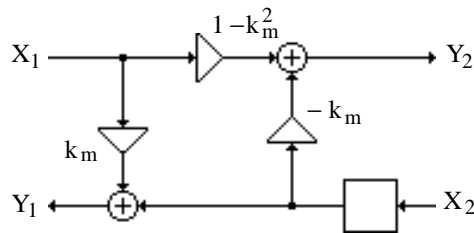


By delay-sharing between adjacent allpass sections we arrive at the following equivalent realization requiring now 7 delays.

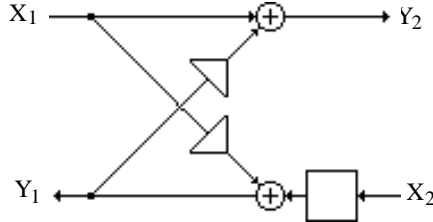


The minimum number of multipliers needed to implement a cascade of  $M$  Type 3H second-order allpass sections is thus  $4 + 3(M-1) = 3M + 1$ .

**6.42** From the transfer parameters given in Eq. (6.52d) we arrive at the input-output relations of the two-pair as (1):  $Y_1 = k_m X_1 + z^{-1} X_2$ , and (2)  $Y_2 = (1 - k_m^2) X_1 - k_m z^{-1} X_2$ . A three-multiplier realization based on these two equations is shown below:



**6.43** Eq. (2) in the solution given above can be rewritten as  $Y_2 = X_1 - k_m^2 X_1 - k_m z^{-1} X_2 = X_1 - k_m (k_m X_1 + z^{-1} X_2)$ . Substituting Eq. (1) of the solution given above in this equation we then get (3):  $Y_2 = X_1 - k_m^2 X_1 - k_m z^{-1} X_2 = X_1 - k_m Y_1$ . A realization based on Eq. (3) and Eq. (1) in the solution given above results in the lattice structure shown below:

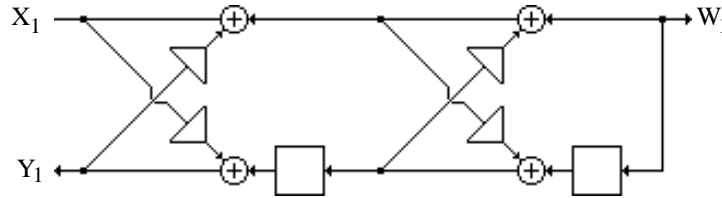


Analyzing the above structure we obtain  $Y_1 = k_m X_1 + z^{-1} X_2$ , and  $Y_2 = X_1 - k_m Y_1 = X_1 - k_m (k_m X_1 + z^{-1} X_2) = (1 - k_m^2) X_1 - k_m z^{-1} X_2$ . Hence, the transfer parameters of the lattice two-pair are given by

$$t_{11} = k_m, \quad t_{12} = z^{-1}, \quad t_{21} = 1 - k_m^2, \quad t_{22} = -k_m z^{-1}.$$

The corresponding chain parameters are obtained using Eq. (4.176b) and are given by

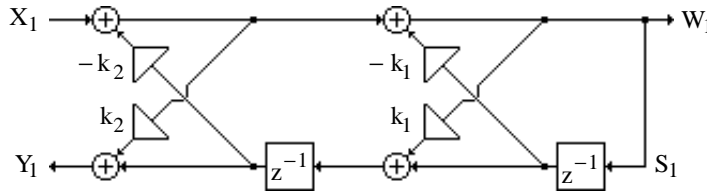
$$A = \frac{1}{1 - k_m^2}, \quad B = \frac{k_m z^{-1}}{1 - k_m^2}, \quad C = \frac{k_m}{1 - k_m^2}, \quad D = \frac{z^{-1}}{1 - k_m^2}.$$



Hence, the input-output relation of the all-pole cascaded lattice structure given above can be

expressed as:  $\begin{bmatrix} X_1(z) \\ Y_1(z) \end{bmatrix} = \frac{1}{(1 - k_2^2)(1 - k_1^2)} \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \begin{bmatrix} W_1(z) \\ S_1(z) \end{bmatrix}$ , from which we

obtain  $H_r(z) = \frac{W_1(z)}{X_1(z)} = \frac{(1 - k_2^2)(1 - k_1^2)}{1 + k_1(1 + k_2)z^{-1} + k_2 z^{-2}}$ .



Likewise, from Section 6.8.1, we arrive at the input-output relation of the all-pole cascaded

lattice structure given above as:  $\begin{bmatrix} X_1(z) \\ Y_1(z) \end{bmatrix} = \begin{bmatrix} 1 & k_2 z^{-1} \\ k_2 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_1 z^{-1} \\ k_1 & z^{-1} \end{bmatrix} \begin{bmatrix} W_1(z) \\ S_1(z) \end{bmatrix}$ , from which we

obtain  $H_s(z) = \frac{W_1(z)}{X_1(z)} = \frac{1}{1 + k_1(1 + k_2)z^{-1} + k_2 z^{-2}}$ .

Now for a second-order all-pole transfer function can also be expressed in terms of its poles as

$H(z) = \frac{1}{1 - 2r \cos \omega_0 z^{-1} + r^2 z^{-2}}$ . Comparing the denominator of  $H(z)$  with that of  $H_s(z)$

and  $H_r(z)$  we observe  $k_2 = r^2$  and  $k_1(1+k_2) = -2r \cos \omega_0$ . As  $r \cong 1$ ,  
 $k_1 = -\frac{2r \cos \omega_0}{1+r^2} = -\cos \omega_0$ .

At the true resonance peak, we have  $|H(e^{j\omega_r})| = \frac{1}{(1-r^2)|\sin \omega_0|}$ . Likewise, at  $\omega = \omega_r$ ,

$$|H_s(e^{j\omega_r})| = \frac{1}{(1-k_2)\sqrt{1-k_1^2}}. \text{ As } r \rightarrow 1, k_2 = r^2 \rightarrow 1, \text{ and hence, the gain } |H_s(e^{j\omega_r})|$$

becomes excessively large. In the case of the all-pole structure realized using the reverse lattice two-pairs, as  $r \rightarrow 1$ ,  $k_2 = r^2 \rightarrow 1$ ,

$$(1-k_1^2)(1-k_2^2) = (1-k_1^2)(1-k_2)(1+k_2) \cong 2(1-k_1^2)(1-k_2). \text{ Hence, at } \omega = \omega_r,$$

$$|H_r(e^{j\omega_r})| \cong \frac{2(1-k_1^2)(1-k_2)}{(1-k_2)\sqrt{1-k_1^2}} = 2\sqrt{1-k_1^2} = 2|\sin \omega_0|. \text{ This indicates that the gain at the}$$

peak resonance is approximately independent of the pole radius as long as the pole angle is constant and the radius is close to 1.

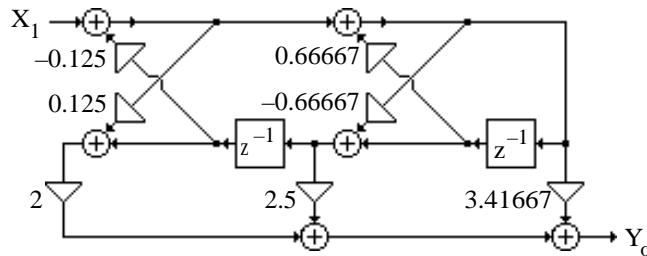
**6.44 (a)**  $H_1(z) = \frac{2+z^{-1}+2z^{-2}}{1-0.75z^{-1}+0.125z^{-2}}$ . Choose,  $A_2(z) = \frac{0.125-0.75z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-2}}$ . Note

$k_2 = A_2(\infty) = d_2 = 0.125 < 1$ . Using Eq. (6.71) we next determine the coefficients of  $A_1(z)$

arriving at  $A_1(z) = \frac{0.66667+z^{-1}}{1-0.66667z^{-1}}$ . Here,  $k_1 = A_1(\infty) = d_1' = -0.66667$ . Hence,

$|k_1| = 0.66667 < 1$ . Therefore,  $A_2(z)$  and hence  $H_1(z)$  is stable.

To determine the feed-forward coefficients we use  $\alpha_1 = p_2 = 2$ ,  $\alpha_2 = p_1 - \alpha_1 d_1 = 2.5$ ,  
 $\alpha_3 = p_0 - \alpha_1 d_2 - \alpha_2 d_1' = 3.41667$ . Final realization of  $H_1(z)$  is thus as shown below:

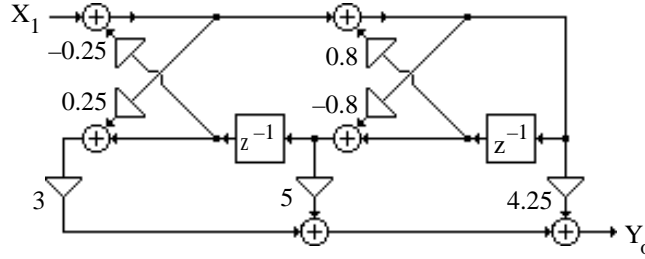


**(b)**  $H_2(z) = \frac{1+2z^{-1}+3z^{-2}}{1-z^{-1}+0.25z^{-2}}$ . Thus,  $A_2(z) = \frac{0.25-z^{-1}+z^{-2}}{1-z^{-1}+0.25z^{-2}}$ . Note  $k_2 = A_2(\infty) = d_2 = 0.25 < 1$ .

Using Eq. (6.71) we next determine the coefficients of  $A_1(z)$  arriving at  $A_1(z) = \frac{-0.8+z^{-1}}{1-0.8z^{-1}}$ .

Here,  $k_1 = A_1(\infty) = -0.8$ . Thus,  $|k_1| = 0.8 < 1$ . Therefore,  $A_2(z)$  and hence  $H_2(z)$  is stable.

To determine the feed-forward coefficients we use  $\alpha_1 = p_2 = 3$ ,  $\alpha_2 = p_1 - \alpha_1 d_1 = 5$ ,  $\alpha_3 = p_0 - \alpha_1 d_2 - \alpha_2 k_1 = 4.25$ . Final realization of  $H_2(z)$  is thus as shown below:



(c)  $H_3(z) = \frac{2+5z^{-1}+8z^{-2}+3z^{-3}}{1+0.75z^{-1}+0.5z^{-2}+0.25z^{-3}}$ . This implies,  $A_3(z) = \frac{0.25+0.5z^{-1}+0.75z^{-2}+z^{-3}}{1+0.75z^{-1}+0.5z^{-2}+0.25z^{-3}}$ .

Note  $k_3 = A_3(\infty) = d_3 = 0.25 < 1$ . Using Eq. (6.71) we next determine the coefficients of  $A_2(z)$

arriving at  $A_2(z) = \frac{\frac{1}{3} + \frac{2}{3}z^{-1} + z^{-2}}{1 + \frac{2}{3}z^{-1} + \frac{1}{3}z^{-2}}$ . Note  $k_2 = A_2(\infty) = d_2' = 1/3 < 1$ . Using Eq. (6.71) we

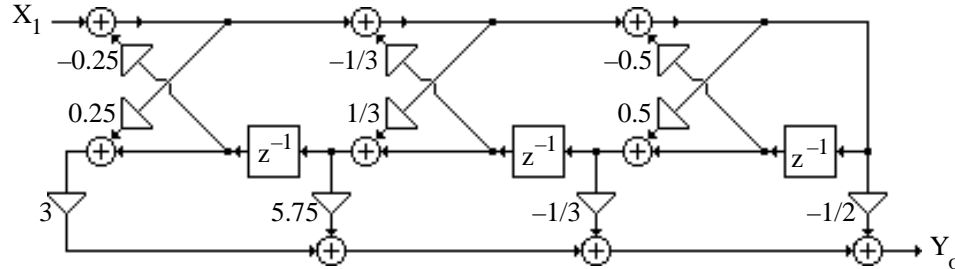
next determine the coefficients of  $A_1(z)$  arriving at  $A_1(z) = \frac{0.5+z^{-1}}{1+0.5z^{-1}}$ . Thus,

$k_1 = A_1(\infty) = d_1'' = 0.5 < 1$ . Since  $|k_i| < 1$  for  $i = 3, 2, 1$ ,  $A_3(z)$  and hence  $H_3(z)$  is stable.

To determine the feed-forward coefficients we make use of Eq. (6.98) and obtain  $\alpha_1 = p_3 = 3$ ,

$\alpha_2 = p_2 - \alpha_1 d_1 = 5.75$ ,  $\alpha_3 = p_1 - \alpha_1 d_2 - \alpha_2 d_1' = -\frac{1}{3}$ .  $\alpha_4 = p_0 - \alpha_1 d_3 - \alpha_2 d_2' - \alpha_3 d_1'' = -\frac{1}{2}$ .

Final realization of  $H_3(z)$  is thus as shown below:



(d)  $H_4(z) = \frac{1+1.6z^{-1}+0.6z^{-2}}{1-z^{-1}-0.25z^{-2}+0.25z^{-3}}$ . This implies  $A_3(z) = \frac{0.25-0.25z^{-1}-z^{-2}+z^{-3}}{1-z^{-1}-0.25z^{-2}+0.25z^{-3}}$ . Note

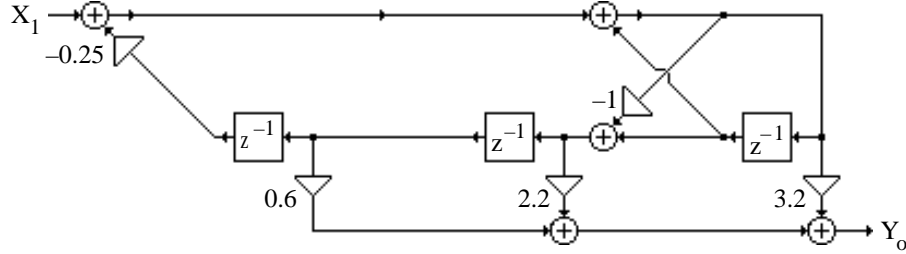
$k_3 = A_3(\infty) = d_3 = 0.25 < 1$ . Using Eq. (6.71) we next determine the coefficients of  $A_2(z)$

arriving at  $A_2(z) = \frac{-1+z^{-1}}{1-z^{-1}}$ . Note  $k_2 = A_2(\infty) = d_2' = 0$ . Using Eq. (6.71) we next determine

the coefficients of  $A_1(z)$  arriving at  $A_1(z) = \frac{-1+z^{-1}}{1-z^{-1}}$ .  $k_1 = A_1(\infty) = d_1'' = -1$ . Since

$|k_1| = 1$ ,  $A_3(z)$  and hence  $H_4(z)$  is unstable. Feed-forward coefficients are next determined using

Eq. (6.98) and are given by  $\alpha_1 = 0, \alpha_2 = 0.6, \alpha_3 = 2.2, \alpha_4 = 3.2$ . Final realization of  $H_4(z)$  is thus as shown below:



(e)  $H_5(z) = \frac{3 + 1.5z^{-1} + z^{-2} + 0.5z^{-3}}{1 - 1.8333z^{-1} + 1.5z^{-2} - 0.5833z^{-3} + 0.0833z^{-4}}$ . Hence,

$$A_4(z) = \frac{0.0833 - 0.5833z^{-1} + 1.5z^{-2} - 1.8333z^{-3} + z^{-4}}{1 - 1.8333z^{-1} + 1.5z^{-2} - 0.5833z^{-3} + 0.0833z^{-4}}. \text{ Note } k_4 = A_4(\infty) = d_4 = 0.0833 < 1.$$

Using Eq. (6.71) we next determine the coefficients of  $A_3(z)$  arriving at

$$A_3(z) = \frac{0.433595 + 1.38466z^{-1} - 1.7972z^{-2} + z^{-3}}{1 - 1.7972z^{-1} + 1.38466z^{-2} - 0.433595z^{-3}}. \text{ Thus, } k_3 = A_3(\infty) = d_3' = 0.433595 < 1.$$

Continuing this process we obtain  $A_2(z) = \frac{0.74558 - 1.4739z^{-1} + z^{-2}}{1 - 1.4739z^{-1} + 0.74558z^{-2}}$ . Thus,

$$k_2 = A_2(\infty) = d_2' = 0.74558 < 1. \text{ Finally we arrive at } A_1(z) = \frac{-0.84436 + z^{-1}}{1 - 0.84436z^{-1}}. \text{ This implies}$$

$k_1 = A_1(\infty) = d_1''' = -0.84436$ . Since  $|k_i| < 1$  for  $i = 4, 3, 2, 1$ ,  $A_4(z)$  and hence  $H_5(z)$  is stable.

Feed-forward coefficients are next determined using Eq. (6.98) and are given by

$$\alpha_1 = 0, \alpha_2 = 0.5, \alpha_3 = 1.89859, \alpha_4 = 3.606, \alpha_5 = 4.846.$$

**6.45 (a)**  $H_1(z) = \frac{0.6 - 0.07z^{-1} - 1.55z^{-2}}{1 + 2.77z^{-1} - 1.593z^{-2} - 2.01z^{-3}}$ . Hence

$$A_3(z) = \frac{-2.01 - 1.593z^{-1} + 2.77z^{-2} + z^{-3}}{1 + 2.77z^{-1} - 1.593z^{-2} - 2.01z^{-3}}. \text{ Note } k_3 = A_3(\infty) = -2.01. \text{ Using Eq. (6.47) we}$$

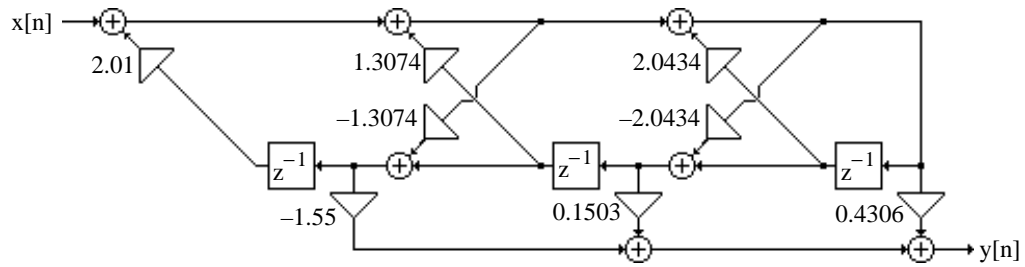
then arrive at  $A_2(z) = \frac{-1.3074 + 0.1421z^{-1} + z^{-2}}{1 + 0.1421z^{-1} - 1.3074z^{-2}}$ . Note  $k_2 = A_2(\infty) = -1.3074$ . Using Eq.

(6.47) again we next arrive at  $A_1(z) = \frac{-2.0434 + z^{-1}}{1 - 2.0434z^{-1}}$ . Note  $k_1 = A_1(\infty) = -2.0434$ . Since

$|k_i| > 1, 1 \leq i \leq 3$ ,  $A_3(z)$  and hence,  $H_1(z)$  is an unstable transfer function. Next using Eq. (6.79) we arrive at the feed-forward multipliers:  $\alpha_1 = 0, \alpha_2 = -1.55, \alpha_3 = 0.1503$ , and  $\alpha_4 = 0.4306$ .

Note that if Program 6\_4 is used to realize  $H_1(z)$ , the following error

message will appear: ??? Error using ==> tf2latc  
The IIR system is unstable



(b)  $H_2(z) = \frac{4.1 - 1.3667z^{-1} + 1.64z^{-3} - 0.5467z^{-4}}{1 - 1.2z^{-1} + 0.15z^{-2} + 0.13z^{-3} - 0.015z^{-4}}$ . Hence,

$A_4(z) = \frac{-0.015 + 0.13z^{-1} + 0.15z^{-2} - 1.2z^{-3} + z^{-4}}{1 - 1.2z^{-1} + 0.15z^{-2} + 0.13z^{-3} - 0.015z^{-4}}$ . Note  $A_4(\infty) = k_4 = -0.015$ . Using Eq.

(6.47) we then arrive at  $A_3(z) = \frac{0.112 + 0.1523z^{-1} - 1.1983z^{-2} + z^{-3}}{1 - 1.1983z^{-1} + 0.1523z^{-2} + 0.112z^{-3}}$ . Note

$A_3(\infty) = k_3 = 0.112$ . Using Eq. (6.47) again we next arrive at

$A_2(z) = \frac{0.2902 - 1.2308z^{-1} + z^{-2}}{1 - 1.2308z^{-1} + 0.2902z^{-2}}$ . Note  $A_2(\infty) = k_2 = 0.2902$ . Using Eq. (6.47) we

finally arrive at  $A_1(z) = \frac{-0.9540 + z^{-1}}{1 - 0.9540z^{-1}}$ . Note  $A_1(\infty) = k_1 = -0.9540$ . Since  $|k_1| < 1$ ,

$1 \leq k_i \leq 4$ ,  $A_4(z)$  and hence,  $H_2(z)$  is a stable transfer function.

The feed-forward multipliers  $\{\alpha_i\}$  are determined using the following equations:

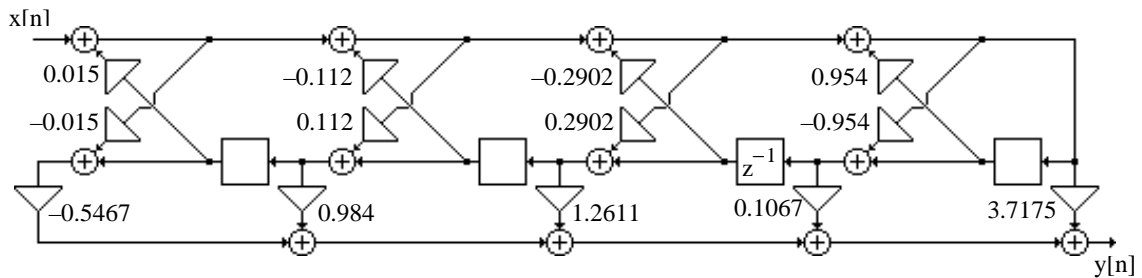
$\alpha_1 = p_4 = -0.5467$ ,  $\alpha_2 = p_3 - \alpha_1 d_1 = 0.9840$ ,  $\alpha_3 = p_2 - \alpha_1 d_2 - \alpha_2 d_1' = 1.2611$ ,

$\alpha_4 = p_1 - \alpha_1 d_3 - \alpha_2 d_2' - \alpha_3 d_3'' = 0.1067$ , and  $\alpha_5 = p_0 - \alpha_1 d_4 - \alpha_2 d_3' - \alpha_3 d_2'' - \alpha_4 d_1''' = 3.7175$ ,

The lattice and feed-forward multipliers can also be obtained using Program 6\_4 yielding

Lattice parameters are  
-0.9540 0.2902 0.1120 -0.0150

Feedforward multipliers are  
3.7175 0.1067 1.2611 0.9840 -0.5467



(c)  $H_3(z) = \frac{8.1 + 6.93z^{-1} - 23.82z^{-2} + 10.5z^{-3}}{1 + 1.52z^{-1} + 0.18z^{-2} - 0.1768z^{-3}}$ . Hence,

$A_3(z) = \frac{-0.1768 + 0.18z^{-1} + 1.52z^{-2} + z^{-3}}{1 + 1.52z^{-1} + 0.18z^{-2} - 0.1768z^{-3}}$ . Note  $A_3(\infty) = k_3 = -0.1768$ . Using Eq. (6.47)

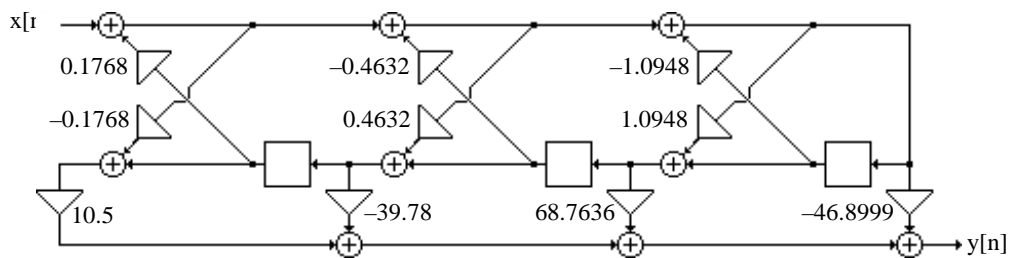
we then arrive at  $A_2(z) = \frac{0.4632 + 1.6019z^{-1} + z^{-2}}{1 + 1.6019z^{-1} + 0.4632z^{-2}}$ . Note  $A_2(\infty) = k_2 = 0.4632$ . Using Eq.

(6.47) again we next arrive at  $A_1(z) = \frac{1.0948 + z^{-1}}{1 + 1.0948z^{-1}}$ . Note  $A_1(\infty) = k_1 = 1.0948$ . Since

$|k_1| > 1$ ,  $A_3(z)$ , and hence,  $H_3(z)$  are unstable transfer functions. Next using Eq. (6.79) we arrive at the feed-forward multipliers:  $\alpha_1 = 10.5$ ,  $\alpha_2 = -39.78$ ,  $\alpha_3 = 68.7636$ , and  $\alpha_4 = -46.8999$ .

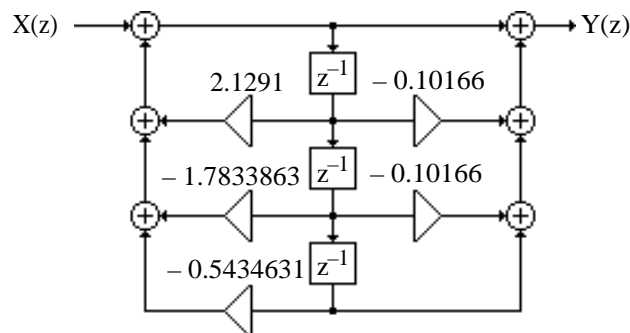
Note that if Program 6\_4 is used to realize  $H_3(z)$ , the following error message will appear:

??? Error using ==> tf2latc  
The IIR system is unstable



6.52  $H(z) = \frac{0.5634(1+z^{-1})(1-1.10166z^{-1}+z^{-2})}{(1-0.683z^{-1})(1-1.4461z^{-1}+0.7957z^{-2})}$ .

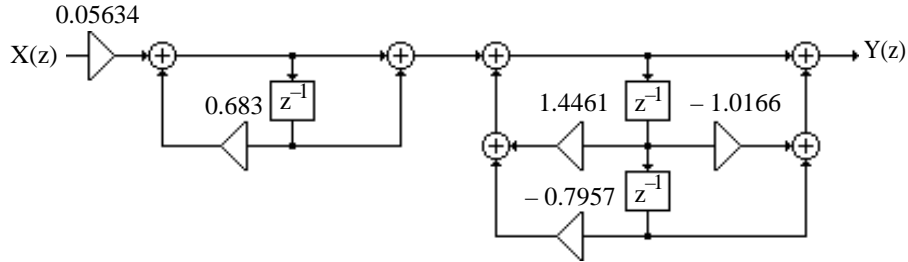
(a) Direct canonic form -  $H(z) = \frac{0.05634(1-0.10166z^{-1}-0.10166z^{-2}+z^{-3})}{1-2.1291z^{-1}+1.78339z^{-2}-0.54346z^{-3}}$ .



Hardware requirements: # of multipliers = 5, # of two-input adders = 6, # of delays = 3.

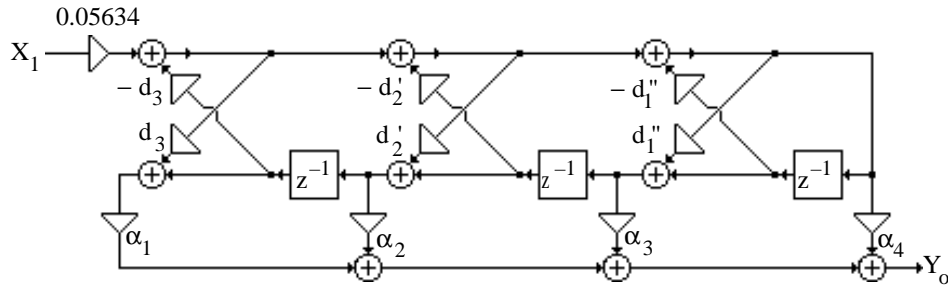
(b) Cascade Form





Hardware requirements: # of multipliers = 5, # of two-input adders = 6, # of delays = 3.

**(c) Gray-Markel Form -**



$$d_3 = -0.5434631, \quad d_2' = 0.8881135, \quad d_1'' = -0.8714813.$$

$$\alpha_1 = p_3 = 1, \quad \alpha_2 = p_2 - \alpha_1 d_1 = 2.02744, \quad \alpha_3 = p_1 - \alpha_1 d_2 - \alpha_2 d_1' = 1.45224,$$

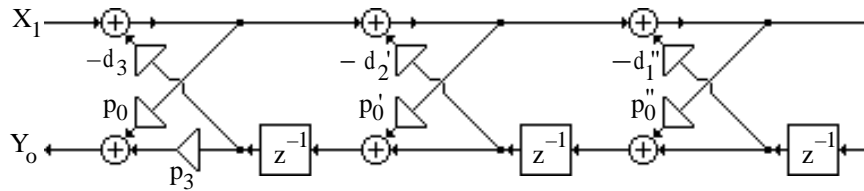
$$\alpha_4 = p_0 - \alpha_1 d_3 - \alpha_2 d_2' - \alpha_3 d_1'' = 1.00702.$$

Hardware requirements: # of multipliers = 9, # of two-input adders = 6, # of delays = 3.

**(d) Cascaded Lattice Structure -**

$$\text{Using Eqn. (6.152) we obtain } H_2(z) = \frac{p_0' + p_1' z^{-1} + p_2' z^{-2}}{1 + d_1' z^{-1} + d_2' z^{-2}} = \frac{1.3136 - 1.2213 z^{-1} + z^{-2}}{1 - 1.4152 z^{-1} + 1.1197 z^{-2}}.$$

$$\text{and } H_1(z) = \frac{p_0'' + p_1'' z^{-1}}{1 + d_1'' z^{-1}} = \frac{-1.3546 + z^{-1}}{1 - 0.1015 z^{-1}}. \text{ The final structure is as shown below:}$$



$$\text{where } p_0 = 0.5634, \quad d_3 = -0.54346, \quad p_3 = 0.5634, \quad p_0' = 1.3136, \quad d_2' = 1.1197, \\ p_0'' = -1.3546, \text{ and } d_1'' = -0.1015.$$

**6.47 (a)** A partial fraction expansion of  $G(z)$  is of the form  $G(z) = d + \sum_{i=1}^{N/2} \frac{v_i}{z - \lambda_i} + \sum_{i=1}^{N/2} \frac{v_i^*}{z - \lambda_i^*}$ . If we

define  $H(z) = \frac{d}{2} + \sum_{i=1}^{N/2} \frac{v_i}{z - \lambda_i}$ , then we can write  $G(z) = H(z) + H_*(z)$ , where  $H_*(z)$  represents the transfer function obtained from  $H(z)$  by conjugating its coefficients.

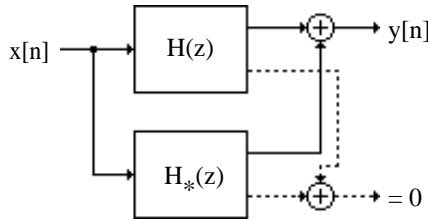
(b) In this case, the partial fraction expansion of  $G(z)$  is of the form

$$G(z) = d + \sum_{i=1}^{N_r} \frac{\rho_i}{z - \xi_i} + \sum_{i=1}^{N_c/2} \frac{v_i}{z - \lambda_i} + \sum_{i=1}^{N_c/2} \frac{v_i^*}{z - \lambda_i^*},$$

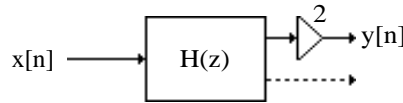
where  $N_r$  and  $N_c$  are the number of real poles  $\xi_i$ 's and complex poles  $\lambda_i$ 's, respectively, with residues  $\rho_i$ 's and  $v_i$ 's. We can thus decompose

$$G(z) \text{ as } G(z) = H(z) + H_*(z), \text{ where } H(z) = \frac{d}{2} + \sum_{i=1}^{N_r} \frac{\rho_i/2}{z - \xi_i} + \sum_{i=1}^{N_c/2} \frac{v_i}{z - \lambda_i}.$$

(c) An implementation of real coefficient  $G(z)$  is thus simply a parallel connection of two complex filters characterized by transfer functions  $H(z)$  and  $H_*(z)$  as indicated in the figure below:



However, for a real valued input  $x[n]$ , the output of  $H(z)$  is the complex conjugate of  $H_*(z)$ . As a result, two times the real part of the output of  $H(z)$  is the desired real-valued sequence  $y[n]$  indicating that single complex filter  $H(z)$  is sufficient to realize  $G(z)$  as indicated below:

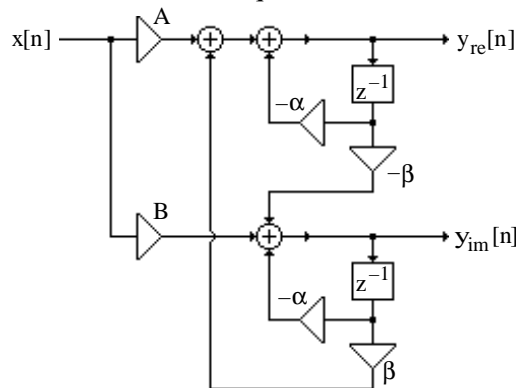


**6.48** From  $H(z) = \frac{Y(z)}{X(z)} = \frac{A + jB}{1 + (\alpha + j\beta)z^{-1}}$ , we arrive at the difference equation representation

$$y_{re}[n] + jy_{im}[n] = -(\alpha + j\beta)(y_{re}[n-1] + jy_{im}[n-1]) + Ax[n] + jBx[n],$$

which is equivalent to a set of two difference equations involving all real variables and real multiplier coefficients:  
 $y_{re}[n] = -\alpha y_{re}[n-1] + \beta y_{im}[n-1] + Ax[n]$ , and  $y_{im}[n] = -\beta y_{re}[n-1] - \alpha y_{im}[n-1] + Bx[n]$ .

A realization of  $H(z)$  based on the last two equations is shown below:



To determine the transfer function  $Y_{re}(z)/X(z)$ , we take the  $z$ -transforms of the last two difference equations and arrive at  $(1 + \alpha z^{-1})Y_{re}(z) - \beta z^{-1}Y_{im}(z) = AX(z)$ , and

$\beta z^{-1} Y_{re}(z) + (1 + \alpha z^{-1}) Y_{im}(z) = BX(z)$ . Solving these two equations we get

$$\frac{Y_{re}(z)}{X(z)} = \frac{A + (A\alpha + B\beta)z^{-1}}{1 + 2\alpha z^{-1} + (\alpha^2 + \beta^2)z^{-2}}, \text{ which is seen to be a second-order transfer function.}$$

**6.49** An  $m$ -th order complex allpass function is given by

$$A_m(z) = \frac{\alpha_m^* + \alpha_{m-1}^* z^{-1} + \mathbf{L} + \alpha_1^* z^{-(m-1)} + \alpha_m^* z^{-m}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \mathbf{L} + \alpha_{m-1} z^{-(m-1)} + \alpha_m z^{-m}}.$$

To generate an  $(m-1)$ -th order allpass we use the recursion

$$A_{m-1}(z) = \frac{P_{m-1}(z)}{D_{m-1}(z)} = z \left[ \frac{A_m(z) - k_m^*}{1 - k_m A_m(z)} \right].$$

Substituting the expression for  $A_m(z)$  in the above we obtain after some algebra

$$\begin{aligned} P_{m-1}(z) &= z[\alpha_m^* + \alpha_{m-1}^* z^{-1} + \alpha_{m-2}^* z^{-2} + \mathbf{L} + \alpha_1^* z^{-(m-1)} + z^{-m} \\ &\quad - \alpha_m^*(1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \mathbf{L} + \alpha_{m-1} z^{-(m-1)} + \alpha_m z^{-m})] \\ &= (\alpha_{m-1}^* - \alpha_m^* \alpha_1) + (\alpha_{m-2}^* - \alpha_m^* \alpha_2) z^{-1} + \mathbf{L} + (\alpha_1^* - \alpha_m^* \alpha_{m-1}) z^{-(m-2)} + (1 - |\alpha_m|^2) z^{-(m-1)}, \end{aligned}$$

$$\begin{aligned} D_{m-1}(z) &= 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \mathbf{L} + \alpha_{m-1} z^{-(m-1)} + \alpha_m z^{-m} \\ &\quad - \alpha_m^*(\alpha_m^* + \alpha_{m-1}^* z^{-1} + \alpha_{m-2}^* z^{-2} + \mathbf{L} + \alpha_1^* z^{-(m-1)} + z^{-m}) \\ &= (1 - |\alpha_m|^2) + (\alpha_1 - \alpha_m \alpha_{m-1}^*) z^{-1} + (\alpha_2 - \alpha_m \alpha_{m-2}^*) z^{-2} + \mathbf{L} + (\alpha_{m-1} - \alpha_m \alpha_1^*) z^{-(m-1)}. \end{aligned}$$

Hence,  $A_{m-1}(z)$  is a complex allpass function of order  $m-1$  given by

$$A_{m-1}(z) = \frac{\beta_{m-1}^* + \beta_{m-2}^* z^{-1} + \mathbf{L} + \beta_1^* z^{-(m-2)} + z^{-(m-1)}}{1 + \beta_1 z^{-1} + \beta_2 z^{-2} + \mathbf{L} + \beta_{m-2} z^{-(m-2)} + \beta_{m-1} z^{-(m-1)}},$$

where  $\beta_k = \frac{\alpha_k - \alpha_m \alpha_{m-k}^*}{1 - |\alpha_m|^2}$ ,  $k = 1, 2, \dots, m-1$ .

To develop a realization of  $A_m(z)$  we express  $A_m(z)$  in terms of  $A_{m-1}(z)$ :

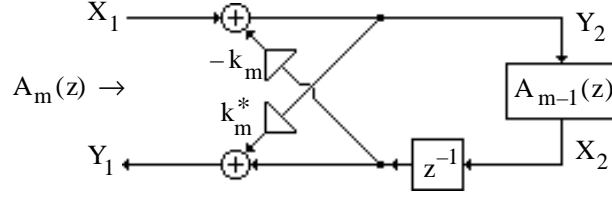
$$A_m(z) = \frac{Y_1}{X_1} = \frac{k_m^* - z^{-1} A_{m-1}(z)}{1 + k_m z^{-1} A_{m-1}(z)},$$

and compare it with Eq. (6.73) resulting in the following expressions for the transfer

parameters of the two-pair:  $t_{11} = k_m^*$ ,  $t_{22} = -k_m z^{-1}$ , and  $t_{12} t_{21} = (1 - k_m k_m^*) z^{-1}$ . As in the case of the realization of a real allpass function, there are many possible choices for  $t_{12}$  and  $t_{21}$ .

We choose  $t_{12} = (1 - k_m k_m^*) z^{-1}$ ,  $t_{21} = 1$ . The corresponding input-output relations of the two-pair  $Y_1 = k_m^* X_1 + (1 - k_m k_m^*) z^{-1} X_2 = k_m^* (X_1 - k_m z^{-1} X_2) + z^{-1} X_2$ , and  $Y_2 = X_1 - k_m z^{-1} X_2$ .

A realization of  $A_m(z)$  based on the above two-pair relations is indicated below:



By continuing this process, we arrive at a cascaded lattice realization of a complex allpass transfer function.

**6.50 (a)**  $H_1(z) = \frac{2+2z^{-1}}{3+z^{-1}} = \frac{1}{2} \left( 1 + \frac{1+3z^{-1}}{3+z^{-1}} \right) = \frac{1}{2} (A_0(z) + A_1(z))$ , where  $A_0(z) = 1$  and

$$A_1(z) = \frac{1+3z^{-1}}{3+z^{-1}}.$$

**(b)**  $H_2(z) = \frac{1-z^{-1}}{4+2z^{-1}} = \frac{1}{2} \left( 1 - \frac{2+4z^{-1}}{4+2z^{-1}} \right) = \frac{1}{2} (A_0(z) - A_1(z))$ , where  $A_0(z) = 1$  and

$$A_1(z) = \frac{2+4z^{-1}}{4+2z^{-1}}.$$

**(c)**  $H_3(z) = \frac{1-z^{-2}}{4+2z^{-1}+2z^{-2}} = \frac{1}{2} \left( 1 - \frac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}} \right) = \frac{1}{2} (A_0(z) - A_1(z))$ , where  $A_0(z) = 1$

and  $A_1(z) = \frac{2+2z^{-1}+4z^{-2}}{4+2z^{-1}+2z^{-2}}.$

**(d)**  $H_4(z) = \frac{3+9z^{-1}+9z^{-2}+3z^{-3}}{12+10z^{-1}+2z^{-2}} = \frac{1}{2} \left( \frac{3+9z^{-1}+9z^{-2}+3z^{-3}}{(3+z^{-1})(2+z^{-1})} \right) = \frac{1}{2} \left[ z^{-1} \left( \frac{1+3z^{-1}}{3+z^{-1}} \right) + \left( \frac{1+2z^{-1}}{2+z^{-1}} \right) \right]$   
 $= \frac{1}{2} (A_0(z) + A_1(z))$ , where  $A_0(z) = z^{-1} \left( \frac{1+3z^{-1}}{3+z^{-1}} \right)$  and  $A_1(z) = \left( \frac{1+2z^{-1}}{2+z^{-1}} \right).$

**6.51 (a)** From the equation given we get

$$y[2l] = h[0]x[2l] + h[1]x[2l-1] + h[2]x[2l-2] + h[3]x[2l-3] + h[4]x[2l-4] + h[5]x[2l-5], \text{ and}$$

$$y[2l+1] = h[0]x[2l+1] + h[1]x[2l] + h[2]x[2l-1] + h[3]x[2l-2] + h[4]x[2l-3] + h[5]x[2l-4].$$

Rewriting the above two equations in a matrix form we arrive at

$$\begin{bmatrix} y[2l] \\ y[2l+1] \end{bmatrix} = \begin{bmatrix} h[0] & 0 \\ h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[2l] \\ x[2l+1] \end{bmatrix} + \begin{bmatrix} h[2] & h[1] \\ h[3] & h[3] \end{bmatrix} \begin{bmatrix} x[2l-2] \\ x[2l-1] \end{bmatrix} \\ + \begin{bmatrix} h[4] & h[3] \\ h[5] & h[4] \end{bmatrix} \begin{bmatrix} x[2l-4] \\ x[2l-3] \end{bmatrix} + \begin{bmatrix} 0 & h[5] \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x[2l-6] \\ x[2l-5] \end{bmatrix},$$

which can be alternately expressed as

$$\mathbf{Y}_1 = \mathbf{H}_0 \mathbf{X}_1 + \mathbf{H}_1 \mathbf{X}_{1-1} + \mathbf{H}_2 \mathbf{X}_{1-2} + \mathbf{H}_3 \mathbf{X}_{1-3},$$

where  $\mathbf{Y}_1 = \begin{bmatrix} y[21] \\ y[21+1] \end{bmatrix}$ ,  $\mathbf{X}_1 = \begin{bmatrix} x[21] \\ x[21+1] \end{bmatrix}$ ,  $\mathbf{H}_0 = \begin{bmatrix} h[0] & 0 \\ h[1] & h[0] \end{bmatrix}$ ,  $\mathbf{H}_1 = \begin{bmatrix} h[2] & h[1] \\ h[3] & h[3] \end{bmatrix}$ ,  
 $\mathbf{H}_2 = \begin{bmatrix} h[4] & h[3] \\ h[5] & h[4] \end{bmatrix}$ , and  $\mathbf{H}_3 = \begin{bmatrix} 0 & h[5] \\ 0 & 0 \end{bmatrix}$ .

- (b) Here  $y[31] = h[0]x[31] + h[1]x[31-1] + h[2]x[31-2] + h[3]x[31-3] + h[4]x[31-4] + h[5]x[31-5]$ ,  
 $y[31+1] = h[0]x[31+1] + h[1]x[31] + h[2]x[31-1] + h[3]x[31-2] + h[4]x[31-3] + h[5]x[31-4]$ , and  
 $y[31+2] = h[0]x[31+2] + h[1]x[31+1] + h[2]x[31] + h[3]x[31-1] + h[4]x[31-2] + h[5]x[31-3]$ .

Rewriting the above two equations in a matrix form we arrive at

$$\begin{bmatrix} y[31] \\ y[31+1] \\ y[31+2] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 \\ h[1] & h[0] & 0 \\ h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[31] \\ x[31+1] \\ x[31+2] \end{bmatrix} + \begin{bmatrix} h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \end{bmatrix} \begin{bmatrix} x[31-3] \\ x[31-2] \\ x[31-1] \end{bmatrix} \\ + \begin{bmatrix} 0 & h[2] & h[1] \\ 0 & 0 & h[2] \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x[31-6] \\ x[31-5] \\ x[31-4] \end{bmatrix},$$

which can be alternately expressed as  $\mathbf{Y}_1 = \mathbf{H}_0\mathbf{X}_1 + \mathbf{H}_1\mathbf{X}_{1-1} + \mathbf{H}_2\mathbf{X}_{1-2}$ , where

$$\mathbf{Y}_1 = \begin{bmatrix} y[31] \\ y[31+1] \\ y[31+2] \end{bmatrix}, \quad \mathbf{X}_1 = \begin{bmatrix} x[31] \\ x[31+1] \\ x[31+2] \end{bmatrix}, \quad \mathbf{H}_0 = \begin{bmatrix} h[0] & 0 & 0 \\ h[1] & h[0] & 0 \\ h[2] & h[1] & h[0] \end{bmatrix}, \quad \mathbf{H}_1 = \begin{bmatrix} h[3] & h[2] & h[1] \\ h[4] & h[3] & h[2] \\ h[5] & h[4] & h[3] \end{bmatrix}, \text{ and} \\ \mathbf{H}_2 = \begin{bmatrix} 0 & h[2] & h[1] \\ 0 & 0 & h[2] \\ 0 & 0 & 0 \end{bmatrix}.$$

- (c) Following a procedure similar to that outlined in Parts (a) and (b) above, we can show that here

$$\mathbf{Y}_1 = \mathbf{H}_0\mathbf{X}_1 + \mathbf{H}_1\mathbf{X}_{1-1} + \mathbf{H}_2\mathbf{X}_{1-2}, \text{ where} \\ \mathbf{Y}_1 = \begin{bmatrix} y[41] \\ y[41+1] \\ y[41+2] \\ y[41+3] \end{bmatrix}, \quad \mathbf{X}_1 = \begin{bmatrix} x[41] \\ x[41+1] \\ x[41+2] \\ x[41+3] \end{bmatrix}, \quad \mathbf{H}_0 = \begin{bmatrix} h[0] & 0 & 0 & 0 \\ h[1] & h[0] & 0 & 0 \\ h[2] & h[1] & h[0] & 0 \\ h[3] & h[2] & h[1] & h[0] \end{bmatrix}, \\ \mathbf{H}_1 = \begin{bmatrix} h[4] & h[3] & h[2] & h[1] \\ h[5] & h[4] & h[3] & h[2] \\ 0 & h[5] & h[4] & h[3] \\ 0 & 0 & h[5] & h[4] \end{bmatrix}, \text{ and } \mathbf{H}_2 = \begin{bmatrix} 0 & 0 & 0 & h[5] \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

**6.52 (a)**  $d_0 y[21] + d_1 y[21-1] + d_2 y[21-2] + d_3 y[21-3] + d_4 y[21-4]$   
 $= p_0 x[21] + p_1 x[21-1] + p_2 x[21-2] + p_3 x[21-3] + p_4 x[21-4],$   
 $d_0 y[21+1] + d_1 y[21] + d_2 y[21-1] + d_3 y[21-2] + d_4 y[21-3]$   
 $= p_0 x[21+1] + p_1 x[21] + p_2 x[21-1] + p_3 x[21-2] + p_4 x[21-3],$

Rewriting the above two equations in a matrix form we arrive at

$$\begin{bmatrix} d_0 & 0 \\ d_1 & d_0 \end{bmatrix} \begin{bmatrix} y[21] \\ y[21+1] \end{bmatrix} + \begin{bmatrix} d_2 & d_1 \\ d_3 & d_2 \end{bmatrix} \begin{bmatrix} y[21-2] \\ y[21-1] \end{bmatrix} + \begin{bmatrix} d_4 & d_3 \\ 0 & d_4 \end{bmatrix} \begin{bmatrix} y[21-4] \\ y[21-3] \end{bmatrix} \\ = \begin{bmatrix} p_0 & 0 \\ p_1 & p_0 \end{bmatrix} \begin{bmatrix} x[21] \\ x[21+1] \end{bmatrix} + \begin{bmatrix} p_2 & p_1 \\ p_3 & p_2 \end{bmatrix} \begin{bmatrix} x[21-2] \\ x[21-1] \end{bmatrix} + \begin{bmatrix} p_4 & p_3 \\ 0 & p_4 \end{bmatrix} \begin{bmatrix} x[21-4] \\ x[21-3] \end{bmatrix},$$

which can be alternately expressed as  $\mathbf{D}_0 \mathbf{Y}_1 + \mathbf{D}_1 \mathbf{Y}_{1-1} + \mathbf{D}_2 \mathbf{Y}_{1-2} = \mathbf{P}_0 \mathbf{X}_1 + \mathbf{P}_1 \mathbf{X}_{1-1} + \mathbf{P}_2 \mathbf{X}_{1-1}$ ,

where  $\mathbf{Y}_1 = \begin{bmatrix} y[21] \\ y[21+1] \end{bmatrix}$ ,  $\mathbf{X}_1 = \begin{bmatrix} x[21] \\ x[21+1] \end{bmatrix}$ ,  $\mathbf{D}_0 = \begin{bmatrix} d_0 & 0 \\ d_1 & d_0 \end{bmatrix}$ ,  $\mathbf{D}_1 = \begin{bmatrix} d_2 & d_1 \\ d_3 & d_2 \end{bmatrix}$ ,  $\mathbf{D}_2 = \begin{bmatrix} d_4 & d_3 \\ 0 & d_4 \end{bmatrix}$ ,

$\mathbf{P}_0 = \begin{bmatrix} p_0 & 0 \\ p_1 & p_0 \end{bmatrix}$ ,  $\mathbf{P}_1 = \begin{bmatrix} p_2 & p_1 \\ p_3 & p_2 \end{bmatrix}$ , and  $\mathbf{P}_2 = \begin{bmatrix} p_4 & p_3 \\ 0 & p_4 \end{bmatrix}$ .

$$(b) \quad \begin{bmatrix} d_0 & 0 & 0 \\ d_1 & d_0 & 0 \\ d_2 & d_1 & d_0 \end{bmatrix} \begin{bmatrix} y[31] \\ y[31+1] \\ y[31+2] \end{bmatrix} + \begin{bmatrix} d_3 & d_2 & d_1 \\ d_4 & d_3 & d_2 \\ 0 & d_4 & d_3 \end{bmatrix} \begin{bmatrix} y[31-3] \\ y[31-2] \\ y[31-1] \end{bmatrix} + \begin{bmatrix} 0 & 0 & d_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y[31-6] \\ y[31-5] \\ y[31-4] \end{bmatrix} \\ = \begin{bmatrix} p_0 & 0 & 0 \\ p_1 & p_0 & 0 \\ p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} x[31] \\ x[31+1] \\ x[31+2] \end{bmatrix} + \begin{bmatrix} p_3 & p_2 & p_1 \\ p_4 & p_3 & p_2 \\ 0 & p_4 & p_3 \end{bmatrix} \begin{bmatrix} x[31-3] \\ x[31-2] \\ x[31-1] \end{bmatrix} + \begin{bmatrix} 0 & 0 & p_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x[31-6] \\ x[31-5] \\ x[31-4] \end{bmatrix},$$

which can be alternately expressed as  $\mathbf{D}_0 \mathbf{Y}_1 + \mathbf{D}_1 \mathbf{Y}_{1-1} + \mathbf{D}_2 \mathbf{Y}_{1-2} = \mathbf{P}_0 \mathbf{X}_1 + \mathbf{P}_1 \mathbf{X}_{1-1} + \mathbf{P}_2 \mathbf{X}_{1-1}$ ,

where  $\mathbf{Y}_1 = \begin{bmatrix} y[31] \\ y[31+1] \\ y[31+2] \end{bmatrix}$ ,  $\mathbf{X}_1 = \begin{bmatrix} x[31] \\ x[31+1] \\ x[31+2] \end{bmatrix}$ ,  $\mathbf{D}_0 = \begin{bmatrix} d_0 & 0 & 0 \\ d_1 & d_0 & 0 \\ d_2 & d_1 & d_0 \end{bmatrix}$ ,  $\mathbf{D}_1 = \begin{bmatrix} d_3 & d_2 & d_1 \\ d_4 & d_3 & d_2 \\ 0 & d_4 & d_3 \end{bmatrix}$ ,

$\mathbf{D}_2 = \begin{bmatrix} 0 & 0 & d_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ,  $\mathbf{P}_0 = \begin{bmatrix} p_0 & 0 & 0 \\ p_1 & p_0 & 0 \\ p_2 & p_1 & p_0 \end{bmatrix}$ ,  $\mathbf{P}_1 = \begin{bmatrix} p_3 & p_2 & p_1 \\ p_4 & p_3 & p_2 \\ 0 & p_4 & p_3 \end{bmatrix}$ , and  $\mathbf{P}_2 = \begin{bmatrix} 0 & 0 & p_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

$$(c) \quad \begin{bmatrix} d_0 & 0 & 0 & 0 \\ d_1 & d_0 & 0 & 0 \\ d_2 & d_1 & d_0 & 0 \\ d_3 & d_2 & d_1 & d_0 \end{bmatrix} \begin{bmatrix} y[41] \\ y[41+1] \\ y[41+2] \\ y[41+3] \end{bmatrix} + \begin{bmatrix} d_4 & d_3 & d_2 & d_1 \\ 0 & d_4 & d_3 & d_2 \\ 0 & 0 & d_4 & d_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix} \begin{bmatrix} y[41-4] \\ y[41-3] \\ y[41-2] \\ y[41-1] \end{bmatrix} \\ = \begin{bmatrix} p_0 & 0 & 0 & 0 \\ p_1 & p_0 & 0 & 0 \\ p_2 & p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 & p_0 \end{bmatrix} \begin{bmatrix} x[41] \\ x[41+1] \\ x[41+2] \\ x[41+3] \end{bmatrix} + \begin{bmatrix} p_4 & p_3 & p_2 & p_1 \\ 0 & p_4 & p_3 & p_2 \\ 0 & 0 & p_4 & p_3 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \begin{bmatrix} x[41-4] \\ x[41-3] \\ x[41-2] \\ x[41-1] \end{bmatrix},$$

which can be alternately expressed as  $\mathbf{D}_0 \mathbf{Y}_1 + \mathbf{D}_1 \mathbf{Y}_{1-1} = \mathbf{P}_0 \mathbf{X}_1 + \mathbf{P}_1 \mathbf{X}_{1-1}$ , where

$\mathbf{Y}_1 = \begin{bmatrix} y[41] \\ y[41+1] \\ y[41+2] \\ y[41+3] \end{bmatrix}$ ,  $\mathbf{X}_1 = \begin{bmatrix} x[41] \\ x[41+1] \\ x[41+2] \\ x[41+3] \end{bmatrix}$ ,  $\mathbf{D}_0 = \begin{bmatrix} d_0 & 0 & 0 & 0 \\ d_1 & d_0 & 0 & 0 \\ d_2 & d_1 & d_0 & 0 \\ d_3 & d_2 & d_1 & d_0 \end{bmatrix}$ ,  $\mathbf{D}_1 = \begin{bmatrix} d_4 & d_3 & d_2 & d_1 \\ 0 & d_4 & d_3 & d_2 \\ 0 & 0 & d_4 & d_3 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$ ,

$\mathbf{P}_0 = \begin{bmatrix} p_0 & 0 & 0 & 0 \\ p_1 & p_0 & 0 & 0 \\ p_2 & p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 & p_0 \end{bmatrix}$ , and  $\mathbf{P}_1 = \begin{bmatrix} p_4 & p_3 & p_2 & p_1 \\ 0 & p_4 & p_3 & p_2 \\ 0 & 0 & p_4 & p_3 \\ 0 & 0 & 0 & p_4 \end{bmatrix}$ .

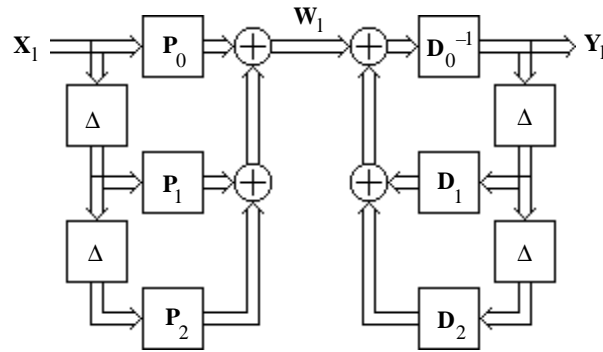
**6.53** We first rewrite the second-order block-difference equation

$$\mathbf{D}_0 \mathbf{Y}_1 + \mathbf{D}_1 \mathbf{Y}_{1-1} + \mathbf{D}_2 \mathbf{Y}_{1-2} = \mathbf{P}_0 \mathbf{X}_1 + \mathbf{P}_1 \mathbf{X}_{1-1} + \mathbf{P}_2 \mathbf{X}_{1-1},$$

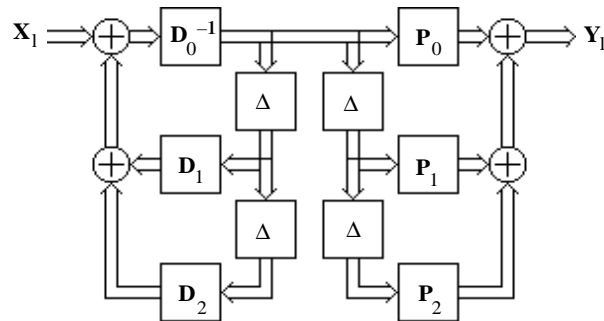
as two separate equations:  $\mathbf{W}_1 = \mathbf{P}_0 \mathbf{X}_1 + \mathbf{P}_1 \mathbf{X}_{1-1} + \mathbf{P}_2 \mathbf{X}_{1-1}$ , and

$$\mathbf{Y}_1 = -\mathbf{D}_0^{-1} \mathbf{D}_1 \mathbf{Y}_{1-1} - \mathbf{D}_0^{-1} \mathbf{D}_2 \mathbf{Y}_{1-2} + \mathbf{D}_0^{-1} \mathbf{W}_1.$$

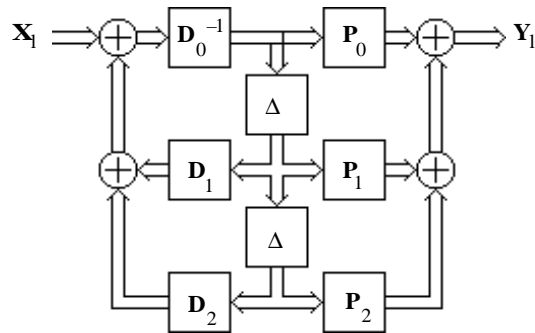
A cascade realization of the IIR block digital filter based on the above two equations is thus as shown below:



By interchanging the locations of the two block sections in the above structure we get an equivalent realization as indicated below:

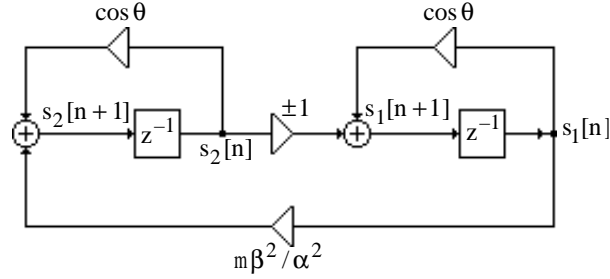


Finally, by delay-sharing the above structure reduces to a canonic realization as shown below:

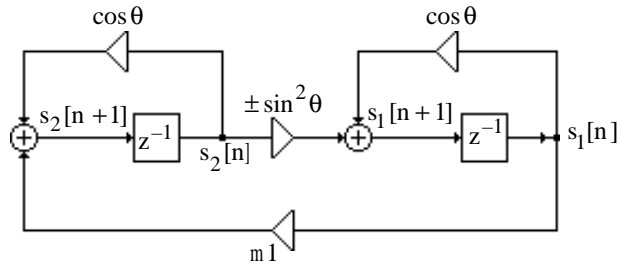


**6.54** By setting  $\alpha \sin \theta = \pm \beta$  in Eq. (6.138), the state-space description of the sine-cosine generator

reduces to 
$$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} \cos \theta & \pm 1 \\ m \frac{\beta^2}{\alpha^2} & \cos \theta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix},$$
 which leads to the three-multiplier structure shown below:



**6.55** By setting  $\alpha = \pm\beta\sin\theta$  in Eq. (6.138), the state-space description of the sine-cosine generator reduces to  $\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} \cos\theta & \pm\sin^2\theta \\ m1 & \cos\theta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}$ , which leads to the three-multiplier structure shown below:

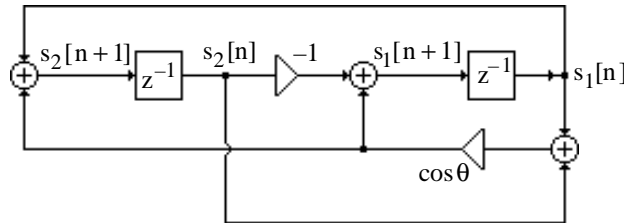


**6.56** Let  $\frac{\beta}{\alpha} = -\frac{1+\cos\theta}{\sin\theta}$ . Then  $\frac{\alpha}{\beta}\sin\theta = \frac{-\sin^2\theta}{\cos\theta+1} = \frac{\cos^2\theta-1}{\cos\theta+1} = \cos\theta-1$ , and,  $-\frac{\beta}{\alpha}\sin\theta = \cos\theta+1$ .

Substituting these values in Eq. (6.138) we arrive at  $\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos\theta-1 \\ \cos\theta+1 & \cos\theta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}$ ,

These equations can be alternately rewritten as

$s_1[n+1] = \cos\theta(s_1[n] + s_2[n]) - s_2[n]$ , and  $s_2[n+1] = \cos\theta(s_1[n] + s_2[n]) + s_1[n]$ . A realization based on the last two equations results in a single-multiplier structure as indicated below:

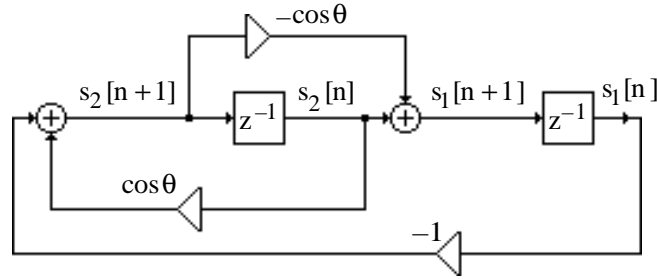


**6.57**  $\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & \frac{\alpha(C-\cos\theta)}{\beta\sin\theta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} + \begin{bmatrix} C & \frac{\alpha(1-C\cos\theta)}{\beta\sin\theta} \\ -\frac{\beta}{\alpha}\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}$ .

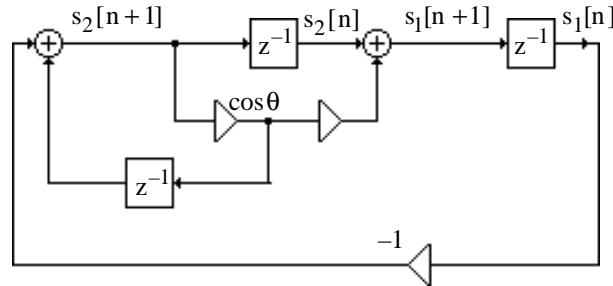
If  $C = 0$ , choose  $\alpha = \beta\sin\theta$ . Then

$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & -\cos\theta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & \cos\theta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}$ , which can be realized with two multipliers as shown below:





The above structure can be modified to yield a single multiplier realization as indicated on next page:

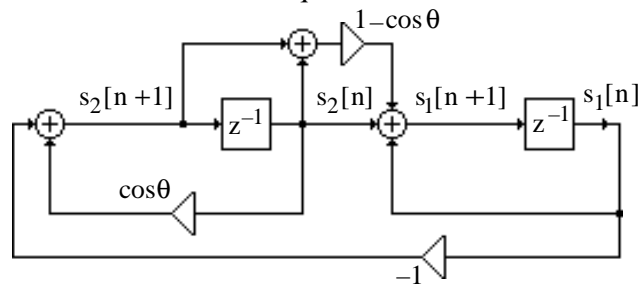


**6.58** If  $C = 1$ , then 
$$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & \frac{\alpha(1-\cos\theta)}{\beta\sin\theta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} + \begin{bmatrix} 1 & \frac{\alpha(1-\cos\theta)}{\beta\sin\theta} \\ -\frac{\beta}{\alpha}\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}.$$

Choose  $\alpha = \beta\sin\theta$ . This leads to

$$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1-\cos\theta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} + \begin{bmatrix} 1 & 1-\cos\theta \\ -1 & \cos\theta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}.$$

A two-multiplier realization of the above equation is shown below:



To arrive at an one-multiplier realization we observe that the two equations describing the sine-cosine generator are given by  $s_1[n+1] = (1-\cos\theta)s_2[n+1] + s_1[n] + (1-\cos\theta)s_2[n]$ , and  $s_2[n+1] = -s_1[n] + \cos\theta s_2[n]$ . Substituting the second equation in the first equation we arrive at an alternate description in the form

$$\begin{aligned} s_1[n+1] &= -\cos\theta s_2[n+1] + s_2[n], \\ s_2[n+1] &= -s_1[n] + \cos\theta s_2[n]. \end{aligned}$$

A realization of above is identical to the single-multiplier structure of Problem 6.64,

**6.59** From Figure P6.17(a), the output-input relation of the channel is given by

$$\begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix} = \begin{bmatrix} 1 & H_{12}(z) \\ H_{21}(z) & 1 \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix}.$$

Likewise, the output-input relation of the channel separation circuit of Figure P6.17(b) is given

by  $\begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} = \begin{bmatrix} 1 & -G_{12}(z) \\ -G_{21}(z) & 1 \end{bmatrix} \begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix}$ . Hence, the overall system is characterized by

$$\begin{bmatrix} V_1(z) \\ V_2(z) \end{bmatrix} = \begin{bmatrix} 1 & -G_{12}(z) \\ -G_{21}(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & H_{12}(z) \\ H_{21}(z) & 1 \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \begin{bmatrix} 1-H_{21}(z)G_{12}(z) & H_{12}(z)-G_{12}(z) \\ H_{21}(z)-G_{21}(z) & 1-H_{12}(z)G_{21}(z) \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix}.$$

The cross-talk is eliminated if  $V_1(z)$  is a function of either  $X_1(z)$  or  $X_2(z)$ , and similarly, if  $V_2(z)$  is a function of either  $X_1(z)$  or  $X_2(z)$ . From the above equation it follows that if  $H_{12}(z) = G_{12}(z)$ , and  $H_{21}(z) = G_{21}(z)$ , then  $V_1(z) = (1 - H_{21}(z)G_{12}(z))X_1(z)$ , and

$V_2(z) = (1 - H_{12}(z)G_{21}(z))X_2(z)$ . Alternately, if  $G_{12}(z) = H_{21}^{-1}(z)$ , and  $G_{21}(z) = H_{12}^{-1}(z)$ , then

$$V_1(z) = \left( \frac{H_{12}(z)H_{21}(z)-1}{H_{21}(z)} \right) X_2(z), \text{ and } V_2(z) = \left( \frac{H_{12}(z)H_{21}(z)-1}{H_{12}(z)} \right) X_1(z).$$

**M6.1 (a)**  $H_1(z) = (1 - 0.3261z^{-1})(1 - 3.0666z^{-1})(1 + 4.748z^{-1} + 7.4388z^{-2})$   
 $\times (1 - 1.0935z^{-1} + z^{-2})(1 + 0.6383z^{-1} + 0.1344z^{-2}).$

**(b)**  $H_2(z) = (1 + 3.5585z^{-1})(1 + 0.281z^{-1})(1 - z^{-1})(1 - 1.4078z^{-1} + 2.0338z^{-2})$   
 $\times (1 + 0.2604z^{-1} + z^{-2})(1 - 0.6922z^{-1} + 0.4917z^{-2})$

**M6.2 (a)**  $G(z) = \frac{1.5(1 + 3z^{-1} + 4z^{-2})(1 + 0.6667z^{-1} + 0.3333z^{-2})}{(1 - z^{-1} + 0.5z^{-2})(1 - z^{-1} + 0.3333z^{-2})}.$

**(b)**  $G(z) = G_a(z)G_b(z)$  where  $G_a(z) = \frac{1.5(1 + 3z^{-1} + 4z^{-2})}{1 - z^{-1} + 0.5z^{-2}}$ , and

$G_b(z) = \frac{1 + 0.6667z^{-1} + 0.3333z^{-2}}{1 - z^{-1} + 0.3333z^{-2}}$ . Alternately, we can write  $G(z) = G_c(z)G_d(z)$  where

$G_c(z) = \frac{1.5(1 + 0.6667z^{-1} + 0.3333z^{-2})}{1 - z^{-1} + 0.5z^{-2}}$ , and  $G_d(z) = \frac{1 + 3z^{-1} + 4z^{-2}}{1 - z^{-1} + 0.3333z^{-2}}.$

**(c)** Parallel form I is obtained using the M-file `residuez` resulting in the partial fraction expansion in  $z^{-1}$  of  $G(z)$  given by

$$\begin{aligned} G(z) &= 12 + \frac{-35.25 - j71.25}{1 - (0.5 - j0.5)z^{-1}} + \frac{-35.25 + j71.25}{1 - (0.5 + j0.5)z^{-1}} + \frac{30 + j147.22}{1 - (0.5 - j0.2887)z^{-1}} + \frac{30 - j147.22}{1 - (0.5 + j0.2887)z^{-1}} \\ &= 12 + \frac{-70.5 - 36z^{-1}}{1 - z^{-1} + 0.5z^{-2}} + \frac{60 + 55z^{-1}}{1 - z^{-1} + 0.3333z^{-2}}. \end{aligned}$$

Parallel form II is obtained using the M-file `residue` resulting in the partial fraction expansion in  $z$  of  $G(z)$  given by

$$G(z) = 1.5 + \frac{(-53.25 + j18)z^{-1}}{1 - (0.5 + j0.5)z^{-1}} + \frac{(-53.25 - j18)z^{-1}}{1 - (0.5 - j0.5)z^{-1}} + \frac{(57.50 - j64.9519)z^{-1}}{1 - (0.5 + j0.2887)z^{-1}} + \frac{(57.50 + j64.9519)z^{-1}}{1 - (0.5 - j0.2887)z^{-1}}$$

$$= 1.5 + \frac{-106.5z^{-1} + 35.25z^{-2}}{1 - z^{-1} + 0.5z^{-2}} + \frac{115z^{-1} - 20z^{-2}}{1 - z^{-1} + 0.3333z^{-2}}.$$

**M6.3 (a)**  $H(z) = \frac{2(1 - 0.6667z^{-1} + 0.3333z^{-2})(1 + 0.5z^{-1} + 0.25z^{-2})}{(1 - 0.5z^{-1} + 0.5z^{-2})(1 + z^{-1} + 0.3333z^{-2})}.$

**(b)**  $H(z) = H_a(z)H_b(z)$  where  $H_a(z) = \frac{2(1 - 0.6667z^{-1} + 0.3333z^{-2})}{1 - 0.5z^{-1} + 0.5z^{-2}}$ , and

$H_b(z) = \frac{1 + 0.5z^{-1} + 0.25z^{-2}}{1 + z^{-1} + 0.3333z^{-2}}.$  Alternately, we can write  $H(z) = H_c(z)H_d(z)$  where

$H_c(z) = \frac{2(1 - 0.6667z^{-1} + 0.3333z^{-2})}{1 + z^{-1} + 0.3333z^{-2}}$ , and  $H_d(z) = \frac{1 + 0.5z^{-1} + 0.25z^{-2}}{1 - 0.5z^{-1} + 0.5z^{-2}}.$

**(c)** Parallel form I is obtained using the M-file `residuez` resulting in the partial fraction expansion in  $z^{-1}$  of  $H(z)$  given by

$$H(z) = 1 + \frac{0.1622 - j0.286}{1 - (0.25 - j0.6614)z^{-1}} + \frac{0.1622 + j0.286}{1 - (0.25 + j0.6614)z^{-1}}$$

$$+ \frac{0.3378 - j1.2093}{1 + (0.5 + j0.2887)z^{-1}} + \frac{0.3378 + j1.2093}{1 + (0.5 - j0.2887)z^{-1}} = 1 + \frac{0.3243 - 0.4595z^{-1}}{1 - 0.5z^{-1} + 0.5z^{-2}} + \frac{0.6757 - 0.3604z^{-1}}{1 + z^{-1} + 0.3333z^{-2}}.$$

Parallel form II is obtained using the M-file `residue` resulting in the partial fraction expansion in  $z$  of  $H(z)$  given by

$$H(z) = 2 + \frac{(-0.5135 + j0.1022)z^{-1}}{1 - (0.25 + j0.6614)z^{-1}} + \frac{(-0.5135 - j0.1022)z^{-1}}{1 - (0.25 - j0.6614)z^{-1}}$$

$$+ \frac{(-0.1532 - j1.2795)z^{-1}}{1 + (0.5 - j0.2887)z^{-1}} + \frac{(-0.1532 + j1.2795)z^{-1}}{1 + (0.5 + j0.2887)z^{-1}}$$

$$= 2 + \frac{-1.027z^{-1} + 0.1216z^{-2}}{1 - 0.5z^{-1} + 0.5z^{-2}} + \frac{-0.3063z^{-1} + 0.5856z^{-2}}{1 + z^{-1} + 0.3333z^{-2}}.$$

**M6.4** Using Program 6\_4 we obtain:

Lattice parameters are

-0.8364      0.7980      -0.5143      0.1667

Feedforward multipliers are

63.6358    170.1598    144.1143    57.0000    12.0000

**M6.5** Using Program 6\_4 we obtain:

Lattice parameters are

0.3481      0.1801      0.2571      0.1667

Feedforward multipliers are

12.7136    -3.8223    2.5810    -2.0000    4.0000

**M6.6 (a)** Using Program 6\_6 we obtain the following error message:

```
??? Error using ==> tf2latc
The FIR system has a zero on the unit circle
```

**(b)** Using Program 6\_6 we obtain the following error message:

```
??? Error using ==> tf2latc
The FIR system has a zero on the unit circle
```

**M6.7 (a)** Using roots we first determine the poles of the denominator of the lowpass transfer function  $G(z)$  which are then paired using the pole interlacing property resulting in the parallel allpass decomposition given by

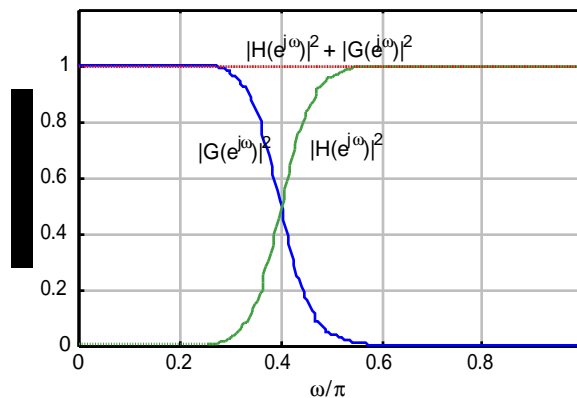
$$G(z) = \frac{1}{2} \left[ \frac{0.1302 - 0.3486z^{-1} + z^{-2}}{1 - 0.3486z^{-1} + 0.1302z^{-2}} + \frac{-0.0868 + 0.6216z^{-1} - 0.6367z^{-2} + z^{-3}}{1 - 0.6367z^{-1} + 0.6216z^{-2} - 0.0868z^{-3}} \right].$$

**(b)** Hence, the power-complementary highpass transfer function  $H(z)$  is given by

$$H(z) = \frac{1}{2} \left[ \frac{0.1302 - 0.3486z^{-1} + z^{-2}}{1 - 0.3486z^{-1} + 0.1302z^{-2}} - \frac{-0.0868 + 0.6216z^{-1} - 0.6367z^{-2} + z^{-3}}{1 - 0.6367z^{-1} + 0.6216z^{-2} - 0.0868z^{-3}} \right]$$

$$= \frac{0.1085(1 - 4.9928z^{-1} + 9.9891z^{-2} - 9.9891z^{-3} + 4.9928z^{-4} + z^{-5})}{1 - 0.9853z^{-1} + 0.9738z^{-2} - 0.3864z^{-3} + 0.1112z^{-4} - 0.0113z^{-5}}.$$

**(c)**



**M6.8 (a)** Using roots we first determine the poles of the denominator of the highpass transfer function  $G(z)$  which are then paired using the pole interlacing property resulting in the parallel allpass decomposition given by

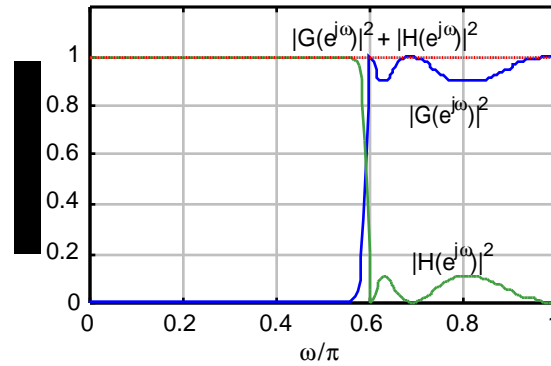
$$G(z) = \frac{1}{2} \left[ \frac{0.5776 + 0.6976z^{-1} + z^{-2}}{1 + 0.6976z^{-1} + 0.5776z^{-2}} - \frac{0.3974 + 1.1635z^{-1} + 1.0052z^{-2} + z^{-3}}{1 + 1.0052z^{-1} + 1.1635z^{-2} + 0.3974z^{-3}} \right].$$

**(b)** Hence, the power-complementary lowpass transfer function  $H(z)$  is given by

$$H(z) = \frac{1}{2} \left[ \frac{0.5776 + 0.6976z^{-1} + z^{-2}}{1 + 0.6976z^{-1} + 0.5776z^{-2}} + \frac{0.3974 + 1.1635z^{-1} + 1.0052z^{-2} + z^{-3}}{1 + 1.0052z^{-1} + 1.1635z^{-2} + 0.3974z^{-3}} \right]$$

$$= \frac{0.4875(1 + 1.3594z^{-1} + 2.2098z^{-2} + 2.2098z^{-3} + 1.3594z^{-4} + z^{-5})}{1 + 1.7028z^{-1} + 2.4423z^{-2} + 1.7896z^{-3} + 0.9492z^{-4} + 0.2295z^{-5}}.$$

**(c)**



**M6.9 (a)** Using roots we first determine the poles of the denominator of the bandpass transfer function  $G(z)$  which are then paired using the pole interlacing property resulting in the parallel allpass decomposition given by

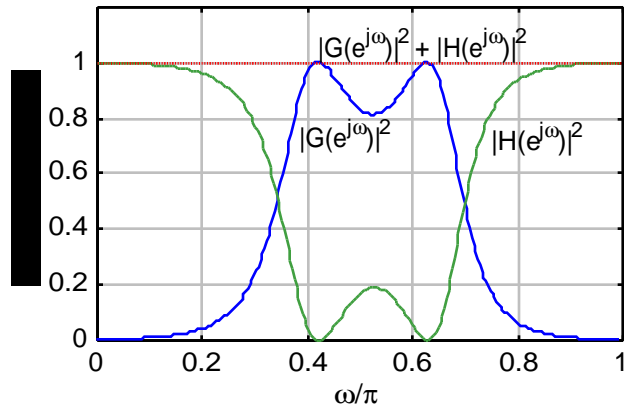
$$G(z) = \frac{1}{2} \left[ \frac{0.5874 - 0.5154z^{-1} + z^{-2}}{1 - 0.5154z^{-1} + 0.5874z^{-2}} - \frac{0.6159 + 0.7531z^{-1} + z^{-2}}{1 + 0.7531z^{-1} + 0.6159z^{-2}} \right]$$

**(b)** Hence, the power-complementary bandstop transfer function  $H(z)$  is given by

$$H(z) = \frac{1}{2} \left[ \frac{0.5874 - 0.5154z^{-1} + z^{-2}}{1 - 0.5154z^{-1} + 0.5874z^{-2}} + \frac{0.6159 + 0.7531z^{-1} + z^{-2}}{1 + 0.7531z^{-1} + 0.6159z^{-2}} \right]$$

$$= 1.2033 \left( \frac{1 + 0.3013z^{-1} + 1.6183z^{-2} + 0.3013z^{-3} + z^{-4}}{1 + 0.2377z^{-1} + 0.8152z^{-2} + 0.1294z^{-3} + 0.3618z^{-4}} \right).$$

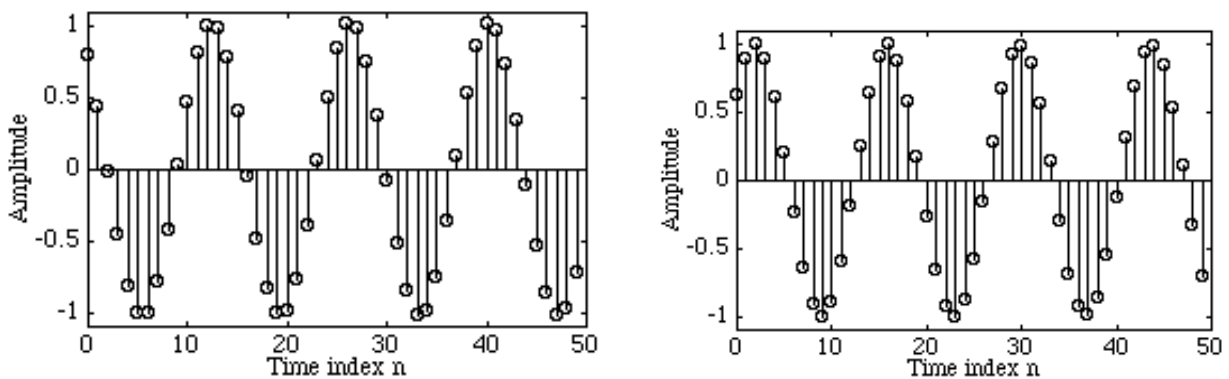
(c)



**M6.10** The MATLAB program for the simulation of the sine-cosine generator of Problem 6.56 is given below:

```
s10 = 0.1; s20 = 0.1; a = 0.9;
y1 = zeros(1,50);y2 = y1;
for n = 1:50;
y1(n) = a*(s10 + s20) - s20;y2(n) = a*(s10 + s20) + s10;
s10 = y1(n);s20 = y2(n);
end
k = 1:1:50;
stem(k-1,y1/abs(y1(7)));axis([0 50 -1.1 1.1]);
xlabel('Time index n');ylabel('Amplitude');
pause
stem(k-1,y2/y2(3));axis([0 50 -1.1 1.1]);
xlabel('Time index n');ylabel('Amplitude');
```

The plots generated by the above program for initial conditions  $s1[-1] = s2[-1] = 0.1$  are shown below:



The outputs are zero for zero initial conditions. Non-zero initial conditions of equal values appear to have no effects on the outputs. However, unequal initial conditions have effects on the amplitudes and phases of the two output sequences.

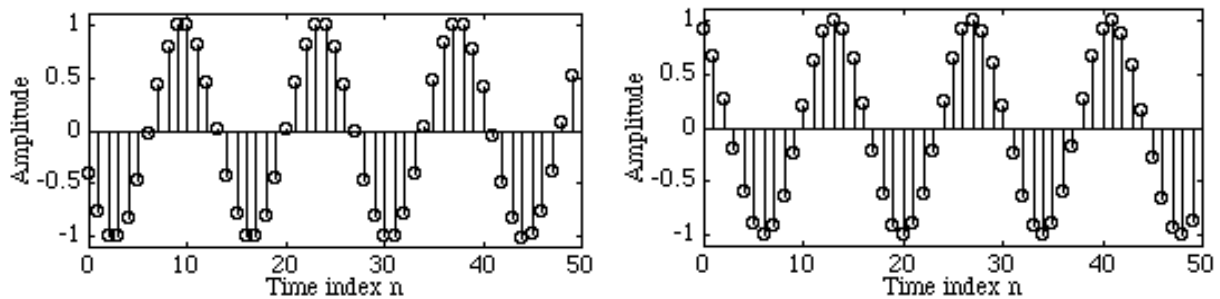
**M6.11** The MATLAB program for the simulation of the sine-cosine generator of Problem 6.57 is given below:

```

s10 = .1; s20 = 1; a = 0.9;
y1 = zeros(1,50);y2 = y1;
for n = 1:50;
y1(n) = -s20 + a*s10;y2(n) = -a*y1(n) + s10;
s10 = y1(n);s20 = y2(n);
end
k = 1:1:50;
stem(k-1,y1/y1(11));axis([0 50 -1.1 1.1]);
xlabel('Time index n');ylabel('Amplitude');
pause
stem(k-1,y2/y2(14));axis([0 50 -1.1 1.1]);
xlabel('Time index n');ylabel('Amplitude');

```

The plots generated by the above program for initial conditions  $s1[-1] = s2[-1] = 0.1$  are shown below:



The outputs are zero for zero initial conditions. Non-zero initial conditions of equal values appear to have no effects on the outputs. However, unequal initial conditions have effects on the amplitudes and phases of the two output sequences.

**M6.12** Since the single multiplier sine-cosine generator of Problem 6.58 is identical to the single-multiplier structure of Problem 6.64, the simulation program given above in the solution of Problem M6.12 also holds here.

## Chapter 7 (2e)

**7.1**  $\delta_p = 1 - 10^{-\alpha_p/20}$  and  $\delta_s = 10^{-\alpha_s/20}$ .

**(a)**  $\alpha_p = 0.15$  and  $\alpha_s = 41$ . Hence  $\delta_p = 1 - 10^{-0.15/20} = 0.017121127$  and  $\delta_s = 10^{-41/20} = 0.0089125$ .

**(b)**  $\alpha_p = 0.23$  and  $\alpha_s = 73$ . Hence,  $\delta_p = 1 - 10^{-0.035/20} = 0.0261322146$  and  $\delta_s = 10^{-73/20} = 0.000223872$ .

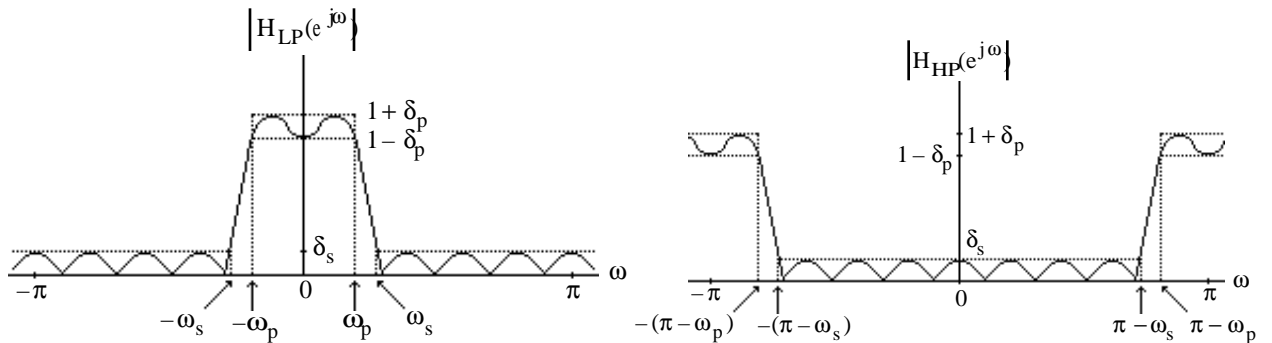
**7.2**  $\alpha_p = -20 \log_{10}(1 - \delta_p)$  and  $\alpha_s = -20 \log_{10}(\delta_s)$ .

**(a)**  $\delta_p = 0.01$  and  $\delta_s = 0.01$ . Hence,  $\alpha_p = -20 \log_{10}(1 - 0.01) = 0.0873$  dB and  $\alpha_s = -20 \log_{10}(0.01) = 40$  dB.

**(b)**  $\delta_p = 0.035$  and  $\delta_s = 0.023$ . Hence,  $\alpha_p = -20 \log_{10}(1 - 0.035) = 0.3094537$  dB and  $\alpha_s = -20 \log_{10}(0.023) = 32.76544$  dB.

**7.3**  $G(z) = H^2(z)$  or equivalently,  $G(e^{j\omega}) = H^2(e^{j\omega})$ .  $|G(e^{j\omega})| = |H^2(e^{j\omega})| = |H(e^{j\omega})|^2$ . Let  $\delta_p$  and  $\delta_s$  denote the passband and stopband ripples of  $H(e^{j\omega})$ , respectively. Also, let  $\delta_{p,2} = 2\delta_p$  and  $\delta_{s,2}$  denote the passband and stopband ripples of  $G(e^{j\omega})$ , respectively. Then  $\delta_{p,2} = 1 - (1 - \delta_p)^2$ , and  $\delta_{s,2} = (\delta_s)^2$ . For a cascade of  $M$  sections,  $\delta_{p,M} = 1 - (1 - \delta_p)^M$ , and  $\delta_{s,M} = (\delta_s)^M$ .

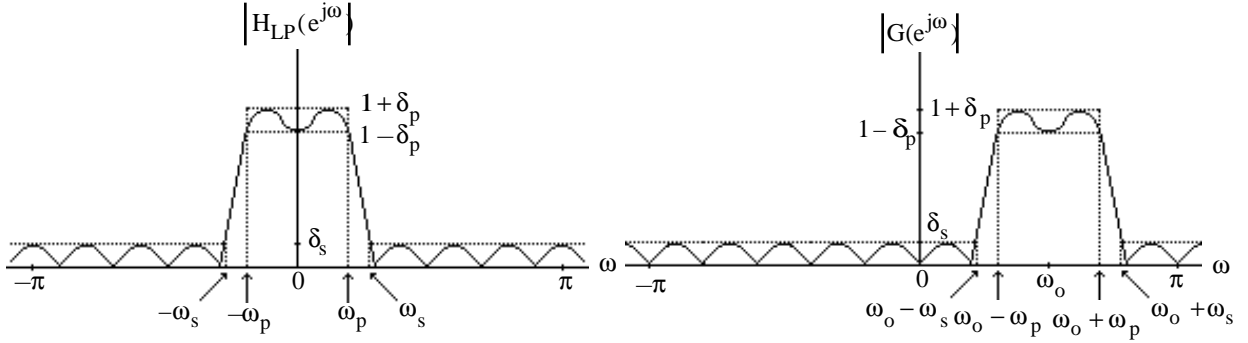
**7.4**



Therefore, the passband edge and the stopband edge of the highpass filter are given by  $\omega_{p,HP} = \pi - \omega_p$ , and  $\omega_{s,HP} = \pi - \omega_s$ , respectively.

**7.5**





Note that  $G(z)$  is a complex bandpass filter with a passband in the range  $0 \leq \omega \leq \pi$ . Its passband edges are at  $\omega_{p,BP} = \omega_o \pm \omega_p$ , and stopband edges at  $\omega_{s,BP} = \omega_o \pm \omega_s$ . A real coefficient bandpass transfer function can be generated according to

$G_{BP}(z) = H_{LP}(e^{j\omega_o}z) + H_{LP}(e^{-j\omega_o}z)$  which will have a passband in the range  $0 \leq \omega \leq \pi$  and another passband in the range  $-\pi \leq \omega \leq 0$ . However because of the overlap of the two spectra a simple formula for the bandedges cannot be derived.

**7.6 (a)**  $h_p(t) = h_a(t) \cdot p(t)$  where  $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ . Thus,  $h_p(t) = \sum_{n=-\infty}^{\infty} h_a(nT) \delta(t - nT)$ .

We also have,  $g[n] = h_a(nT)$ . Now,  $H_a(s) = \int_{-\infty}^{\infty} h_a(t) e^{-st} dt$  and

$$H_p(s) = \int_{-\infty}^{\infty} h_p(t) e^{-st} dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} h_a(nT) \delta(t - nT) e^{-st} dt = \sum_{n=-\infty}^{\infty} h_a(nT) e^{-snT}.$$

Comparing the above expression with  $G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n} = \sum_{n=-\infty}^{\infty} h_a(nT) z^{-n}$ , we conclude

that  $G(z) = H_p(s) \Big|_{s=\frac{1}{T} \ln z}$ .

We can also show that a Fourier series expansion of  $p(t)$  is given by  $p(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{-j(2\pi kt/T)}$ .

Therefore,  $h_p(t) = \left( \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{-j(2\pi kt/T)} \right) h_a(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} h_a(t) e^{-j(2\pi kt/T)}$ . Hence,

$$H_p(s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(s + j \frac{2\pi kt}{T}\right).$$

As a result, we have

(1):  $G(z) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(s + j \frac{2\pi kt}{T}\right) \Big|_{s=\frac{1}{T} \ln z}$ .

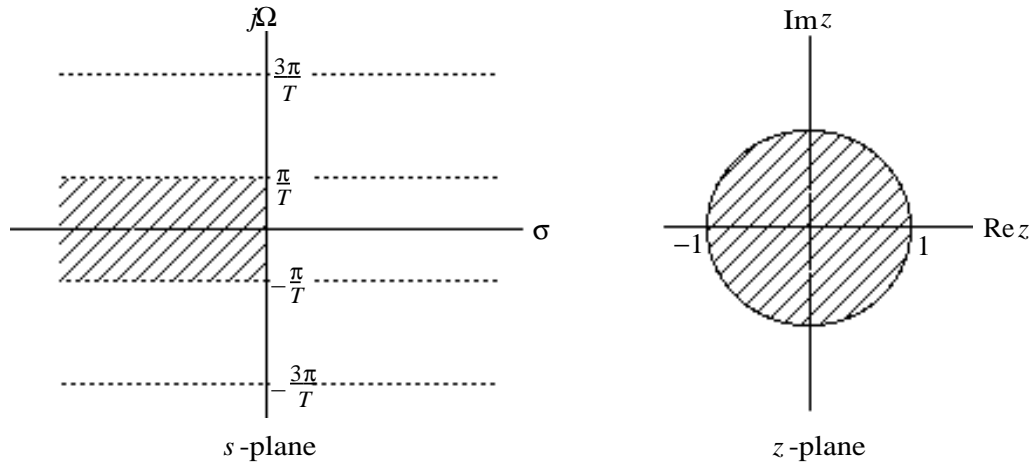
**(b)** The transformation from  $s$ -plane to  $z$ -plane is given by  $z = e^{sT}$ . If we express  $s = \sigma_o + j\omega_o$ , then we can write  $z = re^{j\omega} = e^{\sigma_o T} e^{j\omega_o T}$ . Therefore,

$$|z| = \begin{cases} < 1, & \text{for } \sigma_o < 1, \\ = 1, & \text{for } \sigma_o = 1, \\ > 1, & \text{for } \sigma_o > 1. \end{cases}$$

Or in other words, a point in the left-half s-plane is mapped onto a point inside the unit circle in the z-plane, a point in the right-half s-plane is mapped onto a point outside the unit circle in the z-plane, and a point on the  $j\omega$ -axis in the s-plane is mapped onto a point on the unit circle in the z-plane. As a result, the mapping has the desirable properties enumerated in Section 7.1.3.

(c) However, all points in the s-plane defined by  $s = \sigma_o + j\omega_o \pm j\frac{2\pi k}{T}$ ,  $k = 0, 1, 2, \dots, K$ , are

mapped onto a single point in the z-plane as  $z = e^{\sigma_o T} e^{j(\omega_o \pm \frac{2\pi k}{T})T} = e^{\sigma_o T} e^{j\omega_o T}$ . The mapping is illustrated in the figure below



Note that the strip of width  $2\pi/T$  in the s-plane for values of  $s$  in the range  $-\pi/T \leq \Omega \leq \pi/T$  is mapped into the entire z-plane, and so are the adjacent strips of width  $2\pi/T$ . The mapping is many-to-one with infinite number of such strips of width  $2\pi/T$ .

It follows from the above figure and also from Eq. (1) that if the frequency response

$H_a(j\Omega) = 0$  for  $|\Omega| \geq \frac{\pi}{T}$ , then  $G(e^{j\omega}) = \frac{1}{T} H_a(j\frac{\omega}{T})$ , for  $|\omega| \leq \pi$ , and there is no aliasing.

(d) For  $z = e^{j\omega} = e^{j\Omega T}$ . Or equivalently,  $\omega = \Omega T$ .

**7.7**  $H_a(s) = \frac{A}{s + \alpha}$ . Then  $h_a(t) = Ae^{-\alpha t}\mu(t)$  where  $\mu(t)$  is the unit step function. Hence,

$g[n] = h_a(nT) = Ae^{-\alpha nT}\mu[n]$ . Thus,  $G(z) = A \sum_{n=0}^{\infty} e^{-\alpha nT} z^{-n} = \frac{A}{1 - e^{-\alpha T} z^{-1}}$ , provided  $|e^{-\alpha T}| < 1$ .

**7.8** From Problem 7.7 we recall that if  $H_a(s) = \frac{A}{s + \alpha}$  then  $G(z) = \frac{A}{1 - e^{-\alpha T} z^{-1}}$ . We can express

$$H_a(s) = \frac{\lambda}{\lambda^2 + (s + \beta)^2} = \frac{1}{2j} \left( \frac{1}{s + \beta - j\lambda} \right) - \frac{1}{2j} \left( \frac{1}{s + \beta + j\lambda} \right). \text{ Hence,}$$

$$\begin{aligned}
G(z) &= \frac{1}{2j} \left( \frac{1}{1 - e^{-(\beta - j\lambda)T} z^{-1}} - \frac{1}{1 - e^{-(\beta + j\lambda)T} z^{-1}} \right) = \frac{1}{2j} \left( \frac{-e^{-\beta T} e^{-j\lambda T} z^{-1} + e^{-\beta T} e^{j\lambda T} z^{-1}}{(1 - e^{-(\beta - j\lambda)T} z^{-1})(1 - e^{-(\beta + j\lambda)T} z^{-1})} \right) \\
&= \frac{1}{2j} \left( \frac{z^{-1} e^{-\beta T} (e^{j\lambda T} - e^{-j\lambda T})}{1 - 2z^{-1} e^{-\beta T} \cos(\lambda T) + e^{-2\beta T} z^{-2}} \right) = \frac{z^{-1} e^{-\beta T} \sin(\lambda T)}{1 - 2z^{-1} e^{-\beta T} \cos(\lambda T) + e^{-2\beta T} z^{-2}} \\
&= \frac{ze^{-\beta T} \sin(\lambda T)}{z^2 - 2ze^{-\beta T} \cos(\lambda T) + e^{-2\beta T}}.
\end{aligned}$$

**7.9**  $H_a(s) = \frac{s + \beta}{\lambda^2 + (s + \beta)^2} = \frac{1}{2} \left( \frac{1}{s + \beta + j\lambda} \right) + \frac{1}{2} \left( \frac{1}{s + \beta - j\lambda} \right)$ . Hence

$$\begin{aligned}
G(z) &= \frac{1}{2} \left( \frac{1}{1 - e^{-(\beta + j\lambda)T} z^{-1}} + \frac{1}{1 - e^{-(\beta - j\lambda)T} z^{-1}} \right) = \frac{1}{2} \left( \frac{1 - e^{-\beta T} e^{j\lambda T} z^{-1} + 1 - e^{-\beta T} e^{-j\lambda T} z^{-1}}{1 - 2z^{-1} e^{-\beta T} \cos(\lambda T) + e^{-2\beta T} z^{-2}} \right) \\
&= \frac{1 - z^{-1} e^{-\beta T} \cos(\lambda T)}{1 - 2z^{-1} e^{-\beta T} \cos(\lambda T) + e^{-2\beta T} z^{-2}} = \frac{z^2 - ze^{-\beta T} \cos(\lambda T)}{z^2 - 2ze^{-\beta T} \cos(\lambda T) + e^{-2\beta T}}.
\end{aligned}$$

**7.10** Assume  $h_a(t)$  is causal. Now,  $h_a(t) = \oint H_a(s) e^{st} ds$ . Hence  $g[n] = h_a(nT) = \oint H_a(s) e^{snT} ds$ .

Therefore,

$$G(z) = \sum_{n=0}^{\infty} g[n] z^{-n} = \sum_{n=0}^{\infty} \oint H_a(s) e^{snT} z^{-n} ds = \oint H_a(s) \sum_{n=0}^{\infty} z^{-n} e^{snT} ds = \oint \frac{H_a(s)}{1 - e^{sT} z^{-1}} ds.$$

Hence  $G(z) = \sum_{\text{all poles of } H_a(s)} \text{Residues} \left[ \frac{H_a(s)}{1 - e^{sT} z^{-1}} \right]$ .

**7.11**  $H_a(s) = \frac{A}{s + \alpha}$ . The transfer function has a pole at  $s = -\alpha$ . Now

$$G(z) = \text{Residue} \left[ \frac{A}{(s + \alpha)(1 - e^{sT} z^{-1})} \right]_{s = -\alpha} = \frac{A}{1 - e^{sT} z^{-1}} \Big|_{s = -\alpha} = \frac{A}{1 - e^{-\alpha T} z^{-1}}.$$

**7.12 (a)** Applying partial-fraction expansion we can express

$$\begin{aligned}
H_a(s) &= \frac{16(s+2)}{(s+3)(s^2 + 2s + 5)} = 16 \left[ \frac{-1/8}{s+3} + \frac{0.0625 - j0.1875}{s+1-j2} + \frac{0.0625 + j0.1875}{s+1+j2} \right] \\
&= 16 \left[ \frac{-\frac{1}{8}}{s+3} + \frac{\frac{1}{8}s + \frac{7}{8}}{s^2 + 2s + 5} \right] = \frac{-2}{s+3} + \frac{2s+14}{(s+1)^2 + 2^2} = \frac{-2}{s+3} + \frac{2(s+1)}{(s+1)^2 + 2^2} + \frac{6 \times 2}{(s+1)^2 + 2^2}.
\end{aligned}$$

Using the results of Problems 7.7, 7.8 and 7.9 we thus arrive at

$$G(z) = -\frac{2}{1 - e^{-3T} z^{-1}} + \frac{2(z^2 - ze^{-2T} \cos(2T))}{z^2 - 2ze^{-2T} \cos(2T) + e^{-4T}} + \frac{6ze^{-2T} \sin(2T)}{z^2 - 2ze^{-2T} \cos(2T) + e^{-4T}}. \text{ For } T = 0.2,$$

$$\begin{aligned}
\text{we then get } G_a(z) &= -\frac{2}{1-e^{-0.6}z^{-1}} + \frac{2(z^2 - ze^{-0.4} \cos(0.4))}{z^2 - 2ze^{-0.4} \cos(0.4) + e^{-0.8}} + \frac{6ze^{-0.4T} \sin(0.4)}{z^2 - 2ze^{-0.4} \cos(0.4) + e^{-0.8}} \\
&= -\frac{2}{1-0.5488z^{-1}} + \frac{2(z^2 - 0.6174z)}{z^2 - 1.2348z + 0.4493} + \frac{1.5662z}{z^2 - 1.2348z + 0.4493} \\
&= -\frac{2}{1-0.5488z^{-1}} + \frac{2-1.2348z^{-1}}{1-1.2348z^{-1} + 0.4493z^{-2}} + \frac{1.5662z^{-1}}{1-1.2348z^{-1} + 0.4493z^{-2}} \\
&= -\frac{2}{1-0.5488z^{-1}} + \frac{2+0.3314z^{-1}}{1-1.2348z^{-1} + 0.4493z^{-2}}
\end{aligned}$$

**(b)** Applying partial-fraction expansion we can express

$$\begin{aligned}
H_b(s) &= \frac{4s^2 + 10s + 8}{(s^2 + 2s + 3)(s + 1)} = \frac{1}{s + 1} + \frac{\frac{3}{2} - j\frac{1}{\sqrt{2}}}{s + 1 - j\sqrt{2}} + \frac{\frac{3}{2} + j\frac{1}{\sqrt{2}}}{s + 1 + j\sqrt{2}} \\
&= \frac{1}{s + 1} + \frac{3s + 5}{s^2 + 2s + 3} = \frac{1}{s + 1} + \frac{3(s + 1)}{(s + 1)^2 + (\sqrt{2})^2} + \frac{\sqrt{2}(\sqrt{2})}{(s + 1)^2 + (\sqrt{2})^2}.
\end{aligned}$$

Using the results of Problems 7.7, 7.8 and 7.9 we thus arrive at

$$\begin{aligned}
G_b(z) &= \frac{1}{1-e^{-T}z^{-1}} + \frac{3(z^2 - ze^{-T} \cos(\sqrt{2}T))}{z^2 - 2ze^{-T} \cos(\sqrt{2}T) + e^{-2T}} + \frac{\sqrt{2}(ze^{-T} \sin(\sqrt{2}T))}{z^2 - 2ze^{-T} \cos(\sqrt{2}T) + e^{-2T}} \\
&= \frac{1}{1-e^{-0.2}z^{-1}} + \frac{3(z^2 - ze^{-0.2} \cos(0.2\sqrt{2}))}{z^2 - 2ze^{-0.2} \cos(0.2\sqrt{2}) + e^{-0.4}} + \frac{\sqrt{2}(ze^{-0.2} \sin(0.2\sqrt{2}))}{z^2 - 2ze^{-0.2} \cos(0.2\sqrt{2}) + e^{-0.4}} \\
&= \frac{1}{1-e^{-0.2}z^{-1}} + \frac{3(1 - e^{-0.2} \cos(0.2\sqrt{2})z^{-1})}{1 - 2e^{-0.2} \cos(0.2\sqrt{2})z^{-1} + e^{-0.4}z^{-2}} + \frac{\sqrt{2}(e^{-0.2} \sin(0.2\sqrt{2}))z^{-1}}{1 - 2e^{-0.2} \cos(0.2\sqrt{2})z^{-1} + e^{-0.4}z^{-2}} \\
&= \frac{1}{1-0.81873z^{-1}} + \frac{3-2.3585z^{-1}}{1-1.5724z^{-1} + 0.67032z^{-2}} + \frac{0.32314z^{-1}}{1-1.5724z^{-1} + 0.67032z^{-2}} \\
&= \frac{1}{1-0.81873z^{-1}} + \frac{3-2.03545z^{-1}}{1-1.5724z^{-1} + 0.67032z^{-2}}.
\end{aligned}$$

**(c)** Applying partial-fraction expansion we can express

$$\begin{aligned}
H_c(s) &= \frac{3s^3 + 7s^2 + 10s + 7}{(s^2 + s + 1)(s^2 + 2s + 3)} = \frac{-1.8s + 0.6}{s^2 + s + 1} + \frac{4.8s + 5.2}{s^2 + 2s + 3} \\
&= \frac{-1.8(s + 0.5)}{(s + 0.5)^2 + (\sqrt{3}/2)^2} + \frac{\sqrt{3}(\sqrt{3}/2)}{(s + 0.5)^2 + (\sqrt{3}/2)^2} + \frac{4.8(s + 1)}{(s + 1)^2 + (\sqrt{2})^2} + \frac{0.2828(\sqrt{2})}{(s + 1)^2 + (\sqrt{2})^2}
\end{aligned}$$

Using the results of Problems 7.7, 7.8 and 7.9 we thus arrive at

$$\begin{aligned}
G_c(z) &= \frac{-1.8(z^2 - ze^{-0.5T} \cos(0.866T))}{z^2 - 2ze^{-0.5T} \cos(0.866T) + e^{-T}} + \frac{\sqrt{3}ze^{-0.5T} \sin(0.866T)}{z^2 - 2ze^{-0.5T} \cos(0.866T) + e^{-T}} \\
&+ \frac{4.8(z^2 - ze^{-T} \cos(\sqrt{2}T))}{z^2 - 2ze^{-T} \cos(\sqrt{2}T) + e^{-2T}} + \frac{0.2828ze^{-T} \sin(\sqrt{2}T)}{z^2 - 2ze^{-T} \cos(\sqrt{2}T) + e^{-2T}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{-1.8z^2 + 4.6811z}{z^2 - 9.36226z + 0.8187} + \frac{0.2955z}{z^2 - 9.36226z + 0.8187} \\
&+ \frac{4.8z^2 + 3.77376z}{z^2 - 1.5724z + 0.67032} + \frac{0.071414z}{z^2 - 1.5724z + 0.67032} \\
&= \frac{-1.8z^2 + 4.9766z}{z^2 - 9.36226z + 0.8187} + \frac{4.8z^2 + 3.845174z}{z^2 - 1.5724z + 0.67032}.
\end{aligned}$$

**7.13 (a)** Comparing  $G_a(z)$  with Eq. (7.148) we can write

$$G_a(z) = \frac{2}{1 - e^{-0.9}z^{-1}} + \frac{3}{1 - e^{-1.2}z^{-1}} = \frac{2}{1 - e^{-\alpha T}z^{-1}} + \frac{3}{1 - e^{-\beta T}z^{-1}}. \text{ Hence, } \alpha = 3 \text{ and } \beta = 4.$$

$$\text{Therefore, } H_a(s) = \frac{2}{s+3} + \frac{3}{s+4}.$$

(b) Comparing with Eq. (7.152) we observe  $\beta T = 0.6$  and  $\lambda T = 0.9$ . For  $T = 0.3$  we get

$$\beta = 1/3 \text{ and } \lambda = 3. \text{ Thus, } H_a(s) = \frac{s + \frac{1}{3}}{(s + \frac{1}{3})^2 + 9}.$$

$$\mathbf{7.14 (a)} \quad H_a(s) = G_a(z) \Big|_{z=\frac{1+s}{1-s}} = \frac{5\left(\frac{1+s}{1-s}\right)^2 + 4\left(\frac{1+s}{1-s}\right) - 1}{8\left(\frac{1+s}{1-s}\right)^2 + 4\left(\frac{1+s}{1-s}\right)} = \frac{12s}{7s^2 + 16s + 12}.$$

$$\mathbf{(b)} \quad H_b(s) = G_b(z) \Big|_{z=\frac{1+s}{1-s}} = \frac{8\left[\left(\frac{1+s}{1-s}\right)^3 + 3\left(\frac{1+s}{1-s}\right)^2 + 3\left(\frac{1+s}{1-s}\right) + 1\right]}{\left[3\left(\frac{1+s}{1-s}\right) + 1\right]\left[7\left(\frac{1+s}{1-s}\right)^2 + 6\left(\frac{1+s}{1-s}\right) + 3\right]} = \frac{7s^3 + 21s^2 + 21s + 15}{(2s+4)(4s^2 + 8s + 16)}.$$

**7.15** For the impulse invariance design  $\omega_p = \Omega_p T = 2\pi \times 0.5 \times 10^3 \times 0.5 \times 10^{-3} = 0.5\pi$ . For the

bilinear transformation method design  $\omega_p = 2 \tan^{-1}\left(\frac{\Omega_p T}{2}\right) = 2 \tan^{-1}\left(\frac{0.5\pi}{2}\right) = 2 \tan^{-1}(0.25\pi) = 0.4238447331\pi$ .

**7.16** For the impulse invariance design  $2\pi F_p = \frac{\omega_p}{T} = \frac{0.3\pi}{10^{-4}}$  or  $f_p = 1.5$  kHz. For the bilinear transformation method design  $f_p = 10^4 \tan(0.15\pi) / \pi = 1.62186$  kHz.

**7.17** The passband and the stopband edges of the analog lowpass filter are assumed to  $\Omega_p = 0.25\pi$  and  $\Omega_s = 0.55\pi$ . The requirements to be satisfied by the analog lowpass filter are thus  $20 \log_{10} |H_a(j0.25\pi)| \geq -0.5$  dB and  $20 \log_{10} |H_a(j0.55\pi)| \leq -15$  dB.

From  $\alpha_p = 20 \log_{10}(\sqrt{1+\epsilon^2}) = 0.5$  we obtain  $\epsilon^2 = 0.1220184543$ . From  $\alpha_s = 10 \log_{10}(A^2) = 15$  we obtain  $A^2 = 31.6227766$ . From Eq. (5.??), the inverse discrimination ratio is given by

$$\frac{1}{k_1} = \frac{\sqrt{A^2 - 1}}{\epsilon} = 15.841979 \text{ and from Eq. (5.??) the inverse transition ratio is given by}$$

$$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 2.2. \text{ Substituting these values in Eq. (5.??) we obtain}$$

$$N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = \frac{\log_{10}(15.841979)}{\log_{10}(2.2)} = 3.503885. \text{ We choose } N = 4.$$

From Eq. (5.??) we have  $\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = \epsilon^2$ . Substituting the values of  $\Omega_p$ ,  $N$ , and  $\epsilon^2$  we get  $\Omega_c = 1.3007568(\Omega_p) = 1.021612$ .

Using the statement `[z,p,k] = buttap(4)` we get the poles of the 4-th order Butterworth analog filter with a 3-dB cutoff at 1 rad/s as  $p_1 = -0.3827 + j0.9239$ ,  $p_2 = -0.3827 - j0.9239$ ,  $p_3 = -0.9239 + j0.3827$ , and  $p_4 = -0.9239 - j0.3827$ . Therefore

$$H_{an}(s) = \frac{1}{(s-p_1)(s-p_2)(s-p_3)(s-p_4)} = \frac{1}{(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)}.$$

Next we expand  $H_{an}(s)$  in a partial-fraction expansion using the M-file residue and arrive at

$$H_{an}(s) = \frac{-0.9238729s - 0.7071323}{s^2 + 0.7654s + 1} + \frac{0.9238729s + 1.7071323}{s^2 + 1.8478s + 1}. \text{ We next denormalize } H_{an}(s) \text{ to}$$

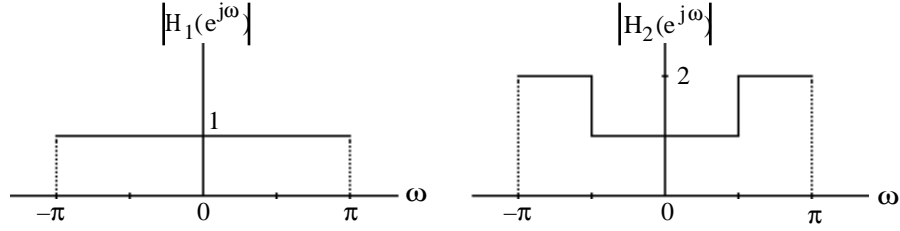
move the 3-dB cutoff frequency to  $\Omega_c = 1.021612$  using the M-file `lp2lp` resulting in

$$\begin{aligned} H_a(s) &= H_{an}\left(\frac{s}{1.021612}\right) \\ &= \frac{-0.943847s - 0.738039}{s^2 + 0.781947948s + 1.0437074244} + \frac{0.943847s + 1.78174665}{s^2 + 1.887749436s + 1.0437074244} \\ &= \frac{-0.943847s - 0.738039}{(s + 0.390974)^2 + (0.9438467)^2} + \frac{0.943847s + 1.78174665}{(s + 0.94387471)^2 + (0.39090656)^2} \end{aligned}$$

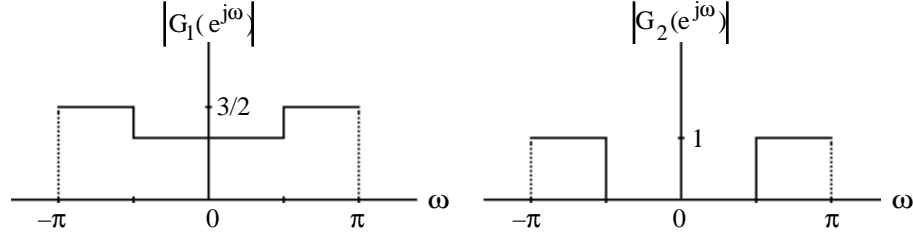
Making use of the M-file bilinear we finally arrive at

$$G(z) = \frac{-0.943847z^2 + 0.68178386z}{z^2 - 1.363567724z + 0.4575139} + \frac{0.943847z^2 - 0.25640047z}{z^2 - 0.77823439z + 0.1514122} \text{ CHECK}$$

**7.18** For no aliasing  $T \leq \frac{\pi}{\Omega_c}$ . Figure below shows the magnitude responses of the digital filters  $H_1(z)$  and  $H_2(z)$ .



(a) The magnitude responses of the digital filters  $G_1(z)$  and  $G_2(z)$  are shown below:



(b) As can be seen from the above  $G_1(z)$  is a multi-band filter, whereas,  $G_2(z)$  is a highpass filter.

7.19 a)  $H_a(s) = \sum_{k=0}^R \frac{A_k}{s - \alpha_k}$ . Hence  $h_a(t) = L^{-1} \left\{ \frac{H_a(s)}{s} \right\} = L^{-1} \left\{ \sum_{k=0}^R \frac{A_k}{s(s - \alpha_k)} \right\}$ .

Thus  $h_a(t) = L^{-1} \left\{ \sum_{k=0}^R \frac{A_k}{\alpha_k} \left( \frac{1}{s - \alpha_k} - \frac{1}{s} \right) \right\} = \sum_{k=0}^R \frac{A_k}{\alpha_k} (e^{\alpha_k t} - 1) \mu(t)$ .

Hence  $g_\mu[n] = h_a(nT) = \sum_{k=0}^R \frac{A_k}{\alpha_k} (e^{\alpha_k nT} - 1) \mu(nT)$ . As a result,

$$G_\mu(z) = \sum_{n=-\infty}^{\infty} g_\mu[n] z^{-n} = \sum_{k=0}^R \frac{A_k}{\alpha_k} \left( \frac{1}{1 - e^{\alpha_k T} z^{-1}} - \frac{1}{1 - z^{-1}} \right) = \sum_{k=0}^R \frac{A_k}{\alpha_k} \left( \frac{z^{-1} (e^{\alpha_k T} - 1)}{(1 - e^{\alpha_k T} z^{-1})(1 - z^{-1})} \right)$$

Now the transfer function  $G(z)$ , which is the  $z$ -transform of the impulse response  $g[n]$ , is related to the  $z$ -transform of the step response  $g_\mu[n]$  by  $G(z) = (1 - z^{-1}) G_\mu(z)$ . Hence

$$G(z) = \sum_{k=0}^R \frac{A_k}{\alpha_k} \left( \frac{z^{-1} (e^{\alpha_k T} - 1)}{1 - e^{\alpha_k T} z^{-1}} \right)$$

(b) Now,  $\frac{H_a(j\omega)}{j\omega} \approx 0$  if  $|\omega| \geq \frac{1}{2T}$ . Consider the digital transfer function obtained by

sampling  $F(s) = \frac{H_a(s)}{s}$ . Now this digital transfer function would correspond to the  $z$ -transform of the sampled step response of  $H_a(s)$ . Thus

$$G_\mu(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F(j\frac{\omega}{T} + j\frac{2\pi k}{T}) = \frac{1}{T} F(j\frac{\omega}{T}) = \frac{H_a(j\omega/T)}{j\omega} \dots$$

Since  $G(z) = (1 - z^{-1})G_\mu(z)$ , hence  $G(e^{j\omega T}) = (1 - e^{-j\omega T})G_\mu(e^{j\omega T}) = (1 - e^{-j\omega T}) \frac{H_a(j\omega)}{j\omega T}$ .

Since  $\omega T \ll 1$ , therefore  $1 - e^{-j\omega T} \approx j\omega T$ . Thus,  $G(e^{j\omega T}) \cong H_a(j\omega)$ .

**7.20** The mapping is given by  $s = \frac{1}{T}(1 - z^{-1})$  or equivalently, by  $z = \frac{1}{1 - sT}$ . For  $s = \sigma_o + j\Omega_o$ ,  $z = \frac{1}{1 - \sigma_o T - j\Omega_o T}$ . Therefore,  $|z|^2 = \frac{1}{(1 - \sigma_o T)^2 + (\Omega_o T)^2}$ . Hence,  $|z| < 1$ , for  $\sigma_o < 0$ . As a result, a stable  $H_a(s)$  results in a stable  $H(z)$  after the transformation. However, for  $\sigma_o = 0$ ,  $|z|^2 = \frac{1}{1 + (\Omega_o T)^2}$  which is equal to 1 only for  $\Omega_o = 0$ . Hence, only the point  $\Omega_o = 0$  on the  $j\Omega$ -axis in the  $s$ -plane is mapped onto the point  $z = 1$  on the unit circle. Consequently, this mapping is not useful for the design of digital filters by analog filter transformation.

**7.21**  $H_a(s)$  is causal and stable and  $|H_a(s)| \leq 1 \forall s$ , Now,  $G(z) = H_a(s) \Big|_{s = \frac{2}{T} \left( \frac{1-z^{-1}}{1+z^{-1}} \right)}$ . Thus,  $G(z)$

is causal and stable. Now,

$$G(e^{j\omega}) = H_a(s) \Big|_{s = \frac{2}{T} \left( \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} \right)} = H_a(s) \Big|_{s = j \frac{2}{T} \tan(\omega/2)} = H_a \left( j \frac{2}{T} \tan(\omega/2) \right). \quad \text{Therefore,}$$

$$|G(e^{j\omega})| = \left| H_a \left( j \frac{2}{T} \tan(\omega/2) \right) \right| \leq 1 \text{ for all values of } \omega. \text{ Hence, } G(z) \text{ is a BR function.}$$

**7.22**  $H_a(j\Omega) = \frac{jB\Omega}{(\Omega_o^2 - \Omega^2) + jB\Omega}$ . Thus,  $|H_a(j\Omega)| = \frac{B\Omega}{\sqrt{(\Omega_o^2 - \Omega^2)^2 + B^2\Omega^2}}$ . It can be seen that

$$|H_a(j0)| = 0, |H_a(j\infty)| = 0 \text{ and } |H_a(j\Omega_o)| = \frac{B\Omega_o}{\sqrt{B^2\Omega_o^2}} = 1. \text{ Hence } H_a(s) \text{ is an analog bandpass transfer function.}$$

Applying bilinear transformation to  $H_a(s)$  we get

$$\begin{aligned} G(z) &= H_a(s) \Big|_{s = \frac{z-1}{z+1}} = \frac{B \left( \frac{z-1}{z+1} \right)}{\left( \frac{z-1}{z+1} \right)^2 + B \left( \frac{z-1}{z+1} \right) + \Omega_o^2} = \frac{B(z^2 - 1)}{(z-1)^2 + B(z^2 - 1) + (z+1)^2} \\ &= \frac{B(z^2 - 1)}{(1 + B + \Omega_o^2)z^2 - 2(1 - \Omega_o^2)z + (1 - B + \Omega_o^2)} \\ &= \frac{B}{1 + B + \Omega_o^2} \cdot \frac{1 - z^{-2}}{1 - \left( \frac{2(1 - \Omega_o^2)}{1 + B + \Omega_o^2} \right) z^{-1} + \left( \frac{1 - B + \Omega_o^2}{1 + B + \Omega_o^2} \right) z^{-2}}. \end{aligned} \text{ From Eqs. (7.36) we have}$$



$$\alpha = \frac{1-B+\Omega_0^2}{1+B+\Omega_0^2} \text{ and } \beta = \frac{1-\Omega_0^2}{1+\Omega_0^2}. \text{ Now } 1+\alpha = 1 + \frac{1-B+\Omega_0^2}{1+B+\Omega_0^2} = \frac{2(1+\Omega_0^2)}{1+B+\Omega_0^2} \text{ and}$$

$$1-\alpha = 1 - \frac{1-B+\Omega_0^2}{1+B+\Omega_0^2} = \frac{2B}{1+B+\Omega_0^2}. \text{ Hence, } G(z) = \frac{1-\alpha}{2} \cdot \frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \text{ which is the same as that given in Eq. (4.113).}$$

**7.23**  $G(z) = \frac{1+\alpha}{2} \cdot \frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$ . For  $\beta = \cos \omega_0$  the numerator of  $G(z)$  becomes

$1-2\cos \omega_0 z^{-1}+z^{-2} = (1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})$  which has roots at  $z = e^{\pm j\omega_0}$ . The numerator of  $G(z^N)$  is then given by  $(1-e^{j\omega_0}z^{-N})(1-e^{-j\omega_0}z^{-N})$  whose roots are obtained by solving the equation  $z^N = e^{\pm j\omega_0}$ , and are given by  $z = e^{j(2\pi n \pm \omega_0)/N}$ ,  $0 \leq n \leq N-1$ . Hence  $G(z^N)$  has  $N$  complex conjugate zero pairs located on the unit circle at angles of  $\frac{2\pi n \pm \omega_0}{N}$  radians,  $0 \leq n \leq N-1$ .

$\omega_0 = \pi/2$ , there are  $2N$  equally spaced zeros on the unit circle starting at  $\omega = \pi/2N$ .

**7.24 (a)**  $H(z) = \frac{1}{2}[1+A_4(z)] = \frac{N(z)}{D(z)}$  where

$$A_4(z) = \left( \frac{\alpha_1 - \beta_1(1+\alpha_1)z^{-1}+z^{-2}}{1-\beta_1(1+\alpha_1)z^{-1}+\alpha_1 z^{-2}} \right) \left( \frac{\alpha_2 - \beta_2(1+\alpha_2)z^{-1}+z^{-2}}{1-\beta_2(1+\alpha_2)z^{-1}+\alpha_2 z^{-2}} \right). \text{ Therefore,}$$

$$\begin{aligned} N(z) &= \frac{1}{2} \left[ (\alpha_1 - \beta_1(1+\alpha_1)z^{-1}+z^{-2})(\alpha_2 - \beta_2(1+\alpha_2)z^{-1}+z^{-2}) \right. \\ &\quad \left. + (1-\beta_1(1+\alpha_1)z^{-1}+\alpha_1 z^{-2})(1-\beta_2(1+\alpha_2)z^{-1}+\alpha_2 z^{-2}) \right] \\ &= \frac{1+\alpha_1\alpha_2}{2} \left\{ 1 - \frac{(1+\alpha_1)(1+\alpha_2)(\beta_1+\beta_2)}{1+\alpha_1\alpha_2} z^{-1} + \frac{2[\alpha_1+\alpha_2+\beta_1\beta_2(1+\alpha_1)(1+\alpha_2)]}{1+\alpha_1\alpha_2} z^{-2} \right. \\ &\quad \left. - \frac{(1+\alpha_1)(1+\alpha_2)(\beta_1+\beta_2)}{1+\alpha_1\alpha_2} z^{-3} + z^{-4} \right\} \text{ which is seen to be a mirror-image} \end{aligned}$$

polynomial. We can express  $N(z) = a(1+b_1z^{-1}+b_2z^{-2}+b_1z^{-3}+z^{-4})$ , where

$$(1): b_1 = -\frac{(1+\alpha_1)(1+\alpha_2)(\beta_1+\beta_2)}{1+\alpha_1\alpha_2}, \quad (2): b_2 = \frac{2[\alpha_1+\alpha_2+\beta_1\beta_2(1+\alpha_1)(1+\alpha_2)]}{1+\alpha_1\alpha_2}, \text{ and}$$

**(b)** (3):  $a = \frac{1+\alpha_1\alpha_2}{2}$ .

**(c)** for  $z = e^{j\omega}$ , we can write

$$N(e^{j\omega}) = a(1+b_1e^{-j\omega}+b_2e^{-j2\omega}+b_1e^{-j3\omega}+e^{-j4\omega}) = ae^{-j2\omega}(b_2+2b_1\cos \omega+2\cos 2\omega).$$

Now  $N(e^{j\omega_i}) = 0$ , for  $i = 1, 2$ . For  $i = 1$  we get (4):  $b_2 + 2b_1 \cos \omega_1 + 2 \cos 2\omega_1 = 0$ , and for  $i = 2$  we get (5):  $b_2 + 2b_1 \cos \omega_2 + 2 \cos 2\omega_2 = 0$ . Solving Eqs. (4) and (5) we get

$$(6): b_1 = -2(\cos \omega_1 + \cos \omega_2), \text{ and } (7): b_2 = 2(2 \cos \omega_1 \cos \omega_2 + 1).$$

From Eqs. (1) and (6) we have (8):  $\frac{(1 + \alpha_1)(1 + \alpha_2)(\beta_1 + \beta_2)}{1 + \alpha_1 \alpha_2} = 2(\cos \omega_1 + \cos \omega_2)$ , and

from Eqs. (2) and (7) we have

$$(9): \frac{2[\alpha_1 + \alpha_2 + \beta_1 \beta_2(1 + \alpha_1)(1 + \alpha_2)]}{1 + \alpha_1 \alpha_2} = 2(2 \cos \omega_1 \cos \omega_2 + 1). \text{ Substituting}$$

$$\alpha_1 = \frac{1 - \tan(B_1 / 2)}{1 + \tan(B_1 / 2)} \text{ and } \alpha_2 = \frac{1 - \tan(B_2 / 2)}{1 + \tan(B_2 / 2)}, \text{ and after rearrangement we get}$$

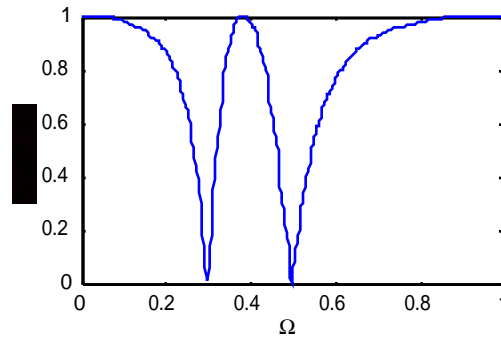
$$(10): \beta_1 + \beta_2 = (\cos \omega_1 + \cos \omega_2) \left[ 1 + \tan\left(\frac{B_1}{2}\right) \tan\left(\frac{B_2}{2}\right) \right] \triangleq \theta_1, \text{ and}$$

$$(11): \beta_1 \beta_2 = \left[ 1 + \tan\left(\frac{B_1}{2}\right) \tan\left(\frac{B_2}{2}\right) \right] \cos \omega_1 \cos \omega_2 + \tan\left(\frac{B_1}{2}\right) \tan\left(\frac{B_2}{2}\right) \triangleq \theta_2.$$

The above two nonlinear equations can be solved resulting in

$$\beta_1 = \frac{\theta_1 \pm \sqrt{\theta_1^2 - 4\theta_2}}{2} \text{ and } \beta_2 = \frac{\theta_2}{\beta_1}.$$

**(d)** For the double notch filter with the following specifications:  $\omega_1 = 0.3\pi$ ,  $\omega_2 = 0.5\pi$ ,  $B_1 = 0.1\pi$ , and  $B_2 = 0.15\pi$ , we get the following values for the parameters of the notch filter transfer function:  $\alpha_1 = 0.7265$ ,  $\alpha_2 = 0.6128$ ,  $\beta_1 = 0.5397$ , and  $\beta_2 = 0.0705$ .



**7.25** A zero (pole) of  $H_{LP}(z)$  is given by the factor  $(z - z_k)$ . After applying the lowpass-to-lowpass transformation, this factor becomes  $\frac{\hat{z} - \alpha}{1 - \alpha \hat{z}} - z_k$ , and hence the new location of the zero (pole) is given by the roots of the equation

$$\hat{z} - \alpha - z_k + \alpha z_k \hat{z} = (1 + \alpha z_k) \hat{z} - (\alpha + z_k) = 0 \text{ or } \hat{z}_k = \frac{\alpha + z_k}{1 + \alpha z_k}. \text{ For } z_k = -1,$$

$$\hat{z}_k = \frac{\alpha - 1}{1 - \alpha} = -1.$$

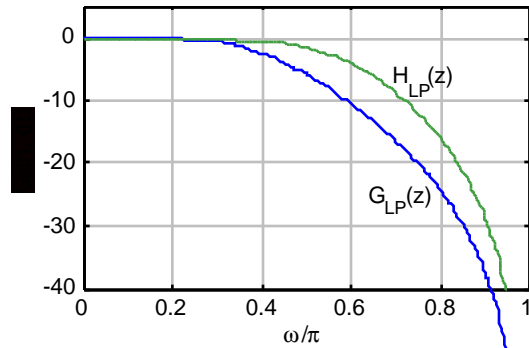
**7.26** The lowpass-to-bandpass transformation is given by  $z \rightarrow \frac{-b + a\hat{z} - \hat{z}^2}{1 - a\hat{z} + b\hat{z}^2}$  where  $a = \frac{2\alpha\beta}{\beta+1}$  and  $b = \frac{\beta-1}{\beta+1}$ . A zero (pole) of  $H_{LP}(z)$  is given by the factor  $(z - z_k)$ . After applying the

lowpass-to-bandpass transformation, this factor becomes  $\frac{-b + a\hat{z} - \hat{z}^2}{1 - a\hat{z} + b\hat{z}^2} - z_k$ , and hence the new location of the zero (pole) of the bandpass transfer function is given by the roots of the equation  $(1 + bz_k)z^2 - a(1 + z_k)z + (b + z_k) = 0$ , or  $z^2 - \frac{a(1 + z_k)}{1 + bz_k}z + \frac{b + z_k}{1 + bz_k} = 0$ , whose solution

$$\text{is given by } \hat{z}_k = \frac{a(1 + z_k)}{2(1 + bz_k)} \pm \sqrt{\left[\frac{a(1 + z_k)}{2(1 + bz_k)}\right]^2 - \left(\frac{b + z_k}{1 + bz_k}\right)}. \text{ For } z_k = -1, \hat{z}_k = \pm 1.$$

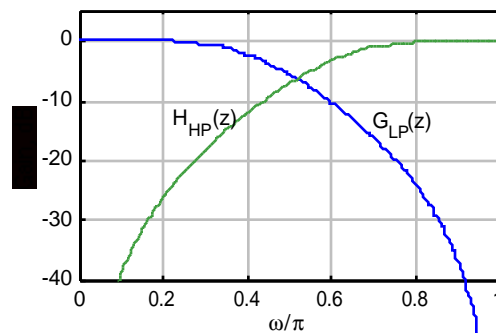
**7.27** For  $\omega_c = 0.42\pi$  and  $\hat{\omega}_c = 0.57\pi$  we have  $\alpha = \frac{\sin\left(\frac{\omega_c - \hat{\omega}_c}{2}\right)}{\sin\left(\frac{\omega_c + \hat{\omega}_c}{2}\right)} = \frac{\sin(-0.075\pi)}{\sin(0.495\pi)} = -0.233474$ . Thus,

$$\begin{aligned} H_{LP}(\hat{z}) &= G_{LP}(z) \Big|_{z^{-1} = \frac{\hat{z}^{-1} - \alpha}{1 - \alpha\hat{z}^{-1}}} = \frac{0.223 \left(1 + \frac{\hat{z}^{-1} - \alpha}{1 - \alpha\hat{z}^{-1}}\right)^2}{1 - 0.2952 \left(\frac{\hat{z}^{-1} - \alpha}{1 - \alpha\hat{z}^{-1}}\right) + 0.187 \left(\frac{\hat{z}^{-1} - \alpha}{1 - \alpha\hat{z}^{-1}}\right)^2} \\ &= \frac{0.223(1 - \alpha)^2(1 + \hat{z}^{-1})^2}{(1 + 0.2952\alpha + 0.187\alpha^2) + [-2\alpha - 0.2952(1 + \alpha^2) - 0.374\alpha]\hat{z}^{-1} + (\alpha^2 + 0.2952\alpha + 0.187)\hat{z}^{-2}} \\ &= \frac{0.33929(1 + \hat{z}^{-1})^2}{0.94127 + 0.24298\hat{z}^{-1} + 0.17259\hat{z}^{-2}} \end{aligned}$$



7.28 For  $\omega_c = 0.42\pi$  and  $\hat{\omega}_c = 0.61\pi$  we have  $\alpha = -\frac{\cos\left(\frac{\omega_c + \hat{\omega}_c}{2}\right)}{\cos\left(\frac{\omega_c - \hat{\omega}_c}{2}\right)} = -\frac{\cos(0.515\pi)}{\cos(-0.95\pi)} = -0.0492852$ .

$$\begin{aligned}
 H_{\text{HP}}(\hat{z}) &= G_{\text{LP}}(z) \Big|_{z^{-1} = -\frac{\hat{z}^{-1} + \alpha}{1 + \alpha \hat{z}^{-1}}} = \frac{0.223 \left(1 - \frac{\hat{z}^{-1} + \alpha}{1 + \alpha \hat{z}^{-1}}\right)^2}{1 + 0.2952 \left(\frac{\hat{z}^{-1} + \alpha}{1 + \alpha \hat{z}^{-1}}\right) + 0.187 \left(\frac{\hat{z}^{-1} + \alpha}{1 + \alpha \hat{z}^{-1}}\right)^2} \\
 &= \frac{0.223(1 - \alpha)^2 (1 - \hat{z}^{-1})^2}{(1 + 0.2952\alpha + 0.187\alpha^2) + [2\alpha + 0.2952(1 + \alpha^2) + 0.374\alpha]\hat{z}^{-1} + (\alpha^2 + 0.2952\alpha + 0.187)\hat{z}^{-2}} \\
 &= \frac{0.20156(1 - \hat{z}^{-1})^2}{1.015 + 0.41292\hat{z}^{-1} + 0.20398\hat{z}^{-2}}.
 \end{aligned}$$



7.29 Since the passband edge frequencies are not specified, we assume the desired 3-dB bandwidth of the bandpass filter  $H_{\text{BP}}(z)$  is  $\hat{\omega}_{c2} - \hat{\omega}_{c1} = \omega_c$ , where  $\omega_c$  is the 3-dB cutoff frequency of  $G_{\text{LP}}(z)$ .

Moreover, the desired center frequency  $\hat{\omega}_c$  is related to the passband edge frequencies through

$$\hat{\omega}_c = \sqrt{\hat{\omega}_{c1}\hat{\omega}_{c2}} \text{ or } \hat{\omega}_c^2 = \hat{\omega}_{c1}\hat{\omega}_{c2}. \text{ Hence,}$$

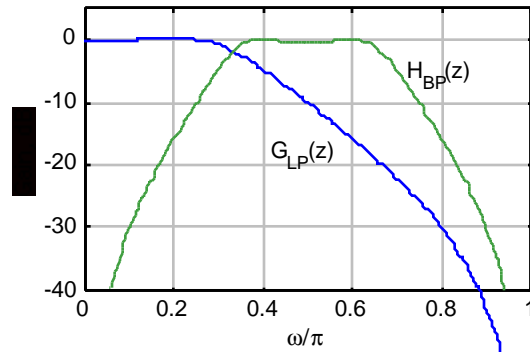
$$(\hat{\omega}_{c2} + \hat{\omega}_{c1})^2 = (\hat{\omega}_{c2} - \hat{\omega}_{c1})^2 + 4\hat{\omega}_{c2}\hat{\omega}_{c1} = \omega_c^2 + 4\hat{\omega}_c^2 \text{ or } \hat{\omega}_{c2} + \hat{\omega}_{c1} = \sqrt{\omega_c^2 + 4\hat{\omega}_c^2}.$$

From Eq. (7.48) the lowpass-to-bandpass transformation is given by

$$z^{-1} \rightarrow -z^{-1} \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} = \frac{\alpha z^{-1} - z^{-2}}{1 - \alpha z^{-1}}, \text{ where } \alpha = \frac{\cos\left(\frac{\hat{\omega}_{c2} + \hat{\omega}_{c1}}{2}\right)}{\cos\left(\frac{\hat{\omega}_{c2} - \hat{\omega}_{c1}}{2}\right)} = \frac{\cos\left(\frac{\sqrt{\omega_c^2 + 4\hat{\omega}_c^2}}{2}\right)}{\cos\left(\frac{\omega_c}{2}\right)}.$$

For  $\omega_c = 0.42\pi$  and  $\hat{\omega}_c = 0.45\pi$ ,  $\alpha = 0.013564$ . Then,  $H_{BP}(z) = G_{LP}(z) \Big|_{z^{-1} \rightarrow \frac{\delta z^{-1} - z^{-2}}{1 - \delta z^{-1}}}$

$$\begin{aligned} &= \frac{0.223(1 - z^{-2})^2}{1 + (-2\alpha - 0.2952\alpha)z^{-1} + [\alpha^2 + 0.2952(1 + \alpha^2) + 0.187\alpha^2]z^{-2} \\ &\quad + (-0.2952\alpha - 0.374\alpha)z^{-3} + 0.187z^{-4}} \\ &= \frac{0.1494(1 - z^{-2})^2}{1 - 0.036718z^{-1} + 0.70738z^{-2} - 0.018832z^{-3} + 0.3407z^{-4}}. \end{aligned}$$



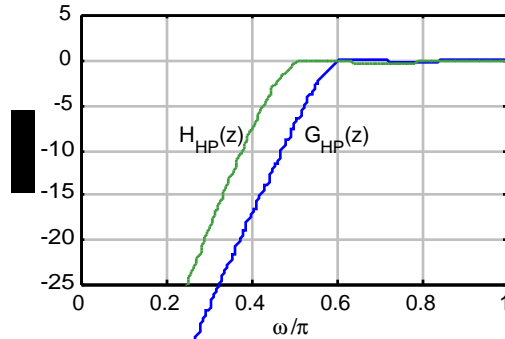
**7.30**  $\omega_p = 0.6\pi$ , and  $\hat{\omega}_p = 0.5\pi$ . Thus,  $\alpha = \frac{\sin\left(\frac{\omega_p - \hat{\omega}_p}{2}\right)}{\sin\left(\frac{\omega_p + \hat{\omega}_p}{2}\right)} = \frac{\sin(0.05\pi)}{\sin(0.55\pi)} = 0.15838444$ .

Therefore,  $H_{HP}(z) = G_{HP}(z) \Big|_{z^{-1} \rightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}} = \frac{N(z)}{D(z)}$  where

$$N(z) = 0.0916(1 + \alpha)^3(1 - z^{-1})^3 = 0.14238(1 - z^{-1})^3 \text{ and}$$

$$D(z) = (1 - 0.7601\alpha + 0.7021\alpha^2 - 0.2088\alpha^3) \\ + [-3\alpha + 0.7601(1 + 2\alpha^2) - 0.7021(2\alpha + \alpha^3) + 0.6264\alpha^2]z^{-1}$$

$$\begin{aligned}
& + [3\alpha^2 - 0.7601(2\alpha + \alpha^3) + 0.7021(1 + 2\alpha^2) - 0.6264\alpha]z^{-2} \\
& + (-\alpha^3 + 0.7601\alpha^2 - 0.7021\alpha + 0.2088)z^{-3} \\
& = 0.896395 + 0.113602498z^{-1} + 0.469574z^{-2} + 0.1126927z^{-3}.
\end{aligned}$$

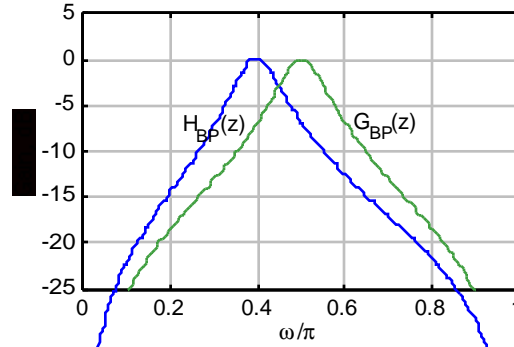


**7.31**  $H_{BP}(z) = \frac{0.136728736(1-z^{-2})}{1-0.53353098z^{-1}+0.726542528z^{-2}}$ . The lowpass-to-lowpass transformation is

given by  $z^{-1} \rightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$  where  $\alpha = \frac{\sin\left(\frac{\omega_o - \hat{\omega}_o}{2}\right)}{\sin\left(\frac{\omega_o + \hat{\omega}_o}{2}\right)}$  where  $\omega_o = 0.4\pi$  and  $\hat{\omega}_o = 0.5\pi$ .

Thus,  $\alpha = -0.15838$ .

$$\begin{aligned}
G_{LP}(z) &= H_{LP}(z) \Big|_{z^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}} = \frac{0.136728736 \left( 1 - \left( \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} \right)^2 \right)}{1 - 0.5335098 \left( \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} \right) + 0.726542528 \left( \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} \right)^2} \\
&= \frac{0.136728736(1 - \alpha)^2 (1 + \hat{z}^{-1})^2}{(1 + 0.533531\alpha + 0.72654253\alpha^2) + [-2\alpha - 0.533531(1 + \alpha^2) - 1.4531\alpha]\hat{z}^{-1} + (\alpha^2 + 0.533531\alpha + 0.72654253)\hat{z}^{-2}} \\
&= \frac{0.1333(1 - \hat{z}^{-2})}{0.93372 + 2.7299 \times 10^{-9} \hat{z}^{-1} + 0.66713\hat{z}^{-2}}.
\end{aligned}$$

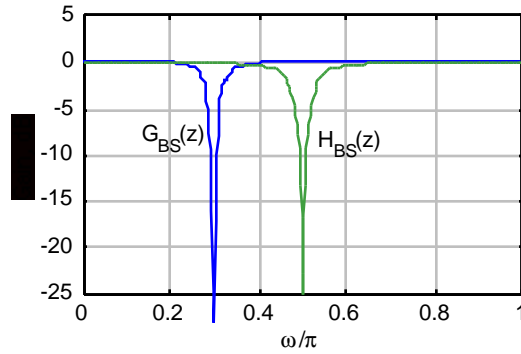


**7.32** The notch filter designed in Example 7.8 has notch frequency at  $\omega_o = 2\pi\left(\frac{60}{400}\right) = 0.3\pi$  and its transfer function is given by  $G_{BS}(z) = \frac{0.940809 - 1.105987z^{-1} + 0.94089z^{-2}}{1 - 1.105987z^{-1} + 0.881618z^{-2}}$ . The desired notch frequency of the transformed filter is  $\hat{\omega}_o = 2\pi\left(\frac{100}{400}\right) = 0.5\pi$ . The lowpass-to-lowpass

transformation to be used is given by  $z^{-1} \rightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$  where  $\alpha = \frac{\sin\left(\frac{\omega_o - \hat{\omega}_o}{2}\right)}{\sin\left(\frac{\omega_o + \hat{\omega}_o}{2}\right)} =$

$-0.32492$ . The desired transfer function is thus given by

$$\begin{aligned}
 H_{BS}(z) &= G_{BS}(z) \Big|_{z^{-1} = \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}} = \frac{0.940809 - 1.105987\left(\frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}\right) + 0.94089\left(\frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}\right)^2}{1 - 1.105987\left(\frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}\right) + 0.881618\left(\frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}\right)^2} \\
 &= \frac{(0.940809 + 1.105987\alpha + 0.940809\alpha^2) + [-3.7632\alpha - 1.105987(1 + \alpha^2)]z^{-1} + (0.940809\alpha^2 + 1.105987\alpha + 0.940809)z^{-2}}{(1 + 1.105987\alpha + 0.940809\alpha^2) + [-2\alpha - 1.105987(1 + \alpha^2) - 1.7632\alpha]z^{-1} + (\alpha^2 + 1.105987\alpha + 0.881618)z^{-2}} \\
 &= \frac{0.68078 + 3.4367 \times 10^{-7} z^{-1} + 0.68078 z^{-2}}{0.73997 + 3.4367 \times 10^{-7} z^{-1} + 0.62783 z^{-2}}.
 \end{aligned}$$

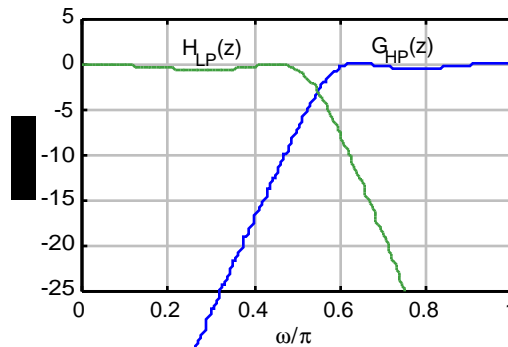


7.33  $G_{HP}(z) = \frac{0.0916(1-z^{-1})^3}{1+0.7601z^{-1}+0.7021z^{-2}+0.2088z^{-3}}$ . Now  $\omega_p = 0.6\pi$ , and  $\hat{\omega}_p = 0.5\pi$ . Thus,

$$\alpha = -\frac{\cos\left(\frac{\omega_p + \hat{\omega}_p}{2}\right)}{\cos\left(\frac{\omega_p - \hat{\omega}_p}{2}\right)} = -\frac{\cos(0.55\pi)}{\cos(0.05\pi)} = 0.15838444. \quad \text{Therefore,}$$

$$H_{HP}(z) = G_{HP}(z)\Big|_{z^{-1} \rightarrow -\frac{z^{-1}-\alpha}{1-\alpha z^{-1}}} = \frac{N(z)}{D(z)}, \text{ where } N(z) = 0.0916(1+\alpha)^3(1+z^{-1})^3, \text{ and}$$

$$D(z) = (1 - 0.7601\alpha + 0.7021\alpha^2 - 0.2088\alpha^3) + [-3\alpha + 0.7601(1 + 2\alpha^2) - 0.7021(2\alpha + \alpha^3) + 0.6264\alpha^2]z^{-1} + [3\alpha^2 - 0.7601(2\alpha + \alpha^3) + 0.7021(1 + 2\alpha^2) - 0.6264\alpha]z^{-2} + (-\alpha^3 + 0.7601\alpha^2 - 0.7021\alpha + 0.2088)z^{-3}$$



7.34 The ideal L-band digital filter  $H_{ML}(z)$  with an ideal frequency response given by  $H_{ML}(e^{j\omega}) = A_k$  for  $\omega_{k-1} \leq \omega \leq \omega_k$ ,  $k = 1, 2, \dots, K, L$ , can be considered as sum of L ideal bandpass filters with cutoff frequencies at  $\omega_{c1}^k = \omega_{k-1}$  and  $\omega_{c2}^k = \omega_k$ , where  $\omega_{c1}^0 = 0$  and  $\omega_{c2}^L = \pi$ . Now from Eq. (7.90) the impulse response of a bandpass filter is given by  $h_{BP}[n] = \frac{\sin(\omega_{c2}n)}{\pi n} - \frac{\sin(\omega_{c1}n)}{\pi n}$ . Therefore,  $h_{BP}^k[n] = A_k \left( \frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1} n)}{\pi n} \right)$ . Hence,



$$\begin{aligned}
h_{\text{ML}}[n] &= \sum_{k=1}^L h_{\text{BP}}^k[n] = \sum_{k=1}^L A_k \left( \frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1} n)}{\pi n} \right) \\
&= A_1 \left( \frac{\sin(\omega_1 n)}{\pi n} - \frac{\sin(0n)}{\pi n} \right) + \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=2}^{L-1} A_k \left( \frac{\sin(\omega_k n)}{\pi n} - \frac{\sin(\omega_{k-1} n)}{\pi n} \right) \\
&\quad + A_L \left( \frac{\sin(\omega_L n)}{\pi n} - \frac{\sin(\omega_{L-1} n)}{\pi n} \right) \\
&= A_1 \frac{\sin(\omega_1 n)}{\pi n} + \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=2}^{L-1} A_k \frac{\sin(\omega_{k-1} n)}{\pi n} - A_L \frac{\sin(\omega_{L-1} n)}{\pi n} \\
&= \sum_{k=1}^{L-1} A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=2}^L A_k \frac{\sin(\omega_{k-1} n)}{\pi n}.
\end{aligned}$$

Since  $\omega_L = \pi$ ,  $\sin(\omega_L n) = 0$ . We add a term  $A_L \frac{\sin(\omega_L n)}{\pi n}$  to the first sum in the above expression and change the index range of the second sum, resulting in

$$h_{\text{ML}}[n] = \sum_{k=1}^L A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=1}^{L-1} A_{k+1} \frac{\sin(\omega_k n)}{\pi n}.$$

Finally, since  $A_{L+1} = 0$ , we can add a term  $A_{L+1} \frac{\sin(\omega_L n)}{\pi n}$  to the second sum. This leads to

$$h_{\text{ML}}[n] = \sum_{k=1}^L A_k \frac{\sin(\omega_k n)}{\pi n} - \sum_{k=1}^L A_{k+1} \frac{\sin(\omega_k n)}{\pi n} = \sum_{k=1}^L (A_k - A_{k+1}) \frac{\sin(\omega_k n)}{\pi n}.$$

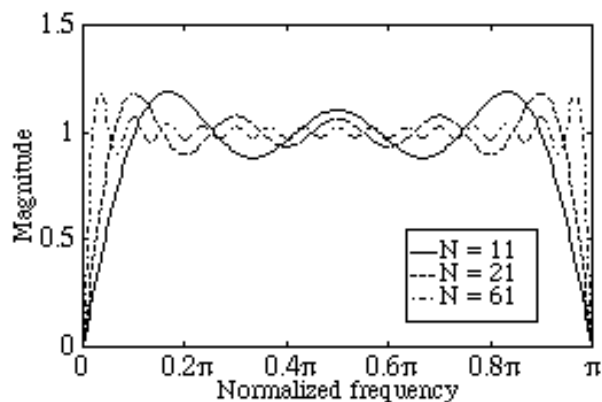
**7.35**  $H_{\text{HT}}(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0, \\ -j, & 0 < \omega < \pi. \end{cases}$  Therefore,

$$\begin{aligned}
h_{\text{HT}}[n] &= \frac{1}{2\pi} \int_{-\pi}^0 H_{\text{HT}}(e^{j\omega}) e^{j\omega n} d\omega + \frac{1}{2\pi} \int_0^{\pi} H_{\text{HT}}(e^{j\omega}) e^{j\omega n} d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^0 j e^{j\omega n} d\omega - \frac{1}{2\pi} \int_0^{\pi} j e^{j\omega n} d\omega = \frac{2}{2\pi n} (1 - \cos(\pi n)) = \frac{2 \sin^2(\pi n / 2)}{\pi n} \quad \text{if } n \neq 0.
\end{aligned}$$

$$\text{For } n = 0, h_{\text{HT}}[0] = \frac{1}{2\pi} \int_{-\pi}^0 j d\omega - \frac{1}{2\pi} \int_0^{\pi} j d\omega = 0.$$

$$\text{Hence, } h_{\text{HT}}[n] = \begin{cases} 0, & \text{if } n = 0, \\ \frac{2 \sin^2(\pi n / 2)}{\pi n}, & \text{if } n \neq 0. \end{cases}$$

Since  $h_{\text{HT}}[n] = -h_{\text{HT}}[-n]$  and the length of the truncated impulse response is odd, it is a Type 3 linear-phase FIR filter.



From the frequency response plots given above, we observe the presence of ripples at the bandedges due to the Gibbs phenomenon caused by the truncation of the impulse response.

$$7.36 \quad H\{x[n]\} = \sum_{k=-\infty}^{\infty} h_{HT}[n-k]x[k].$$

$$\text{Hence } F\{H\{x[n]\}\} = H_{HT}(e^{j\omega})X(e^{j\omega}) = \begin{cases} jX(e^{j\omega}), & -\pi < \omega < 0, \\ -jX(e^{j\omega}), & 0 < \omega < \pi. \end{cases}$$

$$(a) \text{ Let } y[n] = H\{H\{H\{H\{x[n]\}\}\}\}. \text{ Hence } Y(e^{j\omega}) = \begin{cases} j^4 X(e^{j\omega}), & -\pi < \omega < 0, \\ (-j)^4 X(e^{j\omega}), & 0 < \omega < \pi, \end{cases} = X(e^{j\omega}).$$

Therefore,  $y[n] = x[n]$ .

$$(b) \text{ Define } g[n] = H\{x[n]\}, \text{ and } h^*[n] = x[n]. \text{ Then } \sum_{l=-\infty}^{\infty} H\{x[l]\}x[l] = \sum_{l=-\infty}^{\infty} g[l]h^*[l].$$

$$\text{But from the Parseval's relation in Table 3.2, } \sum_{l=-\infty}^{\infty} g[l]h^*[l] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega})G(e^{j\omega})d\omega.$$

$$\text{Therefore, } \sum_{l=-\infty}^{\infty} H\{x[l]\}x[l] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega})d\omega \text{ where}$$

$$H_{HT}(e^{j\omega}) = \begin{cases} j, & -\pi < \omega < 0, \\ -j, & 0 < \omega < \pi. \end{cases} \text{ Since the integrand } H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega}) \text{ is an odd}$$

$$\text{function of } \omega, \int_{-\pi}^{\pi} H_{HT}(e^{j\omega})X(e^{j\omega})X(e^{-j\omega})d\omega = 0. \text{ As a result, } \sum_{l=-\infty}^{\infty} H\{x[l]\}x[l] = 0.$$

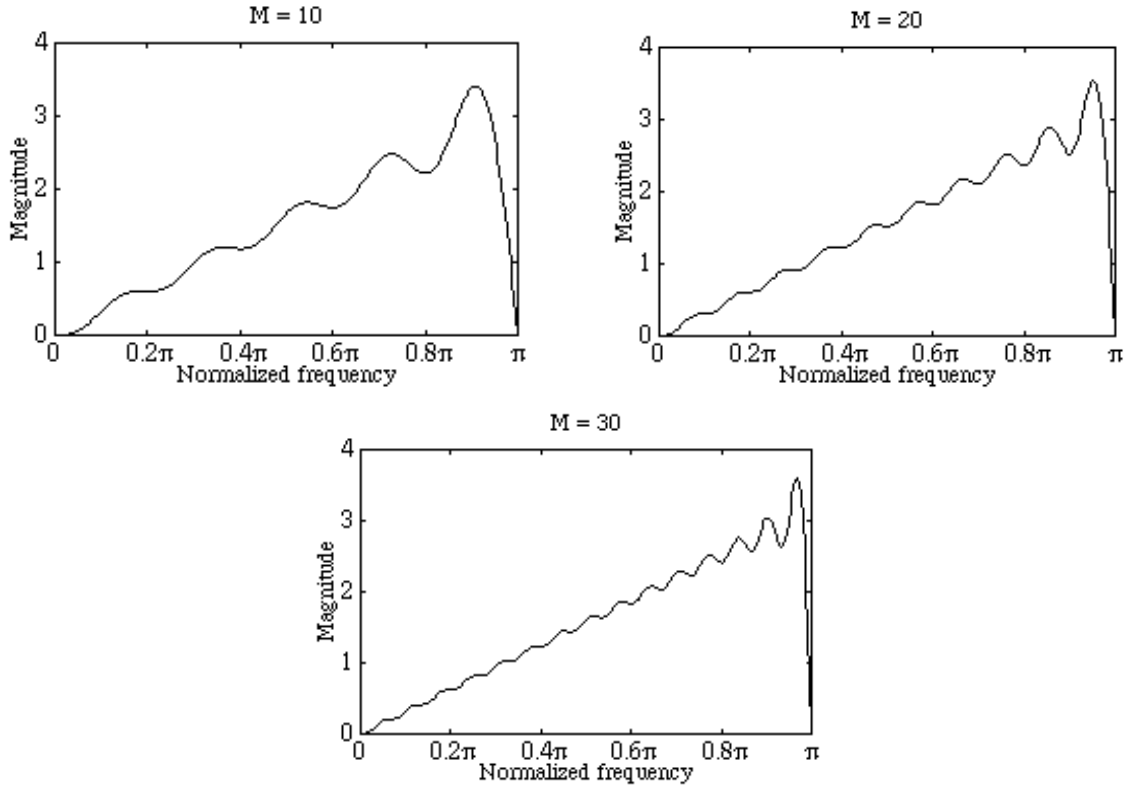
$$7.37 \quad H_{DIF}(e^{j\omega}) = j\omega. \text{ Hence,}$$

$$h_{DIF}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} j\omega e^{j\omega n} d\omega = \frac{j}{2\pi} \int_{-\pi}^{\pi} \omega e^{j\omega n} d\omega = \frac{j}{2\pi} \left( \frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right) \Big|_{-\pi}^{\pi}. \text{ Therefore,}$$

$$h_{DIF}[n] = \frac{\cos(\pi n)}{n} - \frac{\sin(\pi n)}{\pi n^2} = \frac{\cos(\pi n)}{n}, \text{ if } n \neq 0. \text{ For } n = 0, h_{DIF}[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} j\omega d\omega = 0.$$

Hence,  $h_{\text{DIF}}[n] = \begin{cases} 0, & n = 0, \\ \frac{\cos(\pi n)}{n}, & |n| > 0. \end{cases}$  Since  $h_{\text{DIF}}[n] = -h_{\text{DIF}}[-n]$ , the truncated impulse response is a Type 3 linear-phase FIR filter.

The magnitude responses of the above differentiator for several values of  $M$  are given below:



$$7.38 \quad N = 2M + 1. \quad \hat{h}_{\text{HP}}[n] = \begin{cases} 1 - \frac{\omega_c}{\pi}, & \text{for } n = M, \\ \frac{\sin(\omega_c(n-m))}{\pi(n-m)}, & \text{if } n \neq M, 0 \leq n < N, \\ 0. & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \text{Now } \hat{H}_{\text{HP}}(z) + \hat{H}_{\text{LP}}(z) &= \sum_{n=-\infty}^{\infty} \hat{h}_{\text{HP}}[n]z^{-n} + \sum_{n=-\infty}^{\infty} \hat{h}_{\text{LP}}[n]z^{-n} = \sum_{n=0}^{N-1} \hat{h}_{\text{HP}}[n]z^{-n} + \sum_{n=0}^{N-1} \hat{h}_{\text{LP}}[n]z^{-n} \\ &= \sum_{n=0}^{N-1} (\hat{h}_{\text{HP}}[n] + \hat{h}_{\text{LP}}[n])z^{-n}. \end{aligned}$$

$$\text{But } \hat{h}_{\text{HP}}[n] + \hat{h}_{\text{LP}}[n] = \hat{h}_{\text{HP}}[n] + \hat{h}_{\text{LP}}[n] = \begin{cases} 0, & 0 \leq n \leq N-1, n \neq M, \\ 1, & n = M, \end{cases}$$

Hence,  $\hat{H}_{\text{HP}}(z) + \hat{H}_{\text{LP}}(z) = z^{-M}$ , i.e. the two filters are delay-complementary.

**7.39**  $H_{LLP}(e^{j\omega}) = \begin{cases} |\omega|, & |\omega| < \omega_c, \\ 0, & \text{otherwise.} \end{cases}$  Therefore,

$$h_{LLP}[n] = \frac{1}{2\pi} \left( - \int_{-\omega_c}^0 \omega e^{j\omega n} d\omega + \int_0^{\omega_c} \omega e^{j\omega n} d\omega \right) = \frac{1}{2\pi} \left( - \left[ \frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right] \Big|_{-\omega_c}^0 + \left[ \frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right] \Big|_0^{\omega_c} \right)$$

$$= \frac{1}{2\pi} \left( \frac{\omega_c e^{j\omega_c n} - \omega_c e^{-j\omega_c n}}{jn} + \frac{e^{j\omega_c n} + e^{-j\omega_c n} - 2}{n^2} \right) = \frac{\omega_c}{\pi n} \sin(\omega_c n) + \frac{\cos(\omega_c n) - 1}{\pi n^2}.$$

**7.40**  $H_{BLDIF}(e^{j\omega}) = \begin{cases} \omega, & |\omega| < \omega_c, \\ 0, & \text{otherwise.} \end{cases}$  Hence,

$$h_{BLDIF}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \omega e^{j\omega n} d\omega = \frac{1}{2\pi} \left( \left[ \frac{\omega e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right] \Big|_{-\omega_c}^{\omega_c} \right)$$

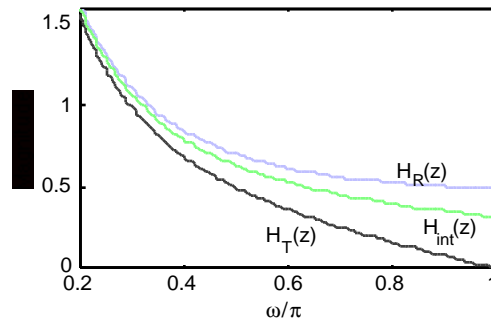
$$= \frac{1}{2\pi} \left( \frac{\omega_c e^{j\omega_c n} + \omega_c e^{-j\omega_c n}}{jn} + \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{n^2} \right) = -j \frac{\omega_c}{\pi n} \cos(\omega_c n) + j \frac{1}{\pi n^2} \sin(\omega_c n).$$

**7.41** From Eq. (2.181)  $y(nT) = y((n-1)T) + \int_{(n-1)T}^{nT} x(\tau) d\tau = y((n-1)T) + T \cdot x((n-1)T)$  which reduces to

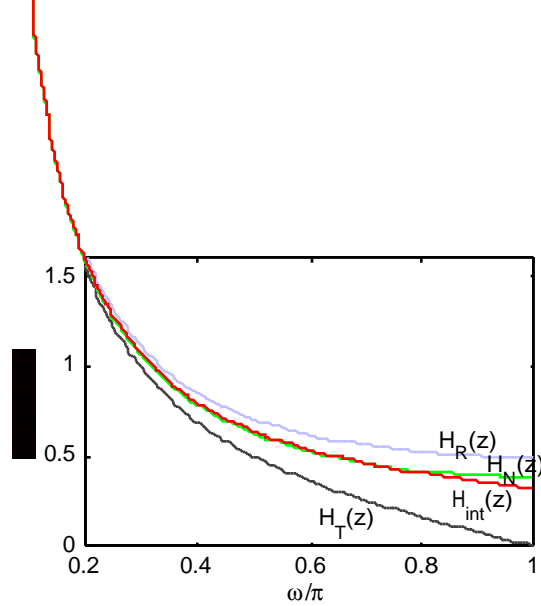
$y[n] = y[n-1] + T \cdot x[n-1]$ . Hence, the corresponding transfer function is given by

$H_R(z) = \frac{Tz^{-1}}{1-z^{-1}}$ . From Eq. (2.99)  $y[n] = y[n-1] + \frac{T}{2}(x[n] + x[n-1])$ . Hence, the corresponding

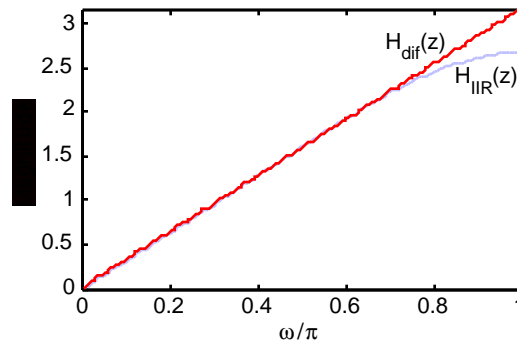
transfer function is given by  $H_T(z) = \frac{T}{2} \cdot \frac{1+z^{-1}}{1-z^{-1}}$ . From the plot given below it can be seen that the magnitude response of  $H_{int}(z)$  lies between that of  $H_R(z)$  and  $H_T(z)$ .



**7.42**  $H_N(z) = \frac{3}{4}H_R(z) + \frac{1}{4}H_T(z)$ . From the plot given below it can be seen that the magnitude response of  $H_N(z)$  lies between that of  $H_R(z)$  and  $H_T(z)$ , and is much closer to that of  $H_{int}(z)$ .

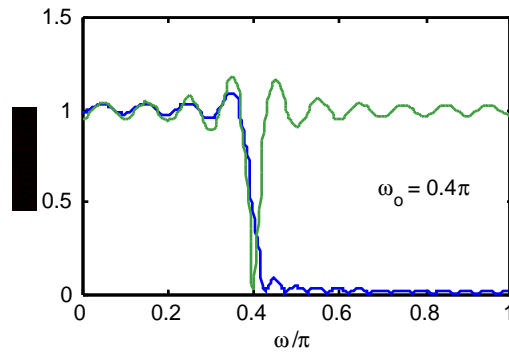


**7.43**  $H_N(z) = \frac{3}{4} \left( \frac{z^{-1}}{1-z^{-1}} \right) + \frac{1}{8} \left( \frac{1+z^{-1}}{1-z^{-1}} \right) = \frac{1+7z^{-1}}{8(1-z^{-1})}$ . Its inverse is given by  $H(z) = \frac{8(1-z^{-1})}{1+7z^{-1}}$ , which is unstable as it has a pole at  $z = -7$ . A stable transfer function with the same magnitude response is obtained by multiplying  $H(z)$  with an allpass function  $(1+7z^{-1}) / (7+z^{-1})$  resulting in  $H_{IIR}(z) = \frac{8(1-z^{-1})}{1+7z^{-1}} \cdot \frac{1+7z^{-1}}{7+z^{-1}} = \frac{8(1-z^{-1})}{7+z^{-1}}$ . A plot of the ideal differentiator  $H_{dif}(z)$  with a frequency response given by Eq. (7.68) and that of the IIR differentiator  $H_{IIR}(z)$  is given below. As can be seen the magnitude response of  $H_{IIR}(z)$  is very close to that  $H_{dif}(z)$ .



**7.44** The frequency response of a causal ideal notch filter can thus be expressed as  $H_{notch}(e^{j\omega}) = \hat{H}_{notch}(\omega)e^{j\theta(\omega)}$  where  $\hat{H}_{notch}(\omega)$  is the amplitude response which can be expressed as  $\hat{H}_{notch}(\omega) = \begin{cases} 1, & 0 \leq \omega < \omega_0 \\ -1, & \omega_0 \leq \omega < \pi. \end{cases}$  It follows then that  $\hat{H}_{notch}(\omega)$  is related to the amplitude response  $\hat{H}_{LP}(\omega)$  of the ideal lowpass filter with a cutoff at  $\omega_0$  through  $\hat{H}_{notch}(\omega) = \pm 2[\hat{H}_{LP}(\omega) - 1]$ . Hence, the impulse response of the ideal notch filter is given by  $h_{notch}[n] = \pm 2[h_{LP}[n] - \delta[n]]$ , where  $h_{LP}[n] = \frac{\sin(\omega_0 n)}{\pi n}$ ,  $-\infty < n < \infty$ . The

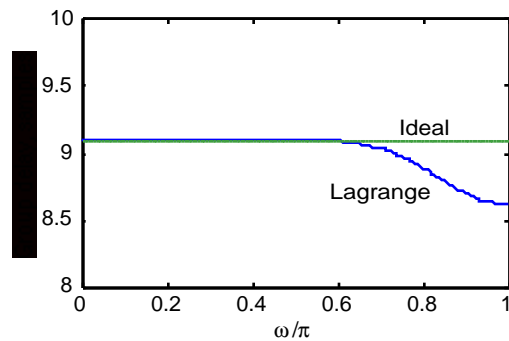
magnitude responses of a length 41 notch filter with a notch frequency at  $\omega_o = 0.4\pi$  and its associated length-41 lowpass filter are shown below.



**7.45 (a)** See Section 10.5.2

**(b)**  $D = 100/11$ . 
$$h[n] = \prod_{\substack{k=0 \\ k \neq n}}^{16} \left( \frac{D-k}{n-k} \right), \quad 0 \leq n \leq 16.$$

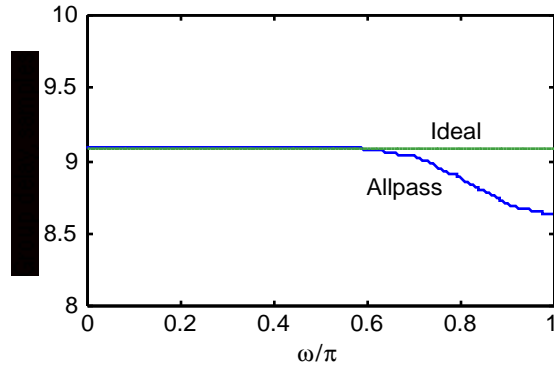
$h[0] = -8.8187e-07$ ,  $h[1] = 1.5854e-05$ ,  $h[2] = -1.3567e-04$ ,  
 $h[3] = 7.3708e-04$ ,  $h[4] = -2.8661e-03$ ,  $h[5] = 8.5600e-03$ ,  
 $h[6] = -2.0771e-02$ ,  $h[7] = 4.3863e-02$ ,  $h[8] = -4580e-02$ ,  $h[9] = 1.0089e+00$ ,  
 $h[10] = 7.0620e-02$ ,  $h[11] = -1.8343e-02$ ,  $h[12] = 5.0156e-03$ ,  
 $h[13] = -1.1485e-03$ ,  $h[14] = 1.9597e-04$ ,  $h[15] = -2.1708e-05$ ,  $h[16] = 1.1604e-06$ .



**7.46** Note that the desired delay  $D = 9.0909$ . Hence  $N = 9$  is the order of the allpass transfer

function. The allpass fractional delay are thus given by  $A(z) = \frac{z^{-9}D(z^{-1})}{D(z)}$  where

$$D(z) = 1 - 0.81081z^{-1} + 0.031901z^{-2} - 0.012872z^{-3} + 0.0045589z^{-4} - 0.0013236z^{-5} \\ + 2.9767 \times 10^{-4} z^{-6} - 4.829 \times 10^{-5} z^{-7} - 5.0088 \times 10^{-6} z^{-8} + 2.489 \times 10^{-7} z^{-9}.$$



**7.47 (a)**  $x[n] = s[n] + \sum_{k=0}^M A_k \sin(k\omega_0 n + \phi_k) = s[n] + r[n]$ , where  $s[n]$  is the desired signal and

$r[n] = \sum_{k=0}^M A_k \sin(k\omega_0 n + \phi_k)$  is the harmonic interference with fundamental frequency  $\omega_0$ .

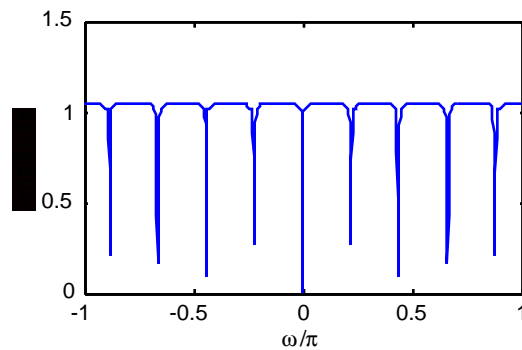
Now,  $r[n-D] = \sum_{k=0}^M A_k \sin[k\omega_0(n-D) + \phi_k] = \sum_{k=0}^M A_k \sin(k\omega_0 n + \phi_k - 2\pi k) = r[n]$ .

**(b)**  $y[n] = x[n] - x[n-D] = s[n] + r[n] - s[n-D] - r[n-D] = s[n] + r[n] - s[n-D] - r[n] = s[n] - s[n-D]$ . Hence,  $y[n]$  does not contain any harmonic disturbances.

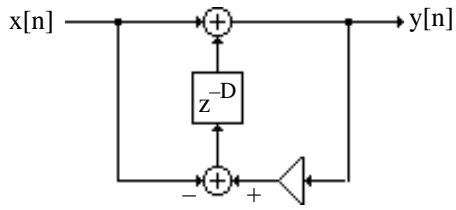
**(c)**  $H_c(z) = \frac{1-z^{-D}}{1-\rho^D z^{-D}}$ . Thus,

$H_c(e^{j\omega}) = \frac{1-e^{-jD\omega}}{1-\rho^D e^{-jD\omega}} = \frac{(1-\cos(D\omega)) + j\sin(D\omega)}{(1-\rho^D \cos(D\omega)) + j\rho^D \sin(D\omega)}$ . Then

$|H_c(e^{j\omega})| = \sqrt{\frac{2(1-\cos(D\omega))}{1-2\rho^D \cos(D\omega) + \rho^{2D}}}$ . A plot of  $|H_c(e^{j\omega})|$  for  $\omega_0 = 0.22\pi$  and  $\rho = 0.99$  is shown below:



**(d)**



**7.48**  $H_c(z) = \frac{P(z)}{Q(z)} = \frac{1 - N(z)}{1 - (0.985)^{(100/11)} N(z)}$  where the coefficients of  $N(z)$  are as given in

Problem 7.45 solution. The transfer function coefficients in ascending powers of  $z^{-1}$  are given by

Numerator Coefficients

Columns 1 through 5

1.0000e+00   -1.5854e-05   1.3567e-04   -7.3708e-04   2.8661e-03

Columns 6 through 10

-8.5600e-03   2.0771e-02   -4.3863e-02   9.4580e-02   -1.0089e+00

Columns 11 through 15

-7.0620e-02   1.8343e-02   -5.0156e-03   1.1485e-03   -1.9597e-04

Columns 16 through 17

2.1708e-05   -1.1604e-06

Denominator Coefficients

Columns 1 through 5

1.0000e+00   -1.3819e-05   1.1826e-04   -6.4246e-04   2.4981e-03

Columns 6 through 10

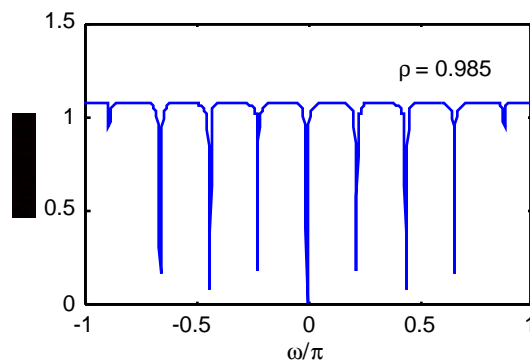
-7.4611e-03   1.8104e-02   -3.8232e-02   8.2439e-02   -8.7934e-01

Columns 11 through 15

-6.1554e-02   1.5988e-02   -4.3717e-03   1.0010e-03   -1.7081e-04

Columns 16 through 17

1.8921e-05   -1.0114e-06

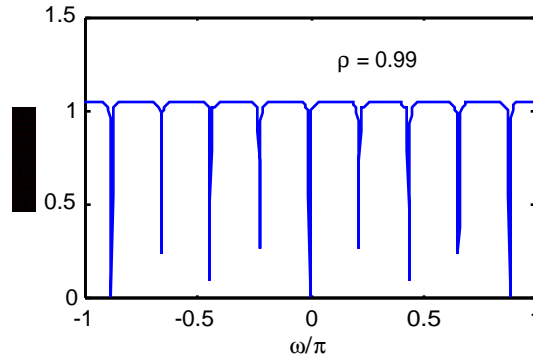




$$7.49 \quad H_c(z) = \frac{P(z)}{Q(z)} = \frac{D(z) - z^{-9}D(z^{-1})}{D(z) - (0.99)^{(100/11)}z^{-9}D(z^{-1})} \text{ where}$$

$$P(z) = 1 - 0.81086z^{-1} + 0.031949z^{-2} - 0.01317z^{-3} + 0.0058825z^{-4} - 0.0058825z^{-5} \\ + 0.01317z^{-6} - 0.031949z^{-7} + 0.0810869z^{-8} - z^{-9}, \text{ and}$$

$$Q(z) = 1 - 0.81086z^{-1} + 0.031945z^{-2} - 0.013144z^{-3} + 0.0057669z^{-4} - 0.0054844z^{-5} \\ + 0.012046z^{-6} - 0.029164z^{-7} + 0.074006z^{-8} - 0.91268z^{-9},$$



$$7.50 \quad H_{LP}(e^{j\omega}) = \begin{cases} 1, & -\omega_p \leq \omega \leq \omega_p, \\ 1 - \left( \frac{\omega - \omega_p}{\omega_s - \omega_p} \right), & \omega_p < \omega \leq \omega_s, \\ 1 + \left( \frac{\omega + \omega_p}{\omega_s - \omega_p} \right), & -\omega_s < \omega \leq -\omega_p, \\ 0, & \text{elsewhere.} \end{cases}$$

$$\text{Now, for } n \neq 0, h_{LP}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[ \int_{-\omega_p}^{\omega_p} e^{j\omega n} d\omega + \int_{\omega_p}^{\omega_s} \left( 1 - \frac{\omega - \omega_p}{\Delta\omega} \right) e^{j\omega n} d\omega + \int_{-\omega_s}^{-\omega_p} \left( 1 + \frac{\omega + \omega_p}{\Delta\omega} \right) e^{j\omega n} d\omega \right]$$

$$= \frac{1}{2\pi} \left[ \int_{-\omega_s}^{\omega_s} e^{j\omega n} d\omega - \int_{\omega_p}^{\omega_s} \frac{\omega - \omega_p}{\Delta\omega} e^{j\omega n} d\omega + \int_{-\omega_s}^{-\omega_p} \frac{\omega + \omega_p}{\Delta\omega} e^{j\omega n} d\omega \right]$$

$$= \frac{1}{2\pi} \left\{ \frac{2\sin(\omega_s n)}{\pi n} - \frac{1}{\Delta\omega} \left[ \frac{(\omega - \omega_p)e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right] \Bigg|_{\omega_p}^{\omega_s} + \frac{1}{\Delta\omega} \left[ \frac{(\omega + \omega_p)e^{j\omega n}}{jn} + \frac{e^{j\omega n}}{n^2} \right] \Bigg|_{-\omega_s}^{-\omega_p} \right\}$$

$$= \frac{1}{2\pi} \left\{ \frac{2\sin(\omega_s n)}{\pi n} - \frac{1}{\Delta\omega} \left[ \frac{\Delta\omega e^{j\omega_s n}}{jn} + \frac{e^{j\omega_s n} - e^{j\omega_p n}}{n^2} + \frac{-\Delta\omega e^{-j\omega_s n}}{jn} - \frac{e^{-j\omega_p n} - e^{-j\omega_s n}}{n^2} \right] \right\}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \left\{ \frac{2\sin(\omega_s n)}{\pi n} - \frac{2}{\Delta\omega} \left[ \frac{\Delta\omega \sin(\omega_s n)}{n} + \frac{\cos(\omega_s n) - \cos(\omega_p n)}{n^2} \right] \right\} \\
&= \frac{1}{\Delta\omega} \left\{ \frac{\cos(\omega_p n)}{\pi n^2} - \frac{\cos(\omega_s n)}{\pi n^2} \right\} = \frac{1}{\Delta\omega} \left\{ \frac{\cos((\omega_c - \Delta\omega/2)n)}{\pi n^2} - \frac{\cos((\omega_c + \Delta\omega/2)n)}{\pi n^2} \right\} \\
&= \frac{2\sin(\Delta\omega n/2)}{\Delta\omega n} \frac{\sin(\omega_c n)}{\pi n}.
\end{aligned}$$

Now for  $n = 0$ ,  $h_{LP}[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) d\omega = \frac{1}{2\pi}$  (area under the curve)

$$= \frac{1}{2\pi} \frac{2(\omega_s + \omega_p)}{2} = \frac{\omega_c}{\pi}.$$

$$\text{Hence, } h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & \text{if } n = 0, \\ \frac{2\sin(\Delta\omega n/2)}{\Delta\omega n} \frac{\sin(\omega_c n)}{\pi n}, & \text{if } n \neq 0. \end{cases}$$

An alternate approach to solving this problem is as follows. Consider the frequency

$$\text{response } G(e^{j\omega}) = \frac{dH_{LP}(e^{j\omega})}{d\omega} = \begin{cases} 0, & -\omega_p \leq \omega \leq \omega_p, \\ -\frac{1}{\Delta\omega}, & \omega_p < \omega < \omega_s, \\ \frac{1}{\Delta\omega}, & -\omega_s < \omega < -\omega_p, \\ 0, & \text{elsewhere.} \end{cases} \quad \text{Its inverse DTFT is given by}$$

$$\begin{aligned}
g[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\omega_s}^{-\omega_p} \frac{1}{\Delta\omega} e^{j\omega n} d\omega - \frac{1}{2\pi} \int_{\omega_p}^{\omega_s} \frac{1}{\Delta\omega} e^{j\omega n} d\omega \\
&= \frac{1}{2\pi\Delta\omega} \left( \frac{e^{j\omega n}}{jn} \Big|_{-\omega_s}^{-\omega_p} - \frac{e^{j\omega n}}{jn} \Big|_{\omega_p}^{\omega_s} \right) = \frac{1}{j\pi n\Delta\omega} (\cos(\omega_p n) - \cos(\omega_s n)).
\end{aligned}$$

$$\begin{aligned}
\text{Thus, } h_{LP}[n] &= \frac{j}{n} g[n] = \frac{1}{\pi n^2 \Delta\omega} (\cos(\omega_p n) - \cos(\omega_s n)) \\
&= \frac{1}{\pi n^2 \Delta\omega} \left[ \cos\left(\left(\omega_c - \frac{\Delta\omega}{2}\right)n\right) - \cos\left(\left(\omega_c + \frac{\Delta\omega}{2}\right)n\right) \right] \\
&= \frac{2\sin(\Delta\omega n/2)}{\Delta\omega n} \frac{\sin(\omega_c n)}{\pi n}, \text{ for } n \neq 0.
\end{aligned}$$

$$\text{For } n = 0, h_{LP}[n] = \frac{\omega_c}{\pi}.$$

**7.51** Consider the case when the transition region is approximated by a second order spline. In this case the ideal frequency response can be constructed by convolving an ideal, no-

transition-band frequency response with a triangular pulse of width  $\Delta\omega = \omega_s - \omega_p$ , which in turn can be obtained by convolving two rectangular pulses of width  $\Delta\omega/2$ . In the time domain this implies that the impulse response of a filter with transition band approximated by a second order spline is given by the product of the impulse response of an ideal low pass filter with no transition region and square of the impulse response of a rectangular pulse. Now,

$$H_{LP(\text{ideal})}[n] = \frac{\sin(\omega_c n)}{\pi n} \quad \text{and} \quad H_{\text{rec}}[n] = \frac{\sin(\Delta\omega n/4)}{\Delta\omega n/4}. \quad \text{Hence,} \quad H_{LP}[n] = H_{LP(\text{ideal})}[n] (H_{\text{rec}}[n])^2.$$

Thus for a lowpass filter with a transition region approximated by a second order spline

$$h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & \text{if } n = 0, \\ \left( \frac{\sin(\Delta\omega n/4)}{\Delta\omega n/4} \right)^2 \frac{\sin(\omega_c n)}{\pi n}, & \text{otherwise.} \end{cases}$$

Similarly the frequency response of a lowpass filter with the transition region specified by a P-th order spline can be obtained by convolving in the frequency domain an ideal filter with no transition region with P rectangular pulses of width  $\Delta\omega/P$ . Hence,

$H_{LP}[n] = H_{LP(\text{ideal})}[n] (H_{\text{rec}}[n])^P$ , where the rectangular pulse is of width  $\Delta\omega/P$ . Thus

$$h_{LP}[n] = \begin{cases} \frac{\omega_c}{\pi}, & \text{if } n = 0, \\ \left( \frac{\sin(\Delta\omega n/2P)}{\Delta\omega n/2P} \right)^P \frac{\sin(\omega_c n)}{\pi n}, & \text{otherwise.} \end{cases}$$

**7.52** Consider another filter with a frequency response  $G(e^{j\omega})$  given by

$$G(e^{j\omega}) = \begin{cases} 0, & 0 \leq \omega \leq \omega_p, \\ \frac{-\pi}{2\Delta\omega} \sin\left(\frac{\pi(\omega - \omega_p)}{\Delta\omega}\right), & \omega_p < \omega \leq \omega_s, \\ \frac{-\pi}{2\Delta\omega} \sin\left(\frac{\pi(\omega + \omega_p)}{\Delta\omega}\right), & -\omega_s \leq \omega \leq -\omega_p, \\ 0, & \text{elsewhere.} \end{cases}$$

Clearly  $G(e^{j\omega}) = \frac{dH(e^{j\omega})}{d\omega}$ . Now,

$$g[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega}) e^{j\omega n} d\omega = \frac{-\pi}{8\pi\Delta\omega j} \left( \int_{\omega_p}^{\omega_s} e^{j\left(\frac{\pi(\omega - \omega_p)}{\Delta\omega}\right)} e^{j\omega n} d\omega - \int_{\omega_p}^{\omega_s} e^{-j\left(\frac{\pi(\omega - \omega_p)}{\Delta\omega}\right)} e^{j\omega n} d\omega \right. \\ \left. + \int_{-\omega_s}^{-\omega_p} e^{j\left(\frac{\pi(\omega + \omega_p)}{\Delta\omega}\right)} e^{j\omega n} d\omega - \int_{-\omega_s}^{-\omega_p} e^{-j\left(\frac{\pi(\omega + \omega_p)}{\Delta\omega}\right)} e^{j\omega n} d\omega \right)$$

$$\begin{aligned}
&= \frac{-1}{8\Delta\omega j} \left( e^{(-j\pi\omega_p/\Delta\omega)} \left[ \frac{e^{j\omega_s(n+\pi/\Delta\omega)} - e^{j\omega_p(n+\pi/\Delta\omega)}}{j(n+\pi/\Delta\omega)} \right] - e^{(j\pi\omega_p/\Delta\omega)} \left[ \frac{e^{j\omega_s(n-\pi/\Delta\omega)} - e^{j\omega_p(n-\pi/\Delta\omega)}}{j(n-\pi/\Delta\omega)} \right] \right. \\
&\quad \left. + e^{(j\pi\omega_p/\Delta\omega)} \left[ \frac{e^{-j\omega_p(n+\pi/\Delta\omega)} - e^{-j\omega_s(n+\pi/\Delta\omega)}}{j(n+\pi/\Delta\omega)} \right] - e^{(-j\pi\omega_p/\Delta\omega)} \left[ \frac{e^{-j\omega_p(n-\pi/\Delta\omega)} - e^{-j\omega_s(n-\pi/\Delta\omega)}}{j(n-\pi/\Delta\omega)} \right] \right) \\
&= \frac{-1}{4\Delta\omega j} \left( \frac{-\sin(\omega_s n) - \sin(\omega_p n)}{(n+\pi/\Delta\omega)} - \frac{-\sin(\omega_s n) - \sin(\omega_p n)}{(n-\pi/\Delta\omega)} \right) = \frac{(\sin(\omega_s n) + \sin(\omega_p n))}{4\Delta\omega j} \left( \frac{-2\pi/\Delta\omega}{n^2 - \pi^2/\Delta\omega^2} \right) \\
&= \frac{\sin(\omega_c n) \cos(\Delta\omega n/2)}{\Delta\omega j} \left( \frac{1}{\pi \left( 1 - (\Delta\omega/\pi)^2 n^2 \right)} \right).
\end{aligned}$$

Now  $h[n] = \frac{j}{n} g[n]$ . Therefore,  $h[n] = \left( \frac{\cos(\Delta\omega n/2)}{1 - (\Delta\omega/\pi)^2 n^2} \right) \left( \frac{\sin(\omega_c n)}{\pi n} \right)$ .

**7.53** Let  $w_R[n] = \frac{1}{k}$ , Since the convolution of two length  $N$  sequences produces a sequence of

length  $2N-1$ , therefore  $2N-1 = 2M+1$  which gives  $N = M+1$ . Therefore,  $w_R[n] = \frac{1}{k}$ ,

$$-\frac{M}{2} \leq n \leq \frac{M}{2}. \text{ Now, } w[n] = w_R[n] \circledast w_R[n] = \begin{cases} \frac{1}{k^2} (M+1-|n|), & -M \leq n \leq M, \\ 0, & \text{elsewhere,} \end{cases} \text{ or}$$

$$w[n] = \begin{cases} \frac{M+1}{k^2} \left( 1 - \frac{|n|}{M+1} \right), & -M \leq n \leq M, \\ 0, & \text{elsewhere.} \end{cases}$$

Now  $\frac{M+1}{k^2} = 1$  which yields  $k = \sqrt{M+1}$ . Hence a Bartlett window of length  $2M+1$  is obtained by the linear convolution of 2 length  $M+1$  rectangular windows scaled by a factor of  $\frac{1}{\sqrt{M+1}}$  each. The DTFT  $\Psi_{\text{BART}}(e^{j\omega})$  of the Bartlett window is thus given by

$$\Psi_{\text{BART}}(e^{j\omega}) = \left( \Psi_R(e^{j\omega}) \right)^2 = \frac{1}{M+1} \frac{\sin^2\left(\frac{\omega(M+1)}{2}\right)}{\sin^2\left(\frac{\omega}{2}\right)}, \text{ where } \Psi_R(e^{j\omega}) \text{ is the DTFT of the}$$

rectangular window. Hence  $\Psi_{\text{BART}}(e^{j\omega}) = 0$  at  $\omega = \pm \frac{2\pi}{M+1}, \pm \frac{4\pi}{M+1}, \dots$

Therefore  $\Delta_{\text{ML,BART}} = \frac{4\pi}{M+1}$ . It should be noted that the main lobe width given here is for a Bartlett window of length  $2M+1 = 2N-1$  as compared to that of a rectangular window of length  $N = M+1$ . The maximas of the DTFT  $\Psi_{\text{BART}}(e^{j\omega})$  of the Bartlett window occur at the same location as the DTFT  $\Psi_R(e^{j\omega})$  of the rectangular window. Since  $\Psi_{\text{BART}}(e^{j\omega}) = \left( \Psi_R(e^{j\omega}) \right)^2$ , it follows then  $A_{\text{sl,BART}} = 2 \times A_{\text{sl,R}} = 2 \times 13.3 = 26.6$  dB.

$$\begin{aligned}
7.54 \quad w_{GC}[n] &= \left[ \alpha + \beta \cos\left(\frac{2\pi n}{2M+1}\right) + \gamma \cos\left(\frac{4\pi n}{2M+1}\right) \right] w_R[n] \\
&= \left[ \alpha + 2\beta \left( e^{j\left(\frac{2\pi n}{2M+1}\right)} + e^{-j\left(\frac{2\pi n}{2M+1}\right)} \right) + 2\gamma \left( e^{j\left(\frac{4\pi n}{2M+1}\right)} + e^{-j\left(\frac{4\pi n}{2M+1}\right)} \right) \right] w_R[n]
\end{aligned}$$

$$\begin{aligned}
\text{Hence, } \Psi_{GC}(e^{j\omega}) &= \alpha \Psi_R(e^{j\omega}) + 2\beta \Psi_R \left( e^{j\left(\omega - \frac{2\pi}{2M+1}\right)} \right) + 2\beta \Psi_R \left( e^{j\left(\omega + \frac{2\pi}{2M+1}\right)} \right) \\
&\quad + 2\gamma \Psi_R \left( e^{j\left(\omega - \frac{4\pi}{2M+1}\right)} \right) + 2\gamma \Psi_R \left( e^{j\left(\omega + \frac{4\pi}{2M+1}\right)} \right).
\end{aligned}$$

For the Hann window :  $\alpha = 0.5$ ,  $\beta = 0.5$  and  $\gamma = 0$ . Hence

$$\begin{aligned}
\Psi_{HANN}(e^{j\omega}) &= 0.5 \Psi_R(e^{j\omega}) + \Psi_R \left( e^{j\left(\omega - \frac{2\pi}{2M+1}\right)} \right) + \Psi_R \left( e^{j\left(\omega + \frac{2\pi}{2M+1}\right)} \right) \\
&= 0.5 \frac{\sin\left(\frac{(2M+1)\omega}{2}\right)}{\sin(\omega/2)} + \frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)} + \frac{\sin\left((2M+1)\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)}.
\end{aligned}$$

For the Hamming window  $\alpha = 0.54$ ,  $\beta = 0.46$ , and  $\gamma = 0$ . Hence

$$\begin{aligned}
\Psi_{HAMMING}(e^{j\omega}) &= 0.54 \Psi_R(e^{j\omega}) + 0.92 \Psi_R \left( e^{j\left(\omega - \frac{2\pi}{2M+1}\right)} \right) + 0.92 \Psi_R \left( e^{j\left(\omega + \frac{2\pi}{2M+1}\right)} \right) \\
&= 0.54 \frac{\sin\left(\frac{(2M+1)\omega}{2}\right)}{\sin(\omega/2)} + 0.92 \frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)} + 0.92 \frac{\sin\left((2M+1)\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)}.
\end{aligned}$$

For the Blackmann window  $\alpha = 0.42$ ,  $\beta = 0.5$  and  $\gamma = 0.08$

$$\begin{aligned}
\Psi_{BLACK}(e^{j\omega}) &= 0.54 \Psi_R(e^{j\omega}) + \Psi_R \left( e^{j\left(\omega - \frac{2\pi}{2M+1}\right)} \right) + \Psi_R \left( e^{j\left(\omega + \frac{2\pi}{2M+1}\right)} \right) \\
&\quad + 0.16 \Psi_R \left( e^{j\left(\omega - \frac{4\pi}{2M+1}\right)} \right) + 0.16 \Psi_R \left( e^{j\left(\omega - \frac{4\pi}{2M+1}\right)} \right) \\
&= 0.42 \frac{\sin\left(\frac{(2M+1)\omega}{2}\right)}{\sin(\omega/2)} + \frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{2M+1}\right)} + \frac{\sin\left((2M+1)\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{2M+1}\right)}
\end{aligned}$$

$$+ 0.16 \frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{2\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{2\pi}{2M+1}\right)} + 0.16 \frac{\sin\left((2M+1)\left(\frac{\omega}{2} - \frac{2\pi}{2M+1}\right)\right)}{\sin\left(\frac{\omega}{2} - \frac{2\pi}{2M+1}\right)}.$$

**7.55 (a)**  $N = 1$  and hence,  $x_a(t) = \alpha_0 + \alpha_1 t$ . Without any loss of generality, for  $L = 5$ , we first fit the data set  $\{x[k]\}$ ,  $-5 \leq k \leq 5$ , by the polynomial  $x_a(t) = \alpha_0 + \alpha_1 t$  with a minimum mean-square error at  $t = -5, -4, \dots, 0, 1, \dots, 5$ , and then replace  $x[0]$  with a new value  $\bar{x}[0] = x_a(0) = \alpha_0$ .

Now, the mean-square error is given by  $e(\alpha_0, \alpha_1) = \sum_{k=-5}^5 \{x[k] - \alpha_0 - \alpha_1 k\}^2$ . We set

$$\frac{\partial \mathcal{E}(\alpha_0, \alpha_1)}{\partial \alpha_0} = 0 \quad \text{and} \quad \frac{\partial \mathcal{E}(\alpha_0, \alpha_1)}{\partial \alpha_1} = 0 \quad \text{which yields} \quad 11\alpha_0 + \alpha_1 \sum_{k=-5}^5 k = \sum_{k=-5}^5 x[k], \quad \text{and}$$

$$\alpha_0 \sum_{k=-5}^5 k + \alpha_1 \sum_{k=-5}^5 k^2 = \sum_{k=-5}^5 k x[k].$$

From the first equation we get  $\bar{x}[0] = \alpha_0 = \frac{1}{11} \sum_{k=-5}^5 x[k]$ . In the general case we thus have

$$\bar{x}[n] = \alpha_0 = \frac{1}{11} \sum_{k=-5}^5 x[n-k], \quad \text{which is a moving average filter of length 11.}$$

**(b)**  $N = 2$  and hence,  $x_a(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2$ . Here, we fit the data set  $\{x[k]\}$ ,  $-5 \leq k \leq 5$ , by the polynomial  $x_a(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2$  with a minimum mean-square error at  $t = -5, -4, \dots, 0, 1, \dots, 5$ , and then replace  $x[0]$  with a new value  $\bar{x}[0] = x_a(0) = \alpha_0$ . The mean-square

error is now given by  $\mathcal{E}(\alpha_0, \alpha_1, \alpha_2) = \sum_{k=-5}^5 (x[k] - \alpha_0 - \alpha_1 k - \alpha_2 k^2)^2$ . We set

$$\frac{\partial \mathcal{E}(\alpha_0, \alpha_1, \alpha_2)}{\partial \alpha_0} = 0, \quad \frac{\partial \mathcal{E}(\alpha_0, \alpha_1, \alpha_2)}{\partial \alpha_1} = 0, \quad \text{and} \quad \frac{\partial \mathcal{E}(\alpha_0, \alpha_1, \alpha_2)}{\partial \alpha_2} = 0, \quad \text{which yields}$$

$11\alpha_0 + 110\alpha_2 = \sum_{k=-5}^5 x[k]$ ,  $110\alpha_1 = \sum_{k=-5}^5 k x[k]$ ,  $110\alpha_0 + 1958\alpha_2 = \sum_{k=-5}^5 k^2 x[k]$ . From the first and the third equations we then get

$$\alpha_0 = \frac{1958 \sum_{k=-5}^5 x[k] - 110 \sum_{k=-5}^5 k^2 x[k]}{(1958 \times 11) - (110)^2} = \frac{1}{429} \sum_{k=-5}^5 (89 - 5k^2) x[k].$$

Hence, here we replace  $x[n]$  with a new value  $\bar{x}[n] = \alpha_0$  which is a weighted combination of the original data set  $\{x[k]\}$ ,  $-5 \leq k \leq 5$ :

$$\begin{aligned} \bar{x}[n] &= \frac{1}{429} \sum_{k=-5}^5 (89 - 5k^2) x[n-k] \\ &= \frac{1}{429} (-36 x[n+5] + 9 x[n+4] + 44 x[n+3] + 69 x[n+2] + 84 x[n+1] + 89 x[n]) \end{aligned}$$

$$+ 84x[n-1] + 69x[n-2] + 44x[n-3] + 9x[n-4] - 36x[n-5].$$

(c) The impulse response of the FIR filter of Part (a) is given by

$$h_1[n] = \frac{1}{11} \{1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1\},$$

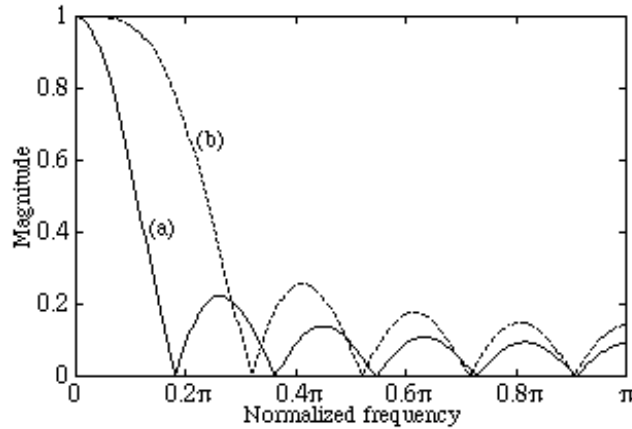
whereas, the impulse response of the FIR filter of Part (b) is given by

$$h_1[n] = \frac{1}{429} \{-36 \quad 9 \quad 44 \quad 69 \quad 84 \quad 89 \quad 84 \quad 69 \quad 44 \quad 9 \quad -36\}.$$

The corresponding frequency responses are given by

$$H_1(e^{j\omega}) = \frac{1}{11} \sum_{k=-5}^5 e^{-j\omega k}, \text{ and } H_2(e^{j\omega}) = \frac{1}{429} \sum_{k=-5}^5 (89 - 5k^2) e^{-j\omega k}.$$

A plot of the magnitude responses of these two filters are shown below from which it can be seen that the filter of Part (b) has a wider passband and thus provides smoothing over a larger frequency range than the filter of Part (a).

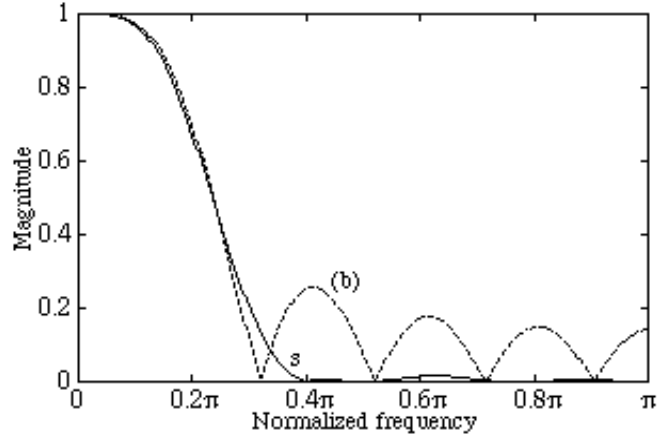


$$\begin{aligned} 7.56 \quad y[n] = & \frac{1}{320} \{-3x[n-7] - 6x[n-6] - 5x[n-5] + 3x[n-4] + 21x[n-3] + 46x[n-2] \\ & + 67x[n-1] + 74x[n] + 67x[n+1] + 46x[n+2] + 21x[n+3] \\ & + 3x[n+4] + 5x[n+5] - 6x[n+6] - 3x[n+7]\}. \end{aligned}$$

$$\begin{aligned} \text{Hence, } \frac{Y(e^{j\omega})}{X(e^{j\omega})} = & \frac{1}{320} \{-3e^{-j7\omega} - 6e^{-j6\omega} - 5e^{-j5\omega} + 3e^{-j4\omega} + 21e^{-j3\omega} + 46e^{-j2\omega} + 67e^{-j\omega} + 74 \\ & + 67e^{j\omega} + 46e^{j2\omega} + 21e^{j3\omega} + 3e^{j4\omega} - 5e^{j5\omega} - 6e^{j6\omega} - 3e^{j7\omega}\}. \end{aligned}$$

$$\begin{aligned} \text{Thus, } H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = & \frac{1}{160} \{74 + 67 \cos \omega + 46 \cos(2\omega) + 21 \cos(3\omega) + 3 \cos(4\omega) \\ & - 5 \cos(5\omega) - 6 \cos(6\omega) - 3 \cos(7\omega)\}. \end{aligned}$$

The magnitude response of the above FIR filter is plotted below (solid line marked 's') along with that of the FIR filter of Part (b) of Problem 7.25 (dashed line marked '(b)'). Note that both filters have roughly the same passband but the Spencer's filter has very large attenuation in the stopband and hence it provides better smoothing than the filter of Part (b).



**7.57** (a)  $L = 3$ .  $P(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$ . Now  $P(0) = 0$  is satisfied by the way  $P(x)$  has been defined. Also to ensure  $P(1) = 1$  we require  $\alpha_1 + \alpha_2 + \alpha_3 = 1$ . Choose  $m = 1$  and  $n = 1$ , Since  $\left. \frac{dP(x)}{dx} \right|_{x=0} = 0$ , hence  $\alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 \Big|_{x=0} = 0$ , implying  $\alpha_1 = 0$ . Also since  $\left. \frac{dP(x)}{dx} \right|_{x=1} = 0$ , hence  $\alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$ . Thus solving the three equations:  

$$\alpha_1 + \alpha_2 + \alpha_3 = 1, \quad \alpha_1 = 0, \quad \text{and} \quad \alpha_1 + 2\alpha_2 + 3\alpha_3 = 0$$
we arrive at  $\alpha_1 = 0$ ,  $\alpha_2 = 3$ , and  $\alpha_3 = -2$ . Therefore,  $P(x) = 3x^2 - 2x^3$ .

(b)  $L = 4$ . Hence,  $P(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4$ . Choose  $m = 2$  and  $n = 1$  (alternatively one can choose  $m = 1$  and  $n = 2$  for better stopband performance). Then,  
 $P(1) = 1 \Rightarrow \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$ ,

$$\left. \frac{dP(x)}{dx} \right|_{x=0} = 0 \Rightarrow \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2 + 4\alpha_4 x^3 \Big|_{x=0} = 0,$$

$$\left. \frac{d^2 P(x)}{dx^2} \right|_{x=0} = 0 \Rightarrow 2\alpha_2 + 6\alpha_3 x + 12\alpha_4 x^2 \Big|_{x=0} = 0,$$

$$\left. \frac{dP(x)}{dx} \right|_{x=1} = 0 \Rightarrow \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 = 0.$$

Solving the above simultaneous equations we get  $\alpha_1 = 0$ ,  $\alpha_2 = 0$ ,  $\alpha_3 = 4$ , and  $\alpha_4 = -3$ . Therefore,  $P(x) = 4x^3 - 3x^4$ .

(c)  $L = 5$ . Hence  $P(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5$ . Choose  $m = 2$  and  $n = 2$ . Following a procedure similar to that in parts (a) and (b) we get  $\alpha_1 = 0$ ,  $\alpha_2 = 0$ ,  $\alpha_3 = 10$ ,  $\alpha_4 = -15$ , and  $\alpha_5 = 6$ .



**7.58** From Eq. (7.102) we have  $\hat{H}(\omega) = \sum_{k=1}^M c[k] \sin(\omega k)$ . Now

$$\hat{H}(\pi - \omega) = \sum_{k=1}^M c[k] \sin((\omega - \pi)k) = - \sum_{k=1}^M c[k] \sin(\omega k) \cos(\pi k) = \sum_{k=1}^M c[k] (-1)^{k+1} \sin(\omega k).$$

Thus,  $\hat{H}(\omega) = \hat{H}(\pi - \omega)$  implies  $\sum_{k=1}^M c[k] \sin(\omega k) = \sum_{k=1}^M c[k] (-1)^{k+1} \sin(\omega k)$ , or equivalently,

$$\sum_{k=1}^M (1 - (-1)^{k+1}) c[k] \sin(\omega k) = 0 \text{ which in turn implies that } c[k] = 0 \text{ for } k = 2, 4, 6, \dots .$$

But from Eq. (7.103) we have  $c[k] = 2h[M - k]$ ,  $1 \leq k \leq M$ . Or,  $h[k] = \frac{1}{2} c[M - k]$ . For  $k$  even, i.e.,  $k = 2R$ ,  $h[2R] = \frac{1}{2} c[M - 2R] = 0$  if  $M$  is even.

**7.59 (a)**  $H[k] = H(e^{j\omega_k}) = H(e^{j2\pi k/M})$ ,  $0 \leq k \leq M - 1$ . Thus,  $h[n] = \frac{1}{M} \sum_{k=0}^{M-1} H[k] W_M^{-kn}$ ,

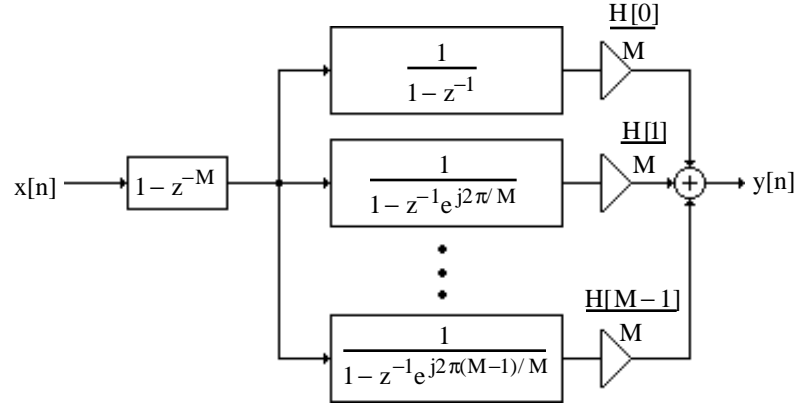
where  $W_M = e^{-j\pi/M}$ .

$$\text{Now, } H(z) = \sum_{n=0}^{M-1} h[n] z^{-n} = \frac{1}{M} \sum_{n=0}^{M-1} \sum_{k=0}^{M-1} H[k] W_M^{-kn} z^{-n} = \frac{1}{M} \sum_{k=0}^{M-1} H[k] \left( \sum_{n=0}^{M-1} W_M^{-kn} z^{-n} \right).$$

$$\begin{aligned} \text{We can write } \sum_{n=0}^{M-1} W_M^{-kn} z^{-n} &= \sum_{n=0}^{\infty} W_M^{-kn} z^{-n} - \sum_{n=M}^{\infty} W_M^{-kn} z^{-n} \\ &= \sum_{n=0}^{\infty} W_M^{-kn} z^{-n} - W_M^{-kM} z^{-M} \sum_{n=0}^{\infty} W_M^{-kn} z^{-n} = (1 - z^{-M}) \sum_{n=0}^{\infty} W_M^{-kn} z^{-n} = \frac{1 - z^{-M}}{1 - W_M^{-k} z^{-1}}. \end{aligned}$$

$$\text{Therefore, } H(z) = \frac{1 - z^{-M}}{M} \sum_{k=0}^{M-1} \frac{H[k]}{1 - W_M^{-k} z^{-1}}.$$

**(b)**



(c) On the unit circle

**7.60 (a)** For Type 1 FIR filter,  $H(e^{j\omega}) = e^{-j\frac{\omega(M-1)}{2}} |H(e^{j\omega})|$ . Since in frequency sampling approach we sample the DTFT  $H(e^{j\omega})$  at  $M$  points given by  $\omega = \frac{2\pi k}{M}$ ,  $k = 0, 1, \dots, M-1$ , therefore  $H[k] = H(e^{j2\pi k/M}) = |H_d(e^{j2\pi k/M})| e^{-j2\pi k(M-1)/2M}$ ,  $k = 0, 1, \dots, M-1$ . Since the filter is of Type 1,  $(M-1)$  is even, thus  $e^{j2\pi(M-1)/2} = 1$ . Moreover,  $h[n]$  being real,  $H(e^{j\omega}) = H^*(e^{j\omega})$ . Thus,  $H(e^{j\omega}) = e^{j\omega(M-1)/2} |H(e^{j\omega})|$ ,  $\pi \leq \omega < 2\pi$ . Hence,

$$H[k] = \begin{cases} |H_d(e^{j2\pi k/M})| e^{-j2\pi k(M-1)/2M}, & k = 0, 1, 2, \dots, \frac{M-1}{2}, \\ |H_d(e^{j2\pi k/M})| e^{j2\pi(M-k)(M-1)/2M}, & k = \frac{M+1}{2}, \frac{M+3}{2}, \dots, M-1. \end{cases}$$

(b) For Type 2 FIR filter

$$H[k] = \begin{cases} |H_d(e^{j2\pi k/M})| e^{-j2\pi k(M-1)/2M}, & k = 0, 1, 2, \dots, \frac{M}{2} - 1, \\ 0, & k = \frac{M}{2}, \\ |H_d(e^{j2\pi k/M})| e^{j2\pi k(M-k)(M-1)/2M}, & k = \frac{M}{2} + 1, \dots, M-1. \end{cases}$$

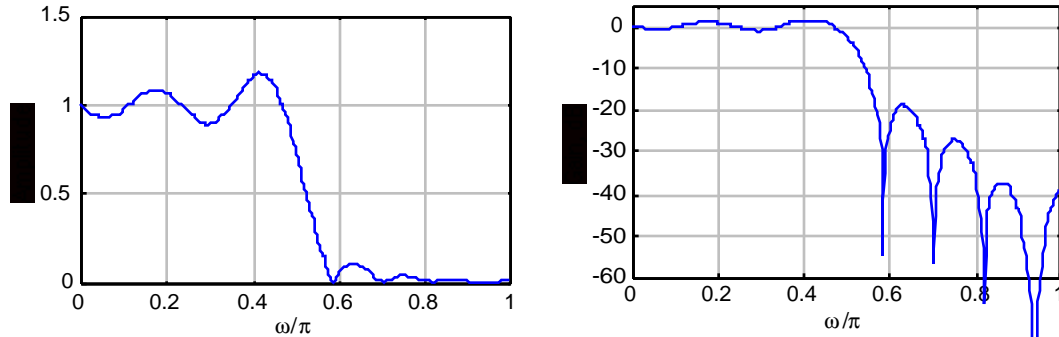
**7.61 (a)** The frequency spacing between two consecutive samples is given by  $\frac{2\pi}{17} = 0.11765\pi$ , and hence the desired passband edge  $\omega_p = 0.5\pi$  is between the frequency samples at  $\omega = \frac{2\pi \times 4}{17} = 0.47059\pi$  and  $\omega = \frac{2\pi \times 5}{17} = 0.58824\pi$ . From Eq. (7.174) the specified DFT samples are thus given by

$$H[k] = \begin{cases} e^{-j(2\pi/17)8}, & k = 0, 1, \dots, 4, 12, \dots, 16, \\ 0, & k = 5, \dots, 11. \end{cases}$$

(b) A 17-point inverse DFT of the above DFT samples yields the impulse response coefficients given below in ascending powers of  $z^{-1}$ :

Columns 1 through 5

4.3498e-03    6.0460e-03    -2.9166e-02    8.1624e-03    5.7762e-02  
 Columns 6 through 10  
 -6.4003e-02    -7.9934e-02    3.0267e-01    5.8824e-01    3.0267e-01  
 Columns 11 through 15  
 -7.9934e-02    -6.4003e-02    5.7762e-02    8.1624e-03    -2.9166e-02  
 Columns 16 through 17  
 6.0460e-03    4.3498e-03



**7.62 (a)**  $\omega_p = 0.3\pi$ . For  $M = 37$ , the frequency spacing between two consecutive frequency samples is  $\frac{2\pi}{37} = 0.054054\pi$ . Hence, the desired passband edge of the lowpass filter is between the frequency sample at  $\omega = \frac{2\pi \times 5}{37} = 0.27027\pi$  and  $\omega = \frac{2\pi \times 6}{37} = 0.3243243\pi$ .  $N = 37$ , and

$$|H_d(e^{j\omega})| = \begin{cases} 1, & 0 < \omega \leq 0.3\pi, \\ 0, & 0.3\pi < \omega \leq 1.7\pi, \\ 1, & 1.7\pi < \omega \leq 2\pi. \end{cases}$$

$$\text{Therefore, } H[k] = \begin{cases} e^{-j(2\pi k/37)18}, & 0 \leq k < 6, \\ 0, & 6 \leq k < 32, \\ e^{j(2\pi(37-k)/37)18}, & 32 \leq k \leq 36. \end{cases}$$

$$\text{Hence, } H[k] = \begin{cases} e^{-j(2\pi k/37)18}, & k = 0, 1, \mathbf{K}, 5, 32, 33, \mathbf{K}, 36, \\ 0, & k = 6, 7, \mathbf{K}, 31. \end{cases}$$

**(b)** A 37-point inverse DFT of the above DFT samples yields the impulse response coefficients given below in ascending powers of  $z^{-1}$ :

Columns 1 through 5  
 -2.4155e-02    -4.6049e-03    1.9130e-02    2.8037e-02    1.4205e-02  
 Columns 6 through 10  
 -1.2467e-02    -3.1027e-02    -2.5234e-02    3.0533e-03    3.3261e-02  
 Columns 11 through 15  
 3.9919e-02    1.2162e-02    -3.4815e-02    -6.5556e-02    -4.5430e-02

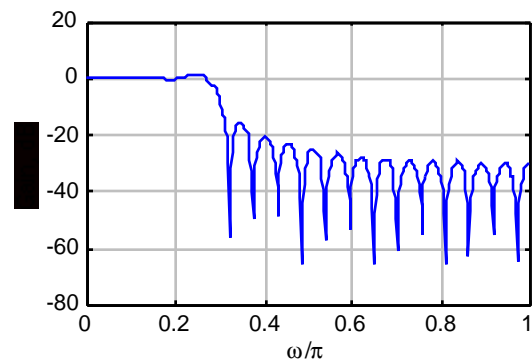
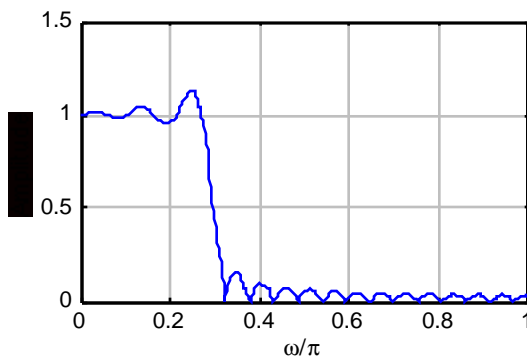
Columns 16 through 20  
 3.5732e-02 1.5291e-01 2.5623e-01 2.9730e-01 2.5623e-01

Columns 21 through 25  
 1.5291e-01 3.5732e-02 -4.5430e-02 -6.5556e-02 -3.4815e-02

Columns 26 through 30  
 1.2162e-02 3.9919e-02 3.3261e-02 3.0533e-03 -2.5234e-02

Columns 31 through 35  
 -3.1027e-02 -1.2467e-02 1.4205e-02 2.8037e-02 1.9130e-02

Columns 36 through 37  
 -4.6049e-03 -2.4155e-02



**7.63** By expressing  $\cos(\omega n) = T_n(\cos \omega)$ , where  $T_n(x)$  is the  $n$ -th order Chebyshev polynomial in  $x$ , we first rewrite Eq. (7.96) in the form:

$$H(\omega) = \sum_{n=0}^M a[n] \cos(\omega n) = \sum_{n=0}^M \alpha_n \cos^n(\omega).$$

Therefore, we can rewrite Eq. (7.118) repeated below for convenience

$$P(\omega_i) [H(\omega_i) - D(\omega_i)] = (-1)^i \varepsilon, \quad 1 \leq i \leq M+2,$$

in a matrix form as

$$\begin{bmatrix} 1 & \cos(\omega_1) & \mathbf{L} & \cos^M(\omega_1) & 1/P(\omega_1) \\ 1 & \cos(\omega_2) & \mathbf{L} & \cos^M(\omega_2) & -1/P(\omega_2) \\ \mathbb{M} & \mathbb{M} & \mathbf{O} & \mathbb{M} & \mathbb{M} \\ 1 & \cos(\omega_{M+1}) & \mathbf{L} & \cos^M(\omega_{M+1}) & (-1)^M/P(\omega_{M+1}) \\ 1 & \cos(\omega_{M+2}) & \mathbf{L} & \cos^M(\omega_{M+2}) & (-1)^{M+1}/P(\omega_{M+2}) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \mathbb{M} \\ \alpha_M \\ \varepsilon \end{bmatrix} = \begin{bmatrix} D(\omega_1) \\ D(\omega_2) \\ \mathbb{M} \\ D(\omega_{M+1}) \\ D(\omega_{M+2}) \end{bmatrix}.$$

Note that the coefficients  $\{\alpha_i\}$  are different from the coefficients  $\{a[i]\}$  of Eq. (7.96). To

determine the expression of  $\varepsilon$  we use Cramer's rule arriving at  $\varepsilon = \frac{\Delta_\varepsilon}{\Delta}$ , where

$$\Delta = \det \begin{bmatrix} 1 & \cos(\omega_1) & \mathbf{L} & \cos^M(\omega_1) & 1/P(\omega_1) \\ 1 & \cos(\omega_2) & \mathbf{L} & \cos^M(\omega_2) & -1/P(\omega_2) \\ \mathbb{M} & \mathbb{M} & \mathbf{O} & \mathbb{M} & \mathbb{M} \\ 1 & \cos(\omega_{M+1}) & \mathbf{L} & \cos^M(\omega_{M+1}) & (-1)^M/P(\omega_{M+1}) \\ 1 & \cos(\omega_{M+2}) & \mathbf{L} & \cos^M(\omega_{M+2}) & (-1)^{M+1}/P(\omega_{M+2}) \end{bmatrix}, \text{ and}$$

$$\Delta_\varepsilon = \det \begin{bmatrix} 1 & \cos(\omega_1) & \mathbf{L} & \cos^M(\omega_1) & D(\omega_1) \\ 1 & \cos(\omega_2) & \mathbf{L} & \cos^M(\omega_2) & D(\omega_2) \\ \mathbb{M} & \mathbb{M} & \mathbf{O} & \mathbb{M} & \mathbb{M} \\ 1 & \cos(\omega_{M+1}) & \mathbf{L} & \cos^M(\omega_{M+1}) & D(\omega_{M+1}) \\ 1 & \cos(\omega_{M+2}) & \mathbf{L} & \cos^M(\omega_{M+2}) & D(\omega_{M+2}) \end{bmatrix}.$$

Expanding both determinants using the last column we get  $\Delta_\varepsilon = \sum_{i=1}^{M+2} b_i D(\omega_{i+1})$ , and

$$\Delta = \sum_{i=1}^{M+2} b_i \frac{(-1)^{i-1}}{P(\omega_i)}, \text{ where}$$

$$b_i = \det \begin{bmatrix} 1 & \cos(\omega_1) & \cos^2(\omega_1) & \mathbf{L} & \cos^M(\omega_1) \\ 1 & \cos(\omega_2) & \cos^2(\omega_2) & \mathbf{L} & \cos^M(\omega_2) \\ \mathbb{M} & \mathbb{M} & \mathbb{M} & \mathbf{O} & \mathbb{M} \\ 1 & \cos(\omega_{i-1}) & \cos^2(\omega_{i-1}) & \mathbf{L} & \cos^M(\omega_{i-1}) \\ 1 & \cos(\omega_{i+1}) & \cos^2(\omega_{i+1}) & \mathbf{L} & \cos^M(\omega_{i+1}) \\ \mathbb{M} & \mathbb{M} & \mathbb{M} & \mathbf{O} & \mathbb{M} \\ 1 & \cos(\omega_{M+2}) & \cos^2(\omega_{M+2}) & \mathbf{L} & \cos^M(\omega_{M+2}) \end{bmatrix}.$$

The above matrix is seen to be a Vandermonde matrix and its determinant is given by

$$b_i = \prod_{\substack{k \neq 1, k > 1 \\ k, l \neq i}} (\cos \omega_k - \cos \omega_l).$$

Define  $c_i = \frac{b_i}{M+2}$ . It can be shown by induction that  $c_i = \prod_{\substack{n=1 \\ n \neq i}}^{M+2} \frac{1}{\cos \omega_i - \cos \omega_n}$ . Therefore,

$$\varepsilon = \frac{\prod_{\substack{r=1 \\ r \neq i}}^{M+2} b_r}{\sum_{i=1}^{M+2} b_i D(\omega_i)} = \frac{c_1 D(\omega_1) + c_2 D(\omega_2) + \mathbf{L} + c_{M+2} D(\omega_{M+2})}{\sum_{i=1}^{M+2} b_i \frac{(-1)^i}{P(\omega_i)}} = \frac{c_1 D(\omega_1) + c_2 D(\omega_2) + \mathbf{L} + c_{M+2} D(\omega_{M+2})}{\frac{c_1}{P(\omega_1)} - \frac{c_2}{P(\omega_2)} + \mathbf{L} + \frac{c_{M+2} (-1)^{M+1}}{P(\omega_{M+2})}}.$$

**7.64** It follows from Eq. (7.67) that the impulse response of an ideal Hilbert transformer is an antisymmetric sequence. If we truncate it to a finite number of terms between  $-M \leq n \leq M$  the impulse response is of length  $(2M + 1)$  which is odd. Hence the FIR Hilbert transformer obtained by truncation and satisfying Eq. (7.143) cannot be satisfied by a Type 4 FIR filter.

$$7.65 \quad (a) \quad X(z) = \sum_{n=0}^{N-1} x[n]z^{-n}. \quad X(\zeta) = X(z) \Big|_{z^{-1} = \frac{-\alpha + \zeta^{-1}}{1 - \alpha \zeta^{-1}}} = \sum_{n=0}^{N-1} x[n] \left( \frac{-\alpha + \zeta^{-1}}{1 - \alpha \zeta^{-1}} \right)^n = \frac{P(\zeta)}{D(\zeta)},$$

where  $P(\zeta) = \sum_{n=0}^{N-1} p[n]\zeta^{-1} = \sum_{n=0}^{N-1} x[n](1 - \alpha \zeta^{-1})^{N-1-n} (-\alpha + \zeta^{-1})^n$ , and

$$D(\zeta) = \sum_{n=0}^{N-1} d[n]\zeta^{-1} = (1 - \alpha \zeta^{-1})^{N-1}.$$

$$(b) \quad \hat{X}[k] = X(\zeta) \Big|_{\zeta = e^{j2\pi k/N}} = \frac{P(\zeta)}{D(\zeta)} \Big|_{\zeta = e^{j2\pi k/N}} = \frac{\hat{P}[k]}{\hat{D}[k]}, \text{ where } \hat{P}[k] = P(\zeta) \Big|_{\zeta = e^{j2\pi k/N}} \text{ is the}$$

$N$ -point DFT of the sequence  $p[n]$  and  $\hat{D}[k] = D(\zeta) \Big|_{\zeta = e^{j2\pi k/N}}$  is the  $N$ -point DFT of the sequence  $d[n]$ .

(c) Let  $\mathbf{P} = [p[0] \quad p[1] \quad \mathbf{K} \quad p[N-1]]^T$  and  $\mathbf{X} = [x[0] \quad x[1] \quad \mathbf{K} \quad x[N-1]]^T$ . Without any loss of generality, assume  $N = 4$  in which case

$$P(\zeta) = \sum_{n=0}^3 p[n]\zeta^{-n} = (x[0] - \alpha x[1] + \alpha^2 x[2] - \alpha^3 x[3]) \\ + (-3\alpha x[0] + (1 + 2\alpha^2)x[1] - \alpha(2 + \alpha^2)x[2] + 3\alpha^2 x[3])\zeta^{-1} \\ + (3\alpha^2 x[0] - \alpha(2 + \alpha^2)x[1] + (1 + 2\alpha^2)x[2] - 3\alpha x[3])\zeta^{-2} \\ + (-\alpha^3 x[0] + \alpha^2 x[1] - \alpha x[2] + \alpha x[3])\zeta^{-3}. \text{ Equating like powers of } \zeta^{-1} \text{ we can write}$$

$$\mathbf{P} = \mathbf{Q} \cdot \mathbf{X} \text{ or } \begin{bmatrix} p[0] \\ p[1] \\ p[2] \\ p[3] \end{bmatrix} = \begin{bmatrix} 1 & -\alpha & \alpha^2 & -\alpha^3 \\ -3\alpha & 1 + 2\alpha^2 & -\alpha(2 + \alpha^2) & 3\alpha^2 \\ 3\alpha^2 & -\alpha(2 + \alpha^2) & 1 + 2\alpha^2 & -3\alpha \\ -\alpha^3 & \alpha^2 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}. \text{ It can be seen that the}$$

elements  $q_{r,s}$ ,  $0 \leq r, s \leq 3$ , of the  $4 \times 4$  matrix  $\mathbf{Q}$  can be determined as follows:

(i) The first row is given by  $q_{0,s} = (-\alpha)^s$ ,

(ii) The first column is given by  $q_{r,0} = {}^3C_r (-\alpha)^r = \frac{3!}{r!(3-r)!} (-\alpha)^r$ , and

(iii) the remaining elements can be obtained using the recurrence relation

$$q_{r,s} = q_{r-1,s-1} - \alpha q_{r,s-1} + \alpha q_{r-1,s}.$$

In the general case, we only change the computation of the elements of the first column using

$$\text{the relation } q_{r,0} = {}^{N-1}C_r (-\alpha)^r = \frac{(N-1)!}{r!(N-1-r)!} (-\alpha)^r,$$

**M7.1** Using the M-file `buttap` we first design a 4-th order analog Bessel filter whose transfer function is given by  $H(z) = \frac{1}{s^4 + 3.1239s^3 + 4.3916s^2 + 3.2011s + 1}$ . A partial-fraction expansion of  $H(z)$  using the M-file `residue` yields  $H(z) = -\frac{1.0393s + 1.8470}{s^2 + 1.3144s + 1.1211} + \frac{1.0393s + 2.3615}{s^2 + 1.8095s + 8.9199}$  where we have used the M-file `residue` again to combine the complex conjugate pole-pairs.

INCOMPLETE

**M7.2** Given specifications:  $F_p = 4$  kHz,  $F_s = 20$  kHz,  $F_T = 80$  kHz,  $\alpha_p = 0.5$  dB and  $\alpha_s = 45$  dB. Using Eqs. (7.7) and (7.8) we obtain the normalized bandedges as  $\omega_p = 0.1\pi = 0.31416$  and  $\omega_s = 0.5\pi = 1.5708$ . Let  $T = 2$ . From Eq. (7.26), the bandedges of the analog prototype are  $\Omega_p = \tan(\omega_p/2) = 0.15838$  and  $\Omega_s = \tan(\omega_s/2) = 1$ . From Eq. (5.29) the inverse transition ratio is  $\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 6.31375$ .

$20 \log_{10} \left( \frac{1}{\sqrt{1 + \epsilon^2}} \right) = -0.5$  which yields  $\epsilon^2 = 0.1220184$ . Similarly,  $20 \log_{10} \left( \frac{1}{A} \right) = -45$  which

yields  $A^2 = 31622.7766$ . From Eq. (5.5.30) the inverse discrimination ratio is given by

$\frac{1}{k_1} = \sqrt{\frac{A^2 - 1}{\epsilon^2}} = \sqrt{\frac{31621.7766}{0.1220184}} = 509.073363$ . Hence, from Eq. (5.33), the order of the lowpass

Butterworth filter is given by  $N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = \frac{\log_{10}(509.073363)}{\log_{10}(6.3137515)} = 3.38226$ . We choose  $N = 4$

as the order. Next we determine the 3-dB cutoff frequency  $\Omega_c$  by meeting exactly the

stopband edge specifications. From Eq. (5.??a), we have  $(\Omega_s / \Omega_c)^8 = A^2 - 1 = 31621.7766$  or

$$\Omega_c = \frac{\Omega_s}{(A^2 - 1)^{1/8}} = 0.273843 \Omega_s = 0.273843.$$

Using `buttap` we determine the normalized analog Butterworth transfer function of 4-th order with a 3-dB cutoff frequency at  $\Omega_c = 1$  which is

$H_{an}(s) = \frac{1}{(s^2 + 0.76536686s + 1)(s^2 + 1.847759s + 1)}$ . We denormalize  $H_{an}(s)$  to move  $\Omega_c$  to

0.273843 leading to  $H_a(s) = H_{an} \left( \frac{s}{0.273843} \right)$

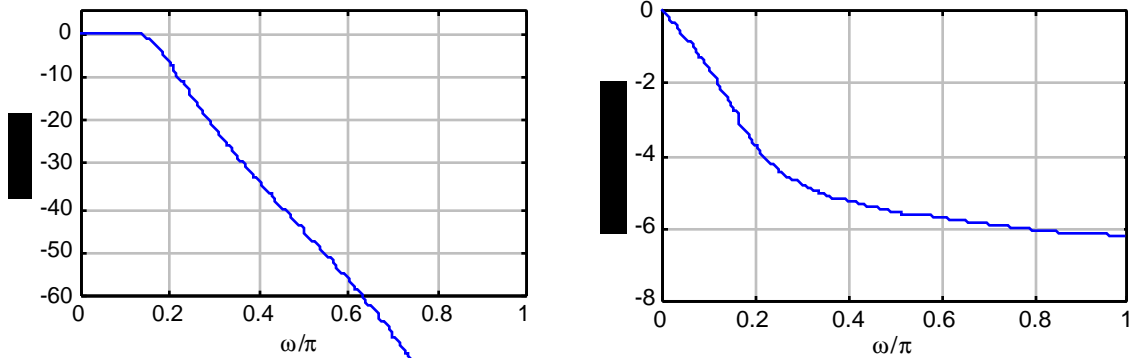
$$= \frac{1}{\left[ \left( \frac{s}{0.273843} \right)^2 + 0.76536686 \left( \frac{s}{0.273843} \right) + 1 \right] \left[ \left( \frac{s}{0.273843} \right)^2 + 1.847759 \left( \frac{s}{0.273843} \right) + 1 \right]}$$

$$= \frac{0.005625}{(s^2 + 0.2096s + 0.075)(s^2 + 0.5059958678s + 0.075)}$$

Applying bilinear transformation to  $H_a(s)$  using the M-file `bilinear` we finally arrive at

$$G(z) = \frac{0.0027696(1+z^{-1})^4}{(1-1.440137z^{-1}+0.67367274z^{-2})(s^2-1.1701485365z^{-1}+0.3599z^{-2})}$$

Plots of the gain and phase responses of this transfer function are shown below.



**M7.3** The modified Program P7\_3 is given below:

```
format short e
Fp = input('Passband edge frequency in Hz = ');
Fs = input('Stopband edge frequency in Hz = ');
FT = input('Sampling frequency in Hz = ');
Rp = input('Passband ripple in dB = ');
Rs = input('Stopband minimum attenuation in dB = ');
Wp = 2*Fp/FT;Ws = 2*Fs/FT;
[N, Wn] = buttord(Wp, Ws, Rp, Rs)
[b,a] = butter(N,Wn);
disp('Numerator polynomial'); disp(b)
disp('Denominator polynomial'); disp(a)
[h,w] = freqz(b,a,512);
plot(w/pi,20*log10(abs(h)));grid;axis([0 1 -60 5]);
xlabel('\omega/\pi');ylabel('Phase, radians')
Pause
plot (w/pi,unwrap(angle(h)));grid;axis([0 1 -8 1]);
xlabel('\omega/\pi');ylabel('Phase, radians')
```

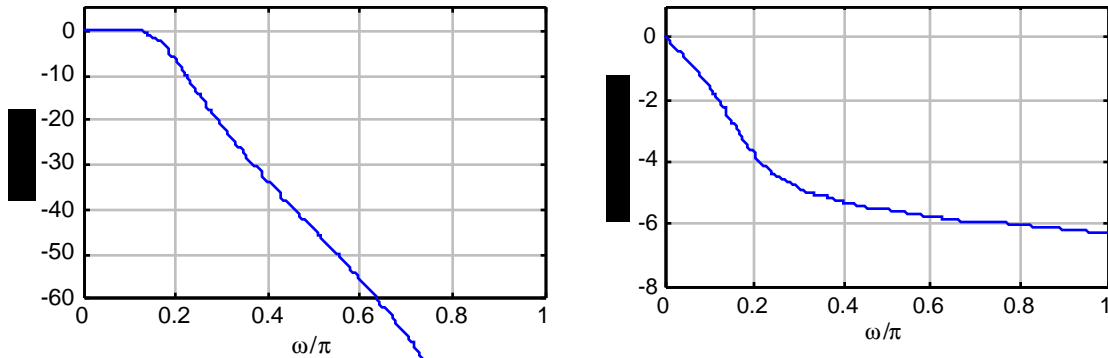
The numerator and the denominator coefficients of the lowpass transfer function obtained are given by

```
Numerator polynomial
  2.7690e-03  1.1076e-02  1.6614e-02  1.1076e-02
 2.7690e-03

Denominator polynomial
 1.0000e+00 -2.6103e+00  2.7188e+00 -1.3066e+00
 2.4246e-01
```

Plots of the gain and phase responses by running this program are shown below.





**M7.4** Given specifications:  $\omega_p = 0.1\pi = 0.31416$   $\omega_s = 0.5\pi = 1.5708$ .

$$20 \log_{10} |G(e^{j0.1\pi})| \geq -0.5, \text{ and } 20 \log_{10} |G(e^{j0.5\pi})| \leq -45.$$

Impulse Invariance Method: Let  $T = 1$ . Assume no aliasing. Then the specifications of  $H_a(s)$  is same as that of  $G(z)$ , i.e.

$$\Omega_p = 0.1\pi, \Omega_s = 0.5\pi, 20 \log_{10} |H_a(j0.1\pi)| \geq -0.5, \text{ and } 20 \log_{10} |H_a(j0.5\pi)| \leq -45. \text{ Now,}$$

$$20 \log_{10} \left( \frac{1}{\sqrt{1 + \epsilon^2}} \right) = -0.5 \text{ which yields } \epsilon^2 = 0.12202. \text{ Similarly, } 20 \log_{10} \left( \frac{1}{A} \right) = -45$$

which yields  $A^2 = 31622.7766$ . From Eq. (5.30) the inverse discrimination ratio is given by

$$\frac{1}{k_1} = \sqrt{\frac{A^2 - 1}{\epsilon^2}} = \sqrt{\frac{31621.7766}{0.12201845}} = 509.073363. \text{ From Eq. (5.29) the inverse transition ratio is}$$

given by  $\frac{1}{k} = \frac{0.5\pi}{0.1\pi} = 5$ . Hence, from Eq. (5.33), the order of the lowpass Butterworth filter is

$$\text{given by } N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = \frac{\log_{10}(509.073363)}{\log_{10}(5)} = 3.8725. \text{ We choose } N = 4 \text{ as the order. Next}$$

we determine the 3-dB cutoff frequency  $\Omega_c$  by meeting exactly the stopband edge

specifications. From Eq. (5.32b), we have  $(\Omega_s / \Omega_c)^8 = A^2 - 1 = 31621.7766$  or

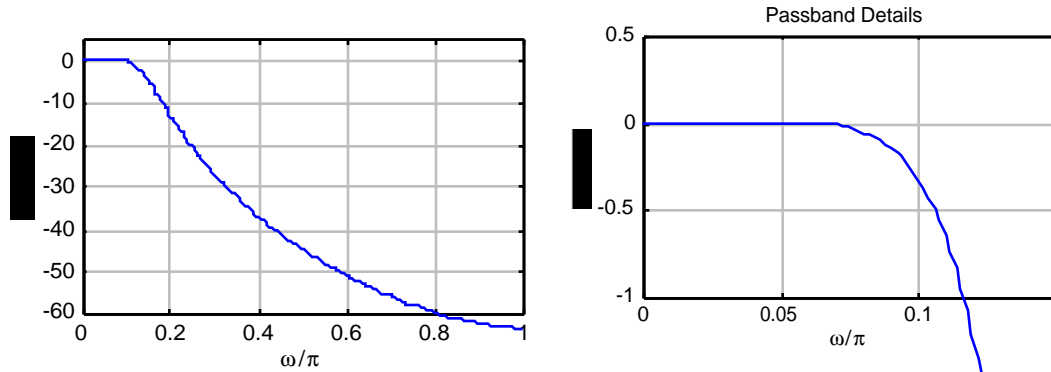
$$\Omega_c = \frac{\Omega_s}{(A^2 - 1)^{1/8}} = 0.273843 \Omega_s = 0.273843 \times 0.5\pi = 0.43015.$$

Using the statement `[B,A] = butter(4,0.43105, 's')` we determine the analog Butterworth transfer function  $H_a(s)$  of 4-th order with a 3-dB cutoff frequency at

$\Omega_c = 0.43105$ .  $H_a(s)$  is then transformed into a digital transfer function using the statement `[num,den]=impinvar(B,A,1)`; which yields

$$G(z) = \frac{0.0043063z^{-1} + 0.012898z^{-2} + 0.0024532z^{-4}}{1 - 2.8879z^{-1} + 3.2407z^{-2} - 1.6573z^{-3} + 0.3242z^{-4}}. \text{ A plot of its gain response is shown}$$

below. It can be seen that the filter meets the given specifications



**M7.5** Given specifications:  $\omega_p = 0.1\pi = 0.31416$   $\omega_s = 0.5\pi = 1.5708$ .

$$20 \log_{10} |G(e^{j0.1\pi})| \geq -0.5, \text{ and } 20 \log_{10} |G(e^{j0.5\pi})| \leq -45.$$

Impulse Invariance Method: Let  $T = 1$ . Assume no aliasing. Then the specifications of  $H_a(s)$  is same as that of  $G(z)$ , i.e.

$$\Omega_p = 0.1\pi, \Omega_s = 0.5\pi, 20 \log_{10} |H_a(j0.1\pi)| \geq -0.5, \text{ and } 20 \log_{10} |H_a(j0.5\pi)| \leq -45. \text{ Now,}$$

$$20 \log_{10} \left( \frac{1}{\sqrt{1 + \epsilon^2}} \right) = -0.5 \text{ which yields } \epsilon^2 = 0.12202. \text{ Similarly, } 20 \log_{10} \left( \frac{1}{A} \right) = -45$$

which yields  $A^2 = 31622.7766$ . From Eq. (5.30) the inverse discrimination ratio is given by

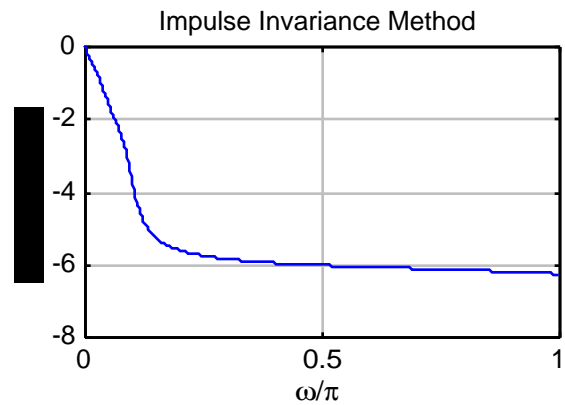
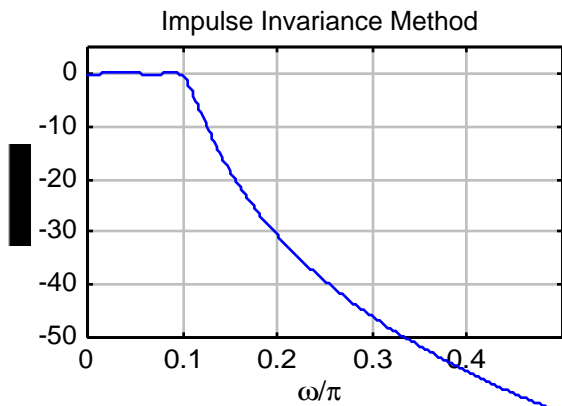
$$\frac{1}{k_1} = \sqrt{\frac{A^2 - 1}{\epsilon^2}} = \sqrt{\frac{31621.7766}{0.12201845}} = 509.073363. \text{ From Eq. (5.29) the inverse transition ratio is}$$

given by  $\frac{1}{k} = \frac{0.5\pi}{0.1\pi} = 5$ . Hence, from Eq. (5.41), the order of the lowpass Type 1 Chebyshev

filter is given by  $N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = \frac{\cosh^{-1}(509.073363)}{\cosh^{-1}(5)} = 3.0211$ . Hence we choose  $N = 4$ .

The MATLAB code fragments used are:

```
[z,p,k] = cheblap(4,0.5);
[B,A] = zp2tf(z,p,k);
[BT,AT] = lp2lp(B,A,0.1*pi);
[num,den] =impinvar(BT,AT,1);
```



**Bilinear Transformation Method:** Let  $T = 2$ . From Eq. (7.26), the bandedges of the analog prototype are  $\Omega_p = \tan(\omega_p/2) = 0.15838$  and  $\Omega_s = \tan(\omega_s/2) = 1$ . From Eq. (5.29) the

inverse transition ratio  $\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 6.31375$ .

$20 \log_{10} \left( \frac{1}{\sqrt{1+\epsilon^2}} \right) = -0.5$  which yields  $\epsilon^2 = 0.1220184$ . Similarly,  $20 \log_{10} \left( \frac{1}{A} \right) = -45$  which

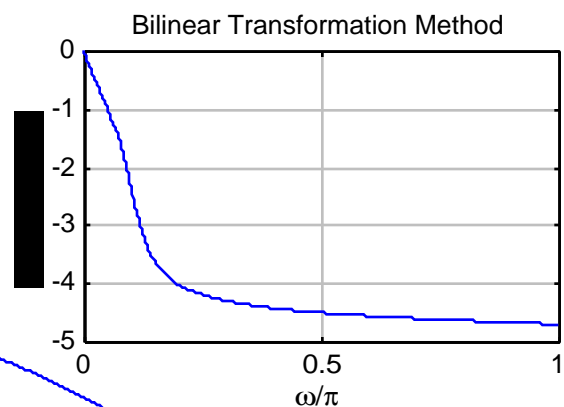
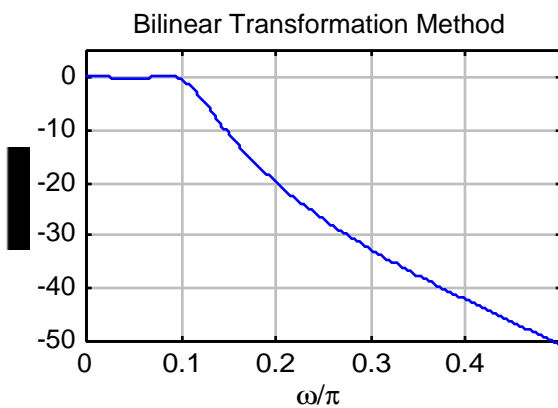
yields  $A^2 = 31622.7766$ . From Eq. (5.5.30) the inverse discrimination ratio is given by

$\frac{1}{k_1} = \sqrt{\frac{A^2 - 1}{\epsilon^2}} = \sqrt{\frac{31621.7766}{0.1220184}} = 509.073363$ . Hence, from Eq. (5.41), the order of the lowpass

Type 1 Chebyshev filter is given by  $N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = \frac{\cosh^{-1}(509.073363)}{\cosh^{-1}(6.31375)} = 2.7379$ .

Hence we choose  $N = 3$ . The MATLAB code fragments used are

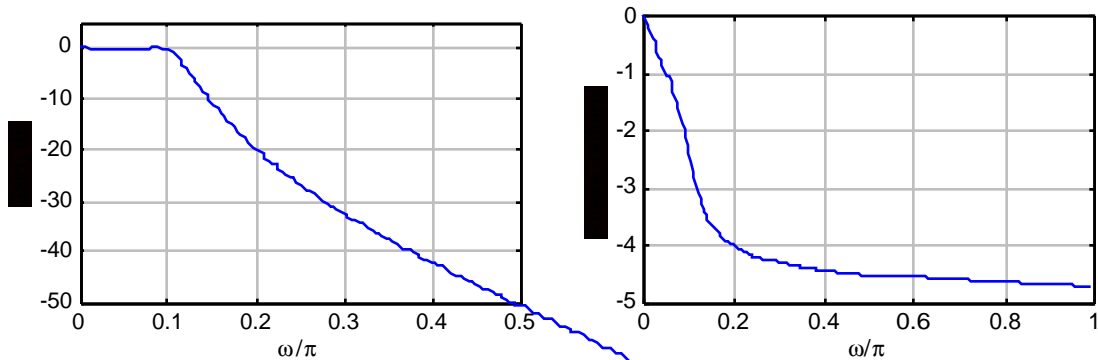
```
[z,p,k] = cheblap(3,0.5);
[B,A] = zp2tf(z,p,k);
[BT,AT] = lp2lp(B,A,0.15838);
[num,den] = bilinear(BT,AT,0.5);
```



Note: Both designs meet the specifications. However, the bilinear transformation method meets with a filter of lower order.

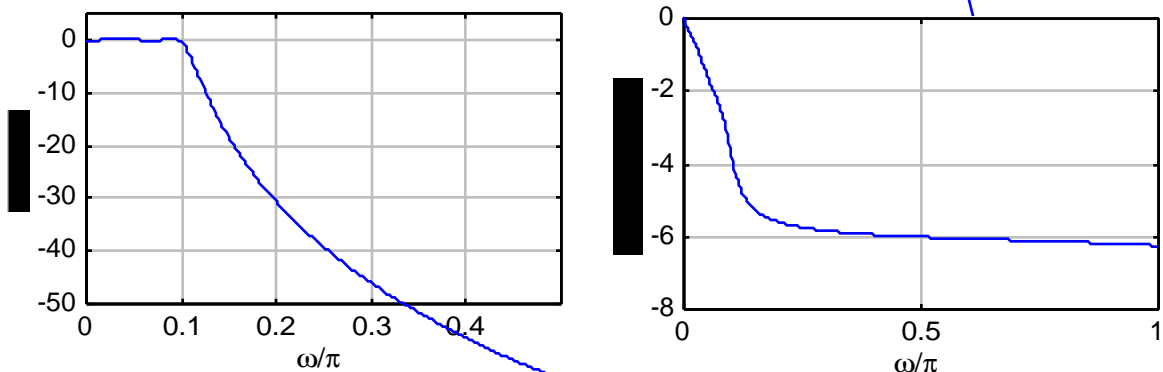
**M7.6** The only change to Program 7\_2 is given by the statement

```
[b,a] = cheby1(N,Rp,Wn);
```



**M7.7** The code fragments used are

```
Wp = input('Passband edge frequency = ');
Ws = input('Stopband edge frequency = ');
Rp = input('Passband ripple in dB = ');
Rs = input('Stopband minimum attenuation in dB = ');
[N, Wn] = cheblord(0.1*pi, 0.5*pi, 0.5, 45, 's');
[B, A] = cheby1(N, 0.5, Wn, 's');
[num, den] =impinvar(B,A,1);
```



**M7.8** Impulse Invariance Method: From Exercise M7.5 we have the inverse selectivity parameter

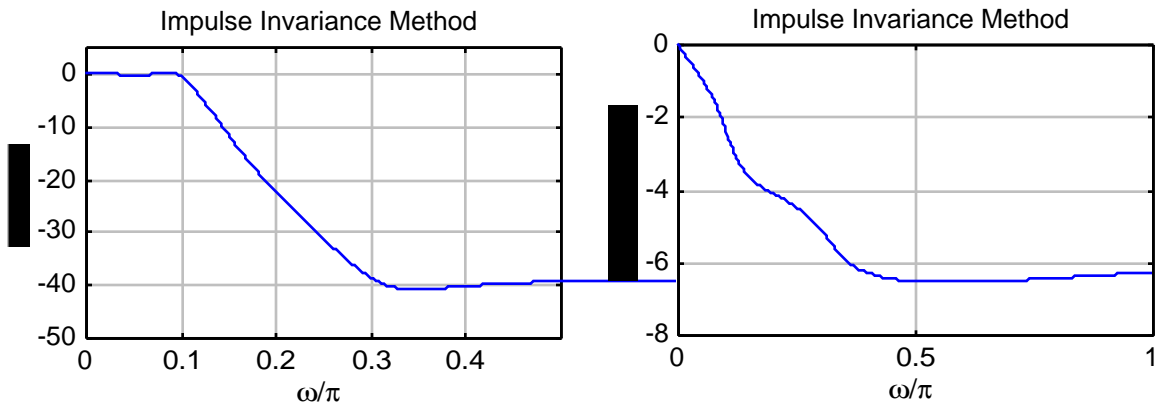
$$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = \frac{0.5\pi}{0.1\pi} = 5, \text{ and the inverse discrimination ratio given by } \frac{1}{k_1} = \sqrt{\frac{A^2 - 1}{\epsilon^2}} = 509.073363.$$

From Eq. (5.51) we have  $N \cong \frac{2 \log_{10}(4/k_1)}{\log_{10}(1/\rho)}$  where  $k' = \sqrt{1-k^2}$ ,  $\rho_0 = \frac{1-\sqrt{k'}}{2(1+\sqrt{k'})}$ , and

$\rho = \rho_0 + 2(\rho_0)^5 + 15(\rho_0)^9 + 150(\rho_0)^{13}$ , which yields  $N = 2.5519$ . We choose  $N = 3$ .

The code fragments used are:

```
[z,p,k] = ellipap(3,0.5,45);
[B,A] = zp2tf(z,p,k);
[BT,AT] = lp2lp(B,A,0.1*pi);
[num,den] =impinvar(BT,AT,1);
```



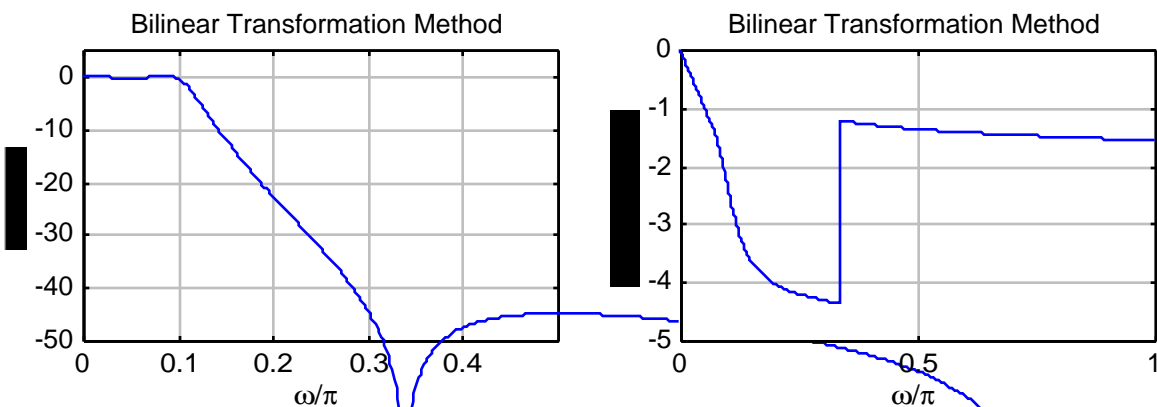
**Bilinear Transformation Method:** From Exercise M7.5, we have  $\Omega_p = \tan(\omega_p/2) = 0.15838$

and  $\Omega_s = \tan(\omega_s/2) = 1$ , the inverse transition ratio  $\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 6.31375$ , and the inverse

discrimination parameter  $\frac{1}{k_1} = \sqrt{\frac{A^2 - 1}{\epsilon^2}} = \sqrt{\frac{31621.7766}{0.12201845}} = 509.073363$ . Using the formula of Eq. (5.51) we get  $N = 2.3641$ . We choose  $N = 3$ .

The code fragments used are:

```
[N,Wn]=ellipord(0.1*pi,0.5*pi,0.5,45,'s');
[z,p,k] = ellipap(3,0.5,45);
[B,A] = zp2tf(z,p,k);
[BT,AT] = lp2lp(B,A,0.15838);
[num,den] = bilinear(BT,AT,0.5);
```



Note that the filter designed using the impulse invariance method does not meet the specifications due to aliasing. However, if the order  $N$  is increased to 4 from 3, it will meet the specifications.

**M7.9** No modifications to Program P7\_1 are necessary. Using this program we get the same results as indicated in the second part of the solution to Problem M7.8

**M7.10** Digital highpass filter specifications:  $F_p = 325$  kHz,  $F_s = 225$  kHz,  $F_T = 1$  MHz,  $\alpha_p = 0.5$  dB, and  $\alpha_s = 50$  dB.

(a) Using Eqs. (7.7) and (7.8) we have  $\omega_p = \frac{2\pi F_p}{F_T} = 2.042$  and  $\omega_s = \frac{2\pi F_s}{F_T} = 1.4137$ . Next

using Eq. (7.26), we get  $\hat{\Omega}_p = \tan\left(\frac{\omega_p}{2}\right) = 1.6319$  and  $\hat{\Omega}_s = \tan\left(\frac{\omega_s}{2}\right) = 0.85408$ .

Analog highpass filter specifications:  $\hat{\Omega}_p = 1.6319$  radians,  $\hat{\Omega}_s = 0.85408$ .  $\alpha_p = 0.5$  dB, and  $\alpha_s = 50$  dB.

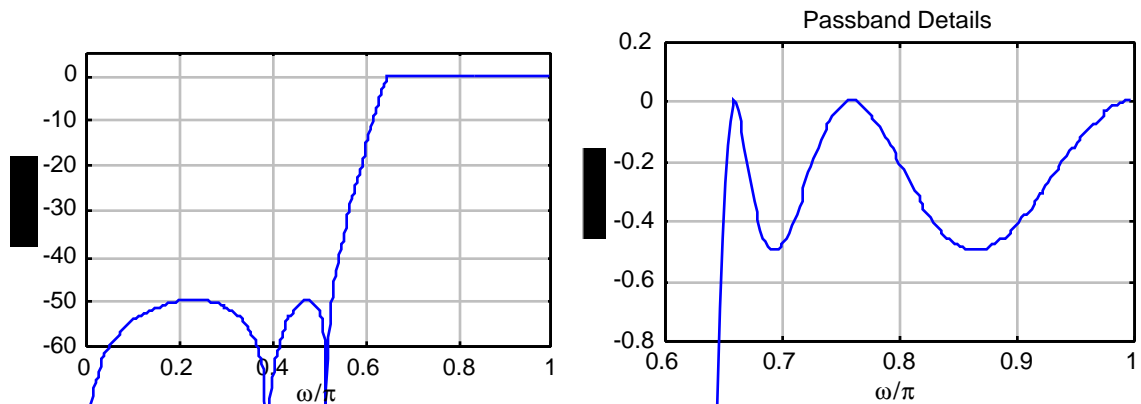
(b) For the prototype analog lowpass filter we choose  $\Omega_p = 1$ . From Eq. (5.60) we then get

$$\Omega_s = \frac{\hat{\Omega}_p}{\hat{\Omega}_s} = 1.9106.$$

Analog lowpass filter specifications:  $\Omega_p = 1$  radians,  $\Omega_s = 1.9106$  radians,  $\alpha_p = 0.5$  dB, and  $\alpha_s = 50$  dB.

Code fragments used are:

```
[N,Wn] = ellipord(1, 1.9106, 0.5, 50, 's');
[B,A] = ellip(N, 0.5, 50, Wn, 's');
[BT,AT] = lp2hp(B,A,1.6319);
[num,den] = bilinear(BT,AT,0.5);
```



(c) Analog lowpass transfer function coefficients:

Numerator

0 0.020293 8.8818e-16 0.15580 1.1102e-15 0.25555

Denominator

1 1.1594 2.0159 1.4100 0.88880 0.25555

Analog highpass transfer function coefficients:

Numerator

1 7.2050e-15 1.62360 1.5710e-14 0.56317 7.2908e-15

Denominator

1 5.6757 14.694 34.282 32.175 45.289

Digital highpass transfer function coefficients:

Numerator

0.023939 -0.037066 0.059191 -0.059191 0.037066 -0.023939

Denominator

1 2.4079 3.3102 2.6346 1.2535 0.28079

**M7.11** Digital bandpass filter specifications:  $F_{p1} = 560$  Hz,  $F_{p2} = 780$  Hz,  $F_{s1} = 375$  Hz,  $F_{s2} = 1000$  Hz,  $F_T = 2500$  Hz,  $\alpha_p = 1.2$  dB, and  $\alpha_s = 25$  dB.

(a) Using Eqs. (7.7) and (7.8) we have  $\omega_{p1} = \frac{2\pi F_{p1}}{F_T} = 0.448\pi = 1.4074$ ,

$\omega_{p2} = \frac{2\pi F_{p2}}{F_T} = 0.624\pi = 1.9604$ ,  $\omega_{s1} = \frac{2\pi F_{s1}}{F_T} = 0.3\pi = 0.94248$ , and

$\omega_{s2} = \frac{2\pi F_{s2}}{F_T} = 0.8\pi = 2.5133$ . Next using Eq. (7.26), we get  $\hat{\Omega}_{p1} = \tan\left(\frac{\omega_{p1}}{2}\right) = 0.84866$ ,

$\hat{\Omega}_{p2} = \tan\left(\frac{\omega_{p2}}{2}\right) = 1.4915$ ,  $\hat{\Omega}_{s1} = \tan\left(\frac{\omega_{s1}}{2}\right) = 0.50953$ , and  $\hat{\Omega}_{s2} = \tan\left(\frac{\omega_{s2}}{2}\right) = 3.0777$ . The

passband width is  $B_w = \hat{\Omega}_{p2} - \hat{\Omega}_{p1} = 0.64287$ . Now  $\hat{\Omega}_o^2 = \hat{\Omega}_{p2}\hat{\Omega}_{p1} = 1.2658$  and

$\hat{\Omega}_{s2}\hat{\Omega}_{s1} = 1.5682 \neq \hat{\Omega}_{p2}\hat{\Omega}_{p1}$ . Hence we adjust the lower stopband edge to

$\hat{\Omega}_{s1} = \frac{\hat{\Omega}_{p2}\hat{\Omega}_{p1}}{\hat{\Omega}_{s2}} = 0.41128$ . Note  $\hat{\Omega}_o = 1.1251$ .

Analog bandpass filter specifications:  $\hat{\Omega}_{p1} = 0.84866$  radians,  $\hat{\Omega}_{p2} = 1.4915$  radians,

$\hat{\Omega}_{s1} = 0.41128$  radians,  $\hat{\Omega}_{s2} = 3.0777$  radians,  $\alpha_p = 1.2$  dB, and  $\alpha_s = 25$  dB.

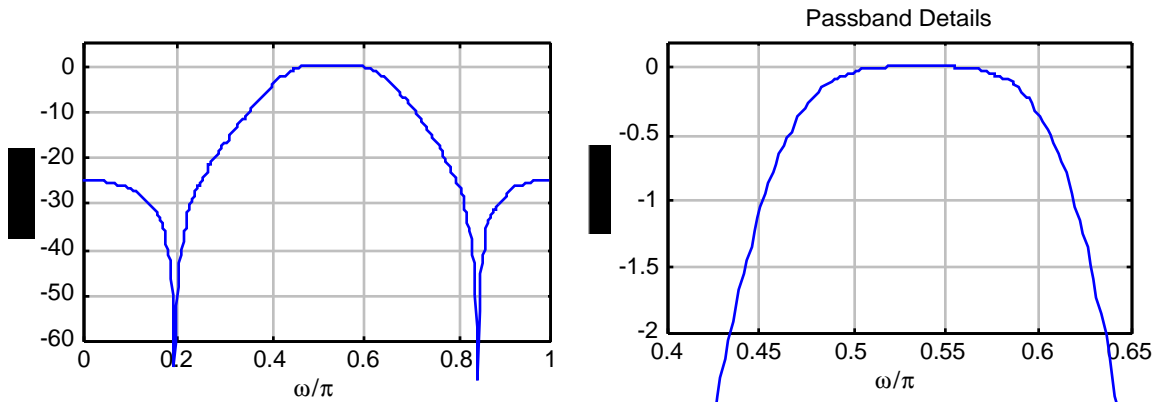
(b) For the prototype analog lowpass filter we choose  $\Omega_p = 1$ . From Eq. (5.62) we then get

$\Omega_s = \frac{\hat{\Omega}_o^2 - \hat{\Omega}_{s1}^2}{\hat{\Omega}_{s1} \cdot B_w} = \frac{1.2658 - 0.16915}{0.41128 \times 0.64287} = 4.1477$ .

Analog lowpass filter specifications:  $\Omega_p = 1$  radians,  $\Omega_s = 4.1477$  radians,  $\alpha_p = 1.2$  dB, and  $\alpha_s = 25$  dB.

Code fragments used are:

```
[N,Wn] = cheb2ord(1, 4.1477, 1.2, 25, 's');
[B,A] = cheby2(N, 25, Wn, 's');
[BT,AT] = lp2bp(B,A,1.1251, 0.64287);
[num,den] = bilinear(BT,AT,0.5);
```



(c) Analog lowpass transfer function coefficients:

Numerator

0.056234 -2.2204e-16 1.8260

Denominator

1 1.8565 1.8260

Analog bandpass transfer function coefficients:

Numerator

0.056234 -2.2204e-16 0.89703 -8.8818e-16 0.090108

Denominator

1 1.1935 3.2864 1.5108 1.6024

Digital bandpass transfer function coefficients:

Numerator

0.12142 0.015768 -0.10660 0.015768 0.12142

Denominator

1 0.35425 1.0522 0.20655 0.37058

**M7.12** Digital bandstop filter specifications:  $F_{p1} = 500$  Hz,  $F_{p2} = 2125$  Hz,  $F_{s1} = 1050$  Hz,  $F_{s2} = 1400$  Hz,  $F_T = 5000$  Hz,  $\alpha_p = 2$  dB, and  $\alpha_s = 40$  dB.

(a) Using Eqs. (7.7) and (7.8) we have  $\omega_{p1} = \frac{2\pi F_{p1}}{F_T} = 0.2\pi = 0.62832$ ,

$\omega_{p2} = \frac{2\pi F_{p2}}{F_T} = 0.85\pi = 2.6704$ ,  $\omega_{s1} = \frac{2\pi F_{s1}}{F_T} = 0.42\pi = 1.3195$ , and



$\omega_{s2} = \frac{2\pi F_{s2}}{F_T} = 0.56\pi = 1.7593$ . Next using Eq. (7.26), we get  $\hat{\Omega}_{p1} = \tan\left(\frac{\omega_{p1}}{2}\right) = 0.32492$ ,  
 $\hat{\Omega}_{p2} = \tan\left(\frac{\omega_{p2}}{2}\right) = 4.1653$ ,  $\hat{\Omega}_{s1} = \tan\left(\frac{\omega_{s1}}{2}\right) = 0.77568$ , and  $\hat{\Omega}_{s2} = \tan\left(\frac{\omega_{s2}}{2}\right) = 1.2088$ . The  
 stopband width is  $B_w = \hat{\Omega}_{s2} - \hat{\Omega}_{s1} = 0.43311$ . Now  $\hat{\Omega}_o^2 = \hat{\Omega}_{s2}\hat{\Omega}_{s1} = 0.93764$  and  
 $\hat{\Omega}_{p2}\hat{\Omega}_{p1} = 1.3534 \neq \hat{\Omega}_{s2}\hat{\Omega}_{s1}$ . Hence we adjust the lower passband edge to  
 $\hat{\Omega}_{p1} = \frac{\hat{\Omega}_{s2}\hat{\Omega}_{s1}}{\hat{\Omega}_{p2}} = 0.22511$ . Note  $\hat{\Omega}_o = 0.96832$ .

Analog bandstop filter specifications:  $\hat{\Omega}_{p1} = 0.22511$  radians,  $\hat{\Omega}_{p2} = 4.1653$  radians,  
 $\hat{\Omega}_{s1} = 0.41128$  radians,  $\hat{\Omega}_{s2} = 1.2088$  radians,  $\alpha_p = 2$  dB, and  $\alpha_s = 40$  dB.

**(b)** For the prototype analog lowpass filter we choose  $\Omega_s = 1$ . From Eq. (5.65) we then get

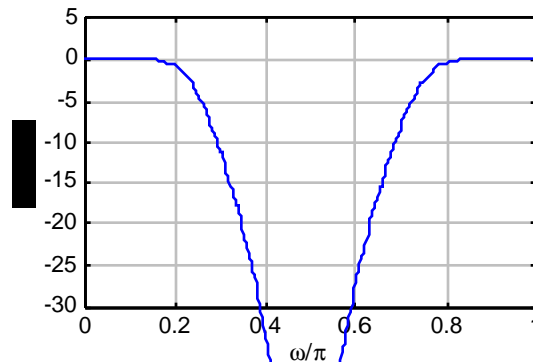
$$\Omega_p = \frac{\hat{\Omega}_{p1} \cdot B_w}{\hat{\Omega}_o^2 - \hat{\Omega}_{p1}^2} = \frac{0.22511 \times 0.43311}{0.93764 - 0.10557} = 0.16913.$$

Analog lowpass filter specifications:  $\Omega_s = 1$  radians,  $\Omega_p = 0.16913$  radians,  $\alpha_p = 2$  dB, and  $\alpha_s = 40$  dB.

Code fragments used are:

```

[N,Wn] = buttord(0.16913, 1, 2, 40, 's');
[B,A] = butter(N, Wn, 's');
[BT,AT] = lp2bs(B,A,0.96832, 0.43311);
[num,den] = bilinear(BT,AT,0.5);
  
```



**(c)** Analog lowpass transfer function coefficients:

Numerator

0                      0                      0                      0.010001

Denominator

```
1 0.43089 0.092835 0.010001
```

Analog bandstop transfer function coefficients:

Numerator

Columns 1 through 6

```
1 -1.5044e-16 2.8129 -2.8211e-16 2.6375 -1.3226e-16
```

Column 7

```
0.82435
```

Denominator

Columns 1 through 6

```
1 4.0206 10.895 15.664 10.216 3.5348
```

Column 7

```
0.82435
```

Digital bandstop transfer function coefficients:

Numerator

Columns 1 through 6

```
0.15762 -0.030434 0.47481 -0.060910 0.47481 -0.030434
```

Column 7

```
0.15762
```

Denominator

Columns 1 through 6

```
10 -0.094374 -0.0641472 -0.017232 0.33514 -0.010173
```

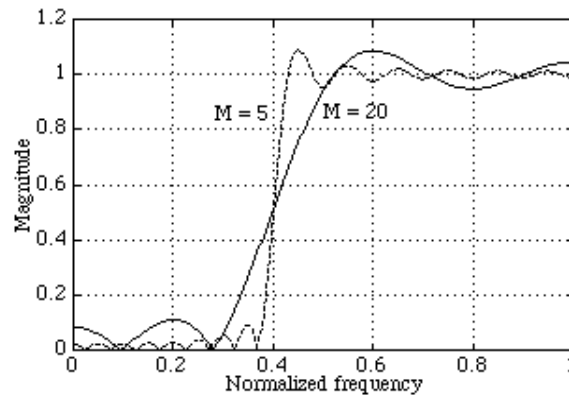
Column 7

```
-0.0061388
```

**M7.13** The impulse response coefficients of the truncated FIR highpass filter with cutoff frequency at  $0.4\pi$  can be generated using the following MATLAB statements:

```
n = -M:M;  
num = -0.4*sinc(0.4*n);  
num(M+1) = 0.6;
```

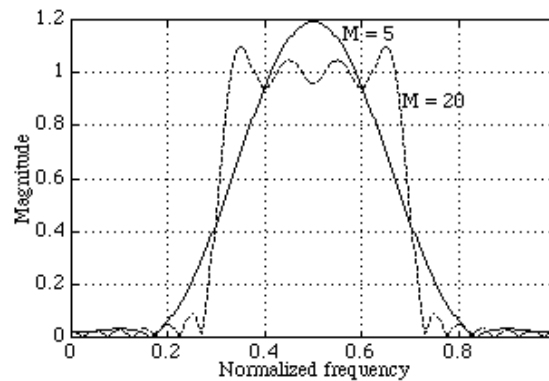
The magnitude responses of the truncated FIR highpass filter for two values of M are shown below:



**M7.14** The impulse response coefficients of the truncated FIR bandpass filter with cutoff frequencies at  $0.7\pi$  and  $0.3\pi$  can be generated using the following MATLAB statements:

```
n = -M:M;
num = 0.7*sinc(0.7*n) - 0.3*sinc(0.3*n);
```

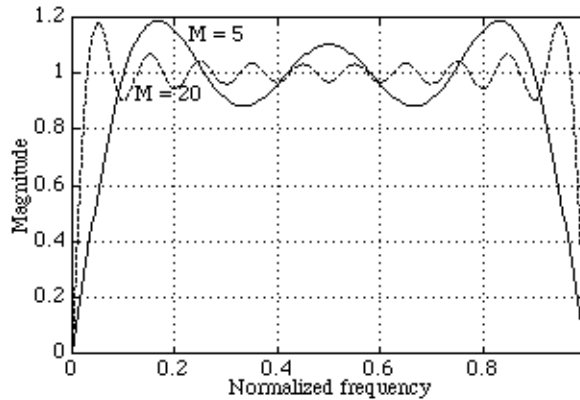
The magnitude responses of the truncated FIR bandpass filter for two values of M are shown below:



**M7.15** The impulse response coefficients of the truncated Hilbert transformer can be generated using the following MATLAB statements:

```
n = 1:M;
c = 2*sin(pi*n/2).*sin(pi*n/2); b = c./(pi*n);
num = [-fliplr(b) 0 b];
```

The magnitude responses of the truncated Hilbert transformer for two values of M are shown below:



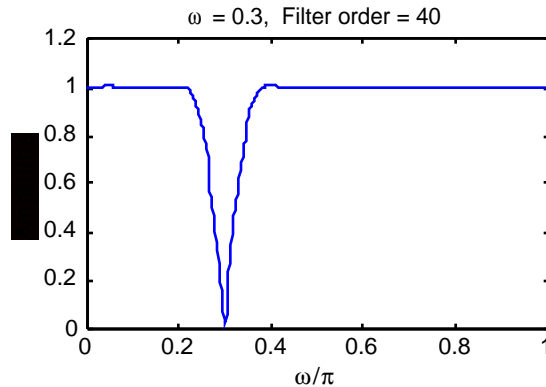
**M7.16** For notch filter design using the Hamming window, the following MATLAB program can be used:

```

fc = input('Normalized notch angular frequency = ');
N = input('Desired notch filter order = '); % must be an even
integer
M = N/2;
n=-M:1:M;
t = fc*n;
lp = fc*sinc(t);
b = 2*[lp(1:M) (lp(M+1)-0.5) lp((M+2):N+1)];
bw = b.*hamming(N+1)';
[h2,w]=freqz(bw,1,512);
plot(w/pi,abs(h2));axis([0 1 0 1.2]);
xlabel('\omega/\pi');ylabel('Amplitude');
title(['\omega = ' num2str(fc), ', Filter order = '
', num2str(N)])

```

For a notch filter of length 41 with a notch frequency at 60-Hz and operating at a 400-Hz sampling rate, the normalized angular notch frequency is given by  $\omega_0 = 0.3\pi$ . The magnitude response of the designed filter is shown below:



**M7.17** The desired function  $D(x) = 2.2x^3 - 3x^2 + 0.5$  is defined for  $-3 \leq x \leq 3.5$ . We wish to approximate  $D(x)$  by a quadratic function  $a_2 x^2 + a_1 x + a_0$  by minimizing the peak value of

the absolute error  $|D(x) - a_2 x^2 - a_1 x - a_0|$ . Since there are four unknowns  $a_0, a_1, a_2$  and  $\varepsilon$ , we need four extremal points on  $x$  in the range  $-3 \leq x \leq 3.5$ , which we arbitrarily choose as  $x_1 = -3, x_2 = -2, x_3 = 1$  and  $x_4 = 3.5$ . We then solve the four linear equations

$a_0 + a_1 x_1 + a_2 x_1^2 - (-1)^1 \varepsilon = D(x_1), 1 = 1, 2, 3, 4$ , which lead to

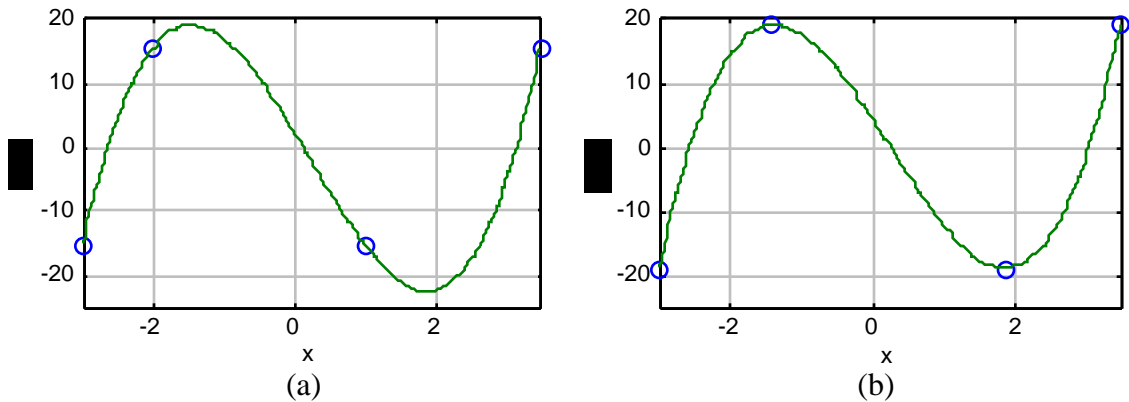
$$\begin{bmatrix} 1 & -3 & 9 & 1 \\ 1 & -2 & 4 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 3.5 & 1.225 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \varepsilon \end{bmatrix} = \begin{bmatrix} -85.9 \\ -29.1 \\ -0.3 \\ -58.075 \end{bmatrix}.$$

Its solution yields  $a_0 = -1.6214, a_1 = 18.229, a_2 = -1.5857$ , and  $\varepsilon = -15.321$ . Figure (a) below shows the plot of the error  $E_1(x) = 2.2x^3 - 1.4143x^2 - 18.229x + 2.1214$  along with the values of the error at the chosen extremal points.

The next set of extremal points are those points where  $E_1(x)$  assumes its maximum absolute values. These extremal points are given by  $x_1 = -3, x_2 = -1.454, x_3 = 1.89$  and  $x_4 = 3.5$ . The new values of the unknowns are obtained by solving

$$\begin{bmatrix} 1 & -3 & 9 & 1 \\ 1 & -1.454 & 2.1141 & -1 \\ 1 & 1.89 & 3.5721 & 1 \\ 1 & 3.5 & 1.225 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \varepsilon \end{bmatrix} = \begin{bmatrix} -85.9 \\ -12.605 \\ 4.6365 \\ 5.8075 \end{bmatrix}$$

whose solution yields  $a_0 = -7.0706, a_1 = -17.021, a_2 = -1.3458$ , and  $\varepsilon = -18.857$ . Figure (b) below shows the plot of the error  $E_2(x) = 2.2x^3 - 1.6542x^2 - 17.021x + 4.3685$  along with the values of the error at the chosen extremal points. This time the algorithm has converged as  $\varepsilon$  is also the maximum value of the absolute error.

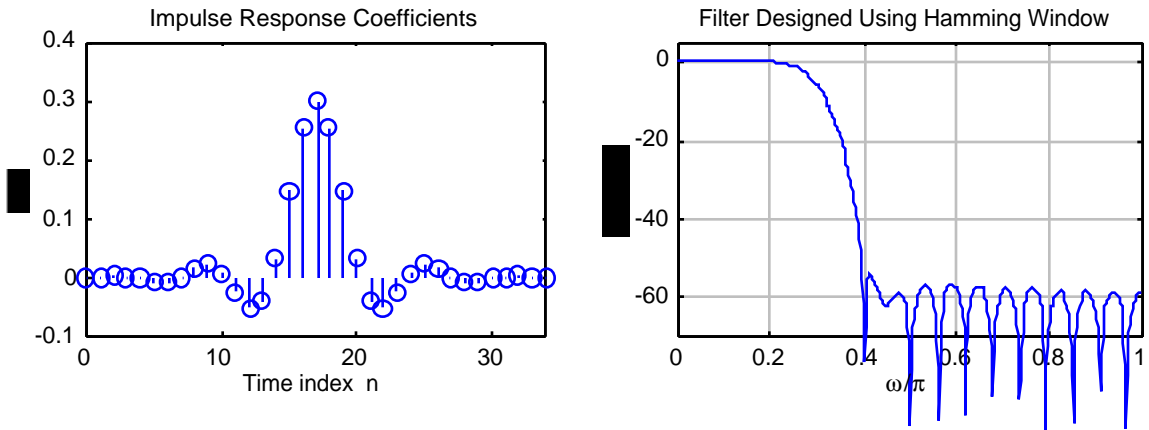


**M7.18** From Eqs. (7.7) and (7.8), the normalized bandedges are given by  $\omega_p = 0.2\pi$  and  $\omega_s = 0.4\pi$ . Therefore,  $\omega_c = 0.3\pi$  and  $\Delta\omega = 0.2\pi$ .

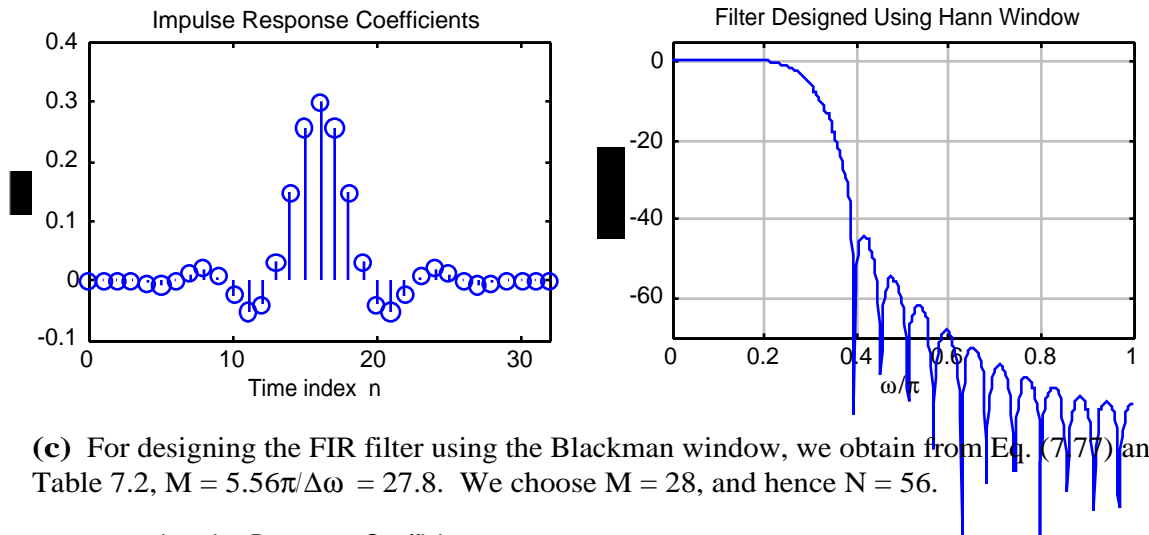
(a) Using Eq. (7.77) and Table 7.2, the estimated order of the FIR filter for designing using the Hamming window is given by  $N = 2M$  where  $M = 3.32\pi/\Delta\omega = 16.6$ . We therefore choose  $M = 17$  or equivalently,  $N = 34$ . The MATLAB program used to generate the windowed filter coefficients are

```
n = -17:17;
num = 0.4*sinc(0.4*n);
wh = hamming(35);
```

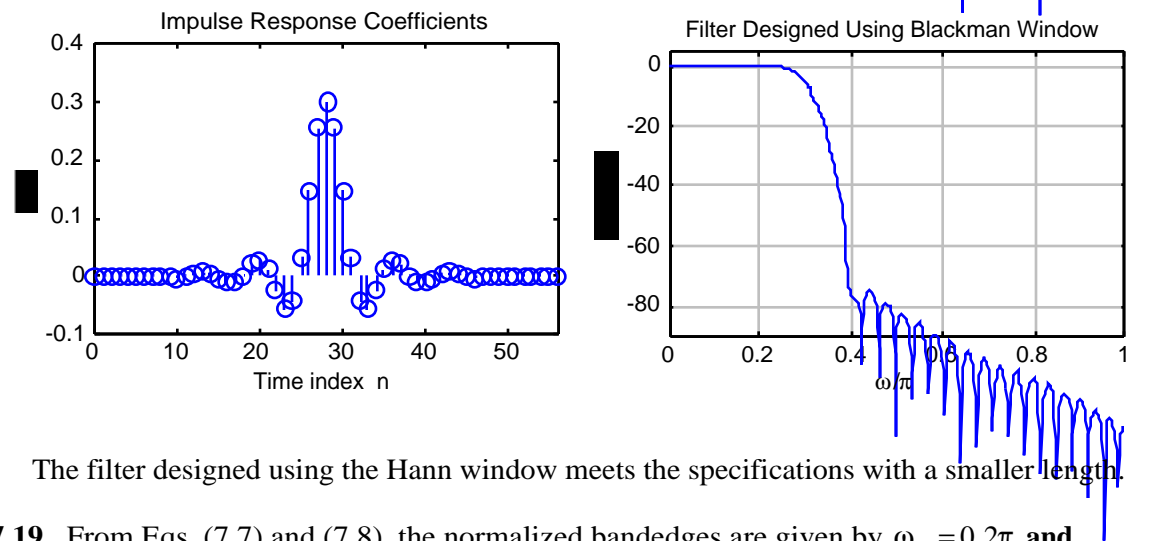
```
b = num.*wh';
```



(b) For designing the FIR filter using the Hann window, we obtain from Eq. (7.77) and Table 7.2,  $M = 3.11\pi/\Delta\omega = 15.55$ . We choose  $M = 16$ , and hence  $N = 32$ .



(c) For designing the FIR filter using the Blackman window, we obtain from Eq. (7.77) and Table 7.2,  $M = 5.56\pi/\Delta\omega = 27.8$ . We choose  $M = 28$ , and hence  $N = 56$ .

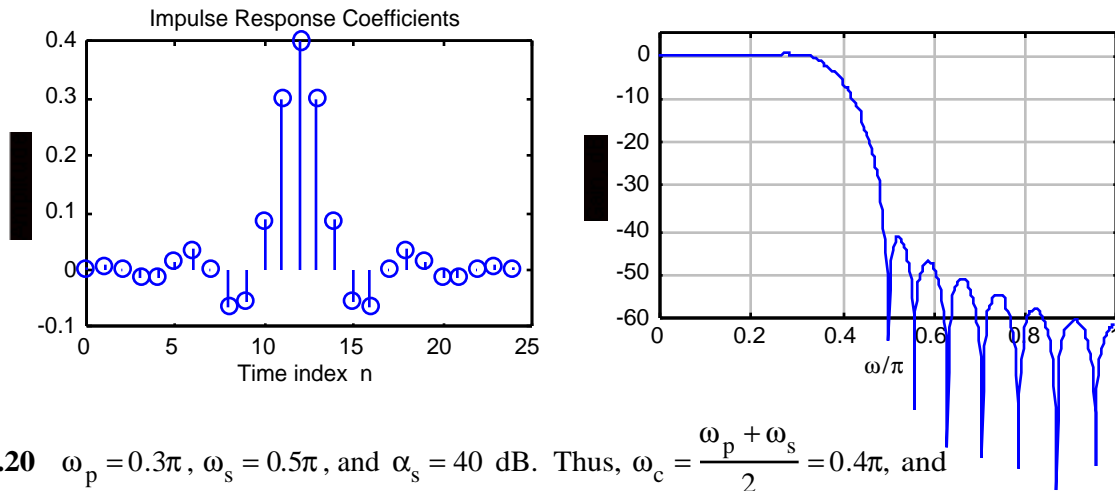


The filter designed using the Hann window meets the specifications with a smaller length.

**M7.19** From Eqs. (7.7) and (7.8), the normalized bandedges are given by  $\omega_p = 0.2\pi$  and  $\omega_s = 0.4\pi$ . Therefore,  $\omega_c = 0.3\pi$  and  $\Delta\omega = 0.2\pi$ .

Substituting  $\alpha_s = 40$  in Eq. (7.85) we obtain  $\beta = 3.3953$ . Next, substituting the value of  $\alpha_s = 40$  and  $\Delta\omega = 0.2\pi$  in Eq. (7.86) we obtain  $N = 22.289$ . We choose the next higher even integer value of 24 as the filter order  $N$ , and hence  $M = 12$ . The code fragments used for the design are given below:

```
n = -N/2:N/2;
num = 0.4*sinc(0.4*n);
wh = kaiser(N+1,3.3953);
b = num.*wh';
```



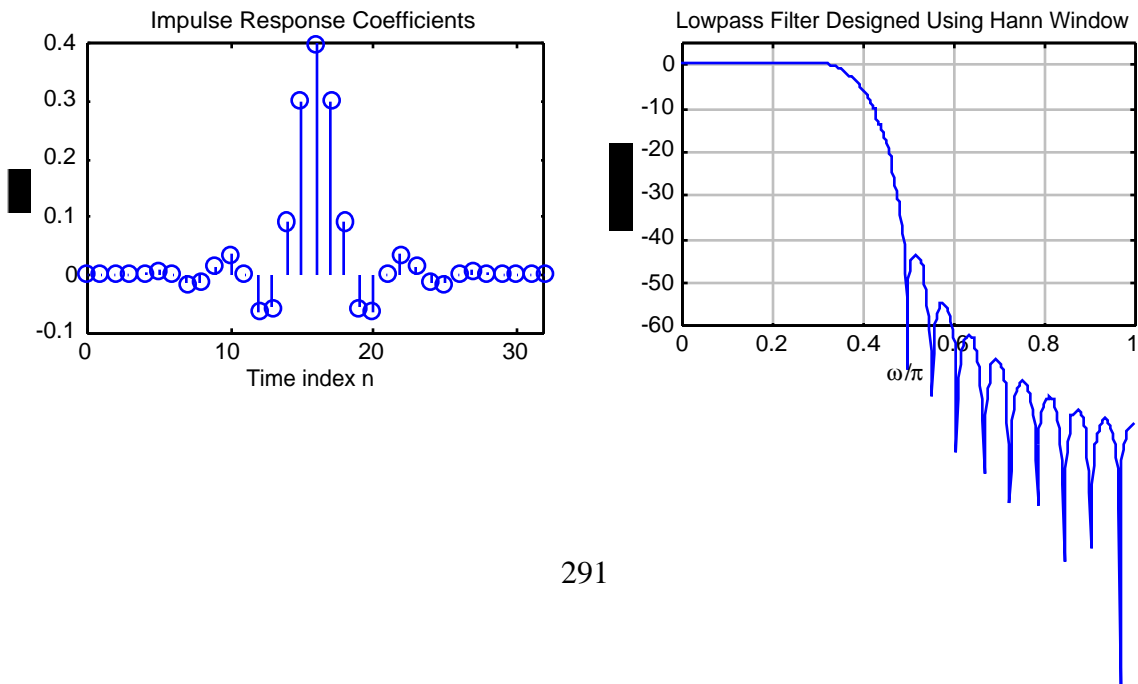
**M7.20**  $\omega_p = 0.3\pi$ ,  $\omega_s = 0.5\pi$ , and  $\alpha_s = 40$  dB. Thus,  $\omega_c = \frac{\omega_p + \omega_s}{2} = 0.4\pi$ , and

$\Delta\omega = \omega_s - \omega_p = 0.2\pi$ . From Table 7.2 we observe that Hann window is appropriate for the filter design as it meets the minimum stopband attenuation requirement with a filter of lowest order.

From Eq. (7.77) and Table 7.2, we get  $M = \frac{3.11\pi}{\Delta\omega} = \frac{3.11\pi}{0.2\pi} = 15.55$ . We choose  $M = 16$  implying a filter of order  $N = 32$ .

Code fragments used in the design are given below:

```
n = -16:1:16;
lp = 0.4*sinc(0.4*n);
wh = hanning(33);
b = lp.*wh';
```

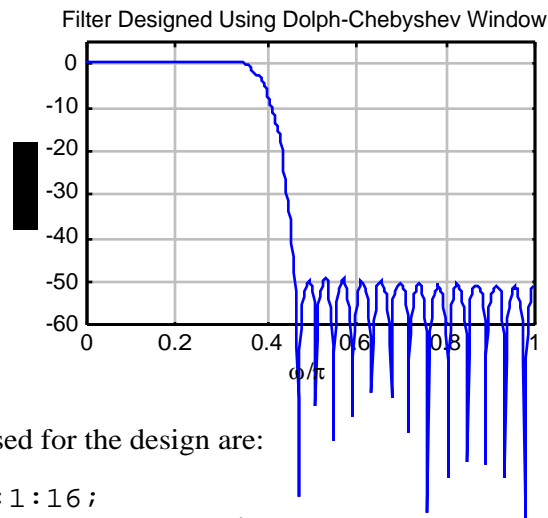


**M7.21**  $\omega_p = 0.3\pi$ ,  $\omega_s = 0.5\pi$ , and  $\alpha_s = 40$  dB. Thus,  $\omega_c = 0.4\pi$  and  $\Delta\omega = 0.2\pi$ . Using Eq. (7.82)

we estimate the filter order  $N = 2M$  which is given by  $N = \frac{2.065 \times 40 - 16.4}{2.285(0.2\pi)} = 45.859$ . Hence we choose  $N = 46$ , i.e.  $M = 23$ .

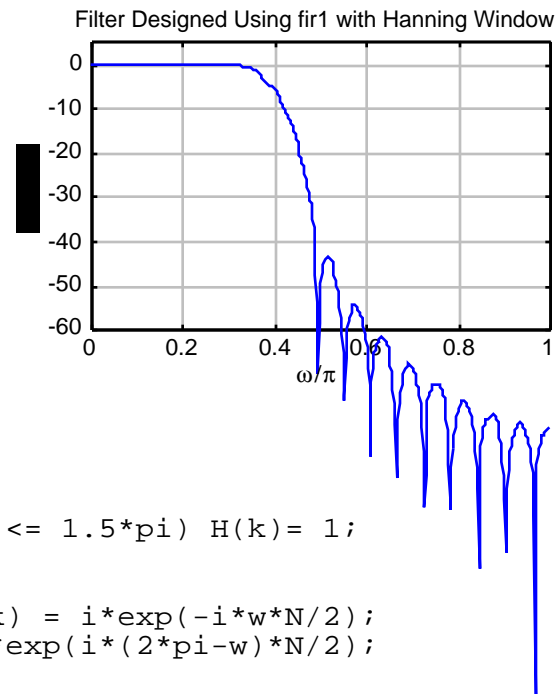
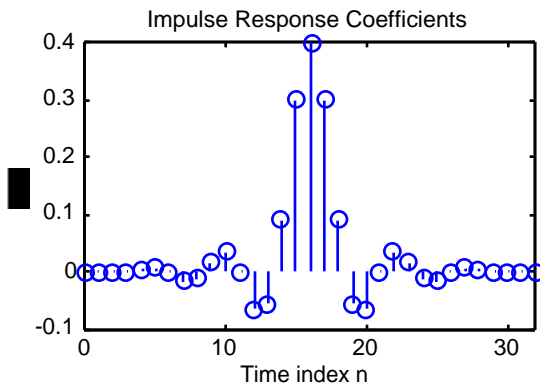
Code fragments used for the design are:

```
n = -23:1:23;
lp = 0.4*sinc(0.4*n);
wh = chebwin(47,40);
b = lp.*wh';
```



**M7.22** Code fragments used for the design are:

```
n = -16:1:16;
b = fir1(32,0.4,hanning(33));
```



**M7.23**

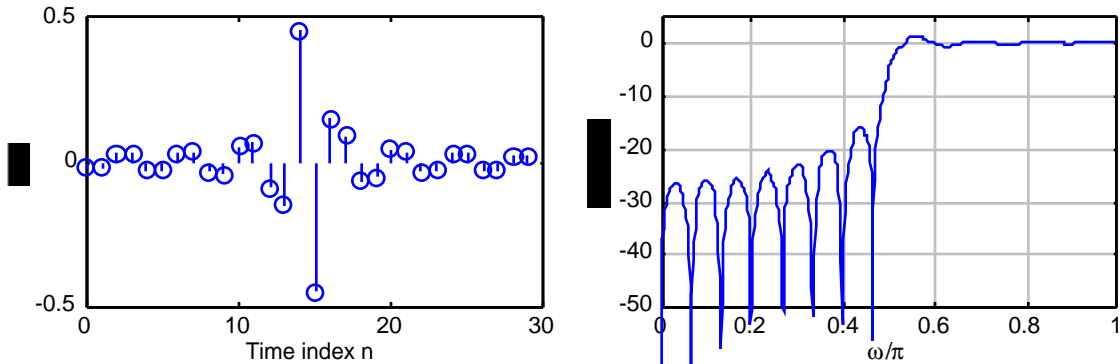
```
N = 29;
for k = 1:N+1
w = 2*pi*(k-1)/30;
if(w >= 0.5*pi & w <= 1.5*pi) H(k) = 1;
else H(k) = 0;
end
if(w <= pi) phase(k) = i*exp(-i*w*N/2);
else phase(k) = -i*exp(i*(2*pi-w)*N/2);
end
end
H = H.*phase;
```



```

f = ifft(H);
[FF,w] = freqz(f,1,512);
k = 0:N;
subplot(211)
stem(k,real(f));
xlabel('Time index n'); ylabel('Amplitude')
subplot(212)
plot(w/pi,20*log10(abs(FF)));
xlabel('\omega/\pi'); ylabel('Gain, dB');
axis([0 1 -50 5]); grid on

```

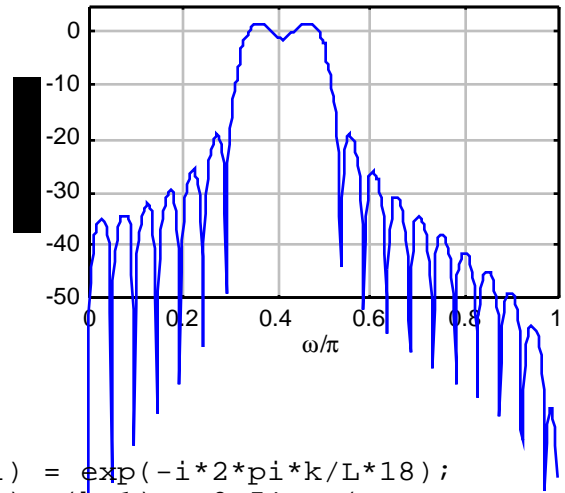
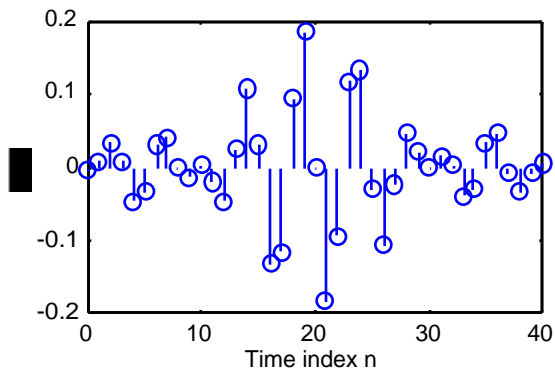


#### M7.24

```

% Length = 41 and bandpass, hence Type 3 filter
N = 40; L = N+1;
for k = 1:L
    w = 2*pi*(k-1)/L;
    if (w >= 0.3*pi & w <= 0.5*pi) H(k) = i*exp(-
i*w*N/2);
    elseif (w >= 1.5*pi & w <= 1.7*pi) H(k) =
-i*exp(i*(2*pi-w)*N/2);
    else H(k) = 0;
    end
end
f = ifft(H);
[FF,w] = freqz(f,1,512);
k = 0:N;
subplot(211);
stem(k,real(f));
xlabel('Time index n'); ylabel('h[n]');
subplot(212);
plot(w/pi,20*log10(abs(FF)));
ylabel('Gain, dB'); xlabel('\omega/\pi');
axis([0 1 -50 5]); grid;

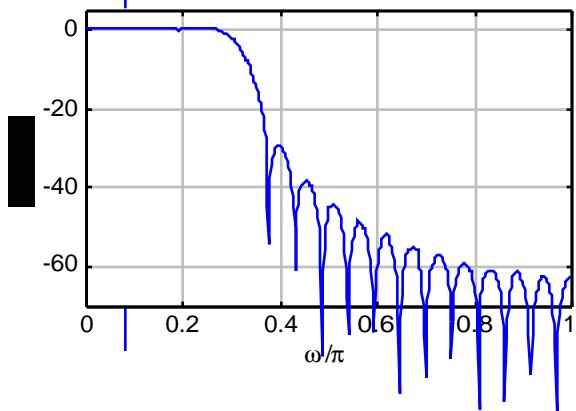
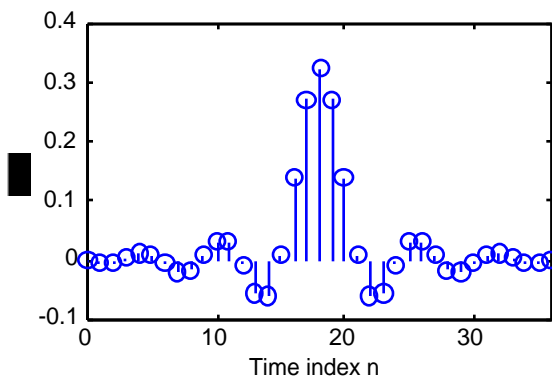
```



```

M7.25 N=36; L=N+1;
for k = 0:36
    if(k <= 5 | k >= 32) H(k+1) = exp(-i*2*pi*k/L*18);
    elseif (k == 6 | k == 31) H(k+1) = 0.5*exp(-
i*2*pi*k/L*18);
    end
end
f = ifft(H);
[FF,w] = freqz(f,1,512);
k = 0:N;
subplot(211);
stem(k,real(f));axis([0 36 -0.1 0.4]);
xlabel('Time index n'); ylabel(h[n]);
subplot(212);
plot(w/pi,20*log10(abs(FF)));
xlabel('\omega/\pi'); ylabel('Gain, dB');
axis([0 1 -70 5]);grid;

```



```

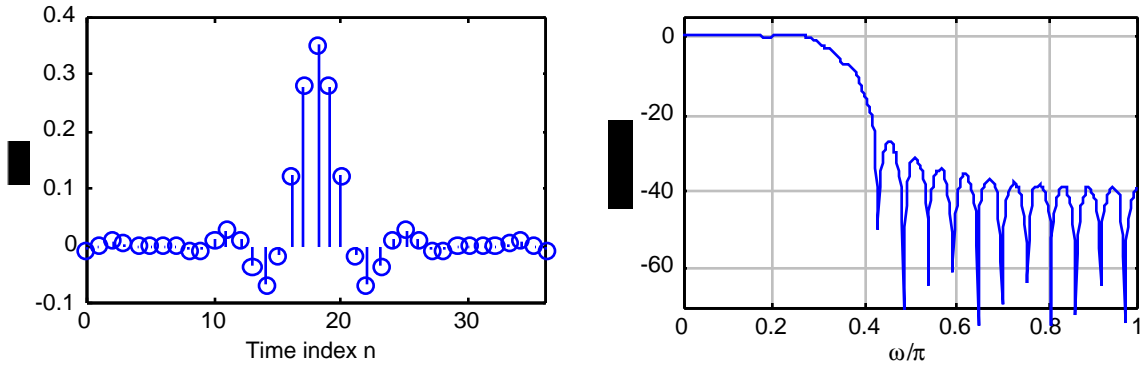
M7.26 N = 36; L=N+1;
for k = 0:36
    if(k <= 5 | k >= 32) H(k+1) = exp(-i*2*pi*k/L*18);
    elseif (k ==6 | k == 31) H(k+1) = 2/3*exp(-i*2*pi*k/L*18);
    elseif (k ==7 | k == 30) H(k+1) = 1/3*exp(-i*2*pi*k/L*18);
    end
end
f = ifft(H);
[FF,w] = freqz(f,1,512);
k = 0:N;

```

```

subplot(211);
stem(k,real(f));
xlabel('Time index n'); ylabel('Amplitude');
subplot(212);
plot(w/pi,20*log10(abs(FF)));
xlabel('\omega/\pi'); ylabel('Gain, dB');
axis([0 1 -70 5]); grid;

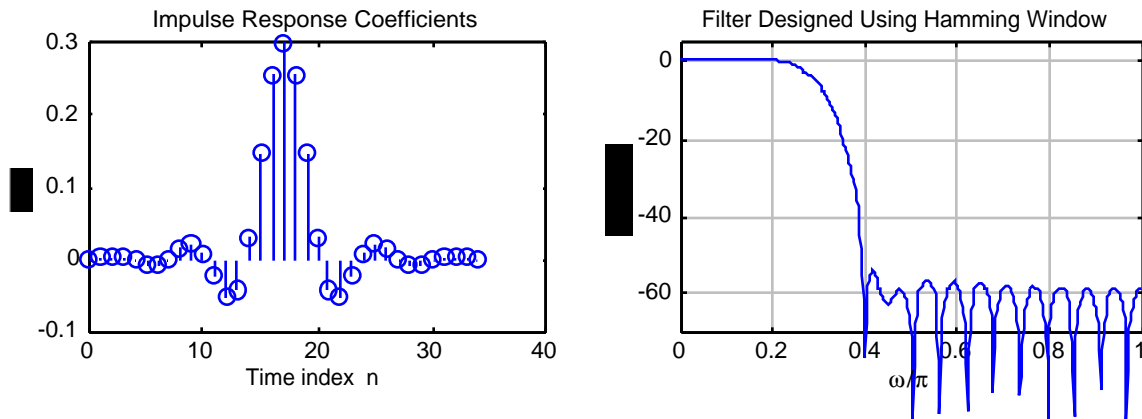
```



**M7.27**  $\omega_c = 0.3\pi$  and  $\Delta\omega = 0.2\pi$ .

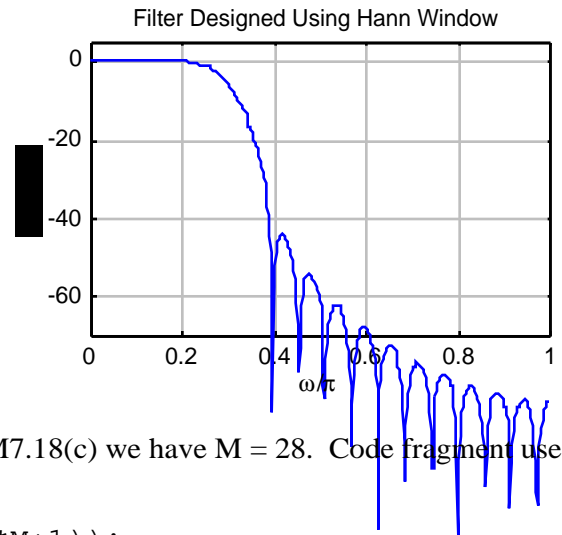
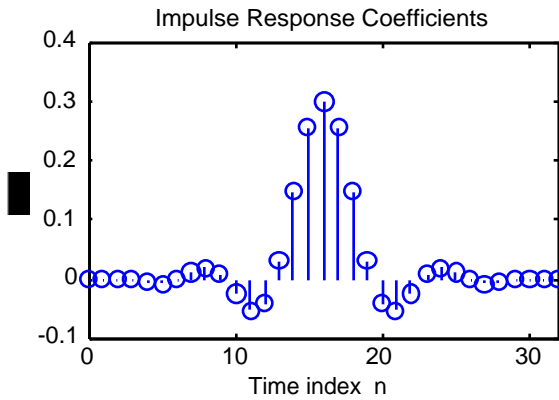
**(a) Hamming Window:** From Exercise M7.18(a) we have  $M = 17$ . Code fragment used for the design are:

```
b = fir1(2*M,0.3);
```



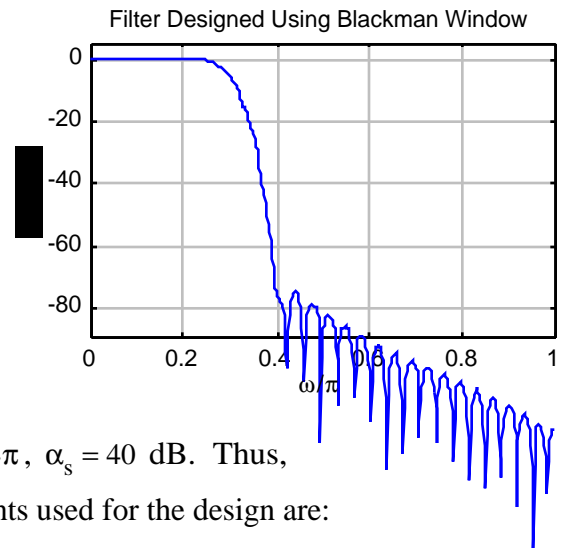
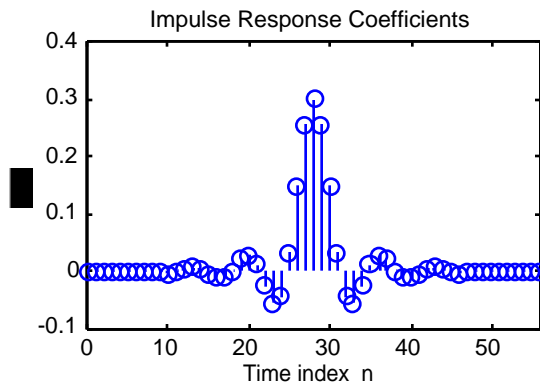
**(b) Hann Window:** From Exercise M7.18(b) we have  $M = 16$ . Code fragment used for the design are:

```
b = fir1(2*M,0.3, hanning(2*M+1));
```



(c) **Blackman Window:** From Exercise M7.18(c) we have  $M = 28$ . Code fragment used for the design are:

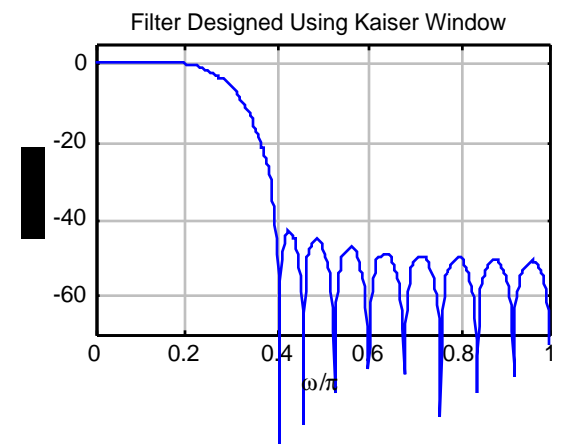
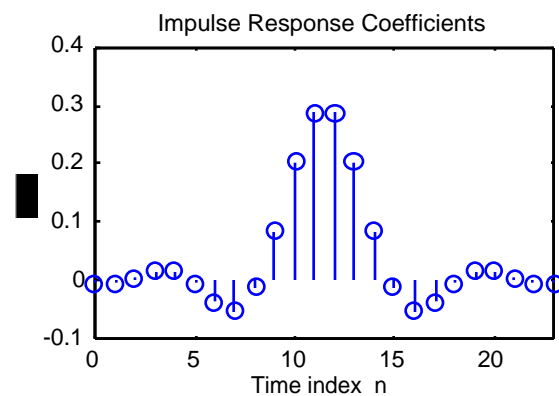
```
b = fir1(2*M,0.3, blackman(2*M+1));
```



(d) **Kaiser Window:**  $\omega_p = 0.2\pi$ ,  $\omega_s = 0.4\pi$ ,  $\alpha_s = 40$  dB. Thus,

$\delta_s = 10^{-\alpha_s/20} = 10^{-2} = 0.01$ . Code fragments used for the design are:

```
[N,Wn,beta,type] = kaiserord([0.2 0.4],[1 0], [0.01 0.01]);
b = fir1(N,0.3,kaiser(N+1,beta));
```



**M7.28**  $\omega_p = 0.6\pi$ ,  $\omega_s = 0.45\pi$ ,  $\alpha_p = 0.2$  dB and  $\alpha_s = 45$ . Thus,  $\Delta\omega = 0.15\pi$  and

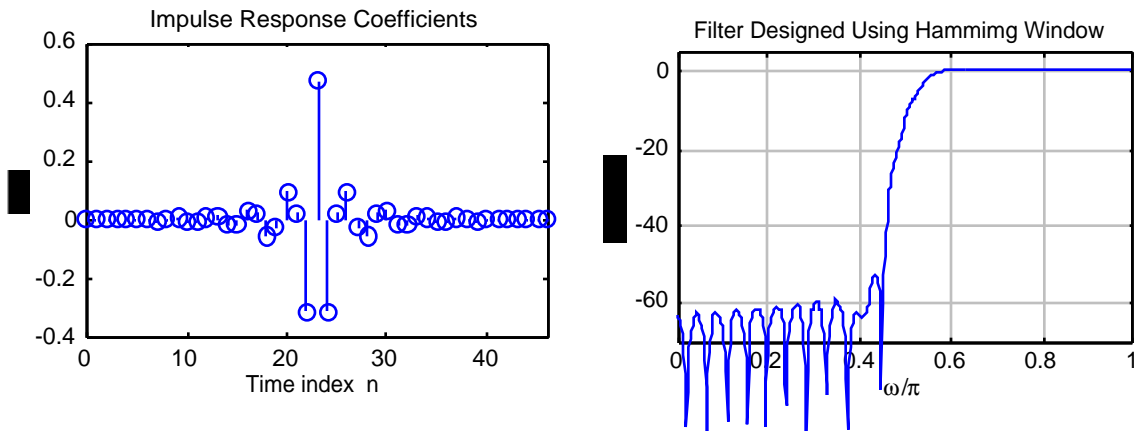
$$\omega_c = \frac{0.6\pi + 0.45\pi}{2} = 0.525\pi.$$

(a) Hamming Window: From Table 7.2 and Eq. (7.77) we have

$$M = \frac{3.32\pi}{\Delta\omega} = \frac{3.32\pi}{0.15\pi} = 22.133, \text{ We choose } M = 23, \text{ i.e. filter order } N = 2M = 46.$$

Code fragment used is:

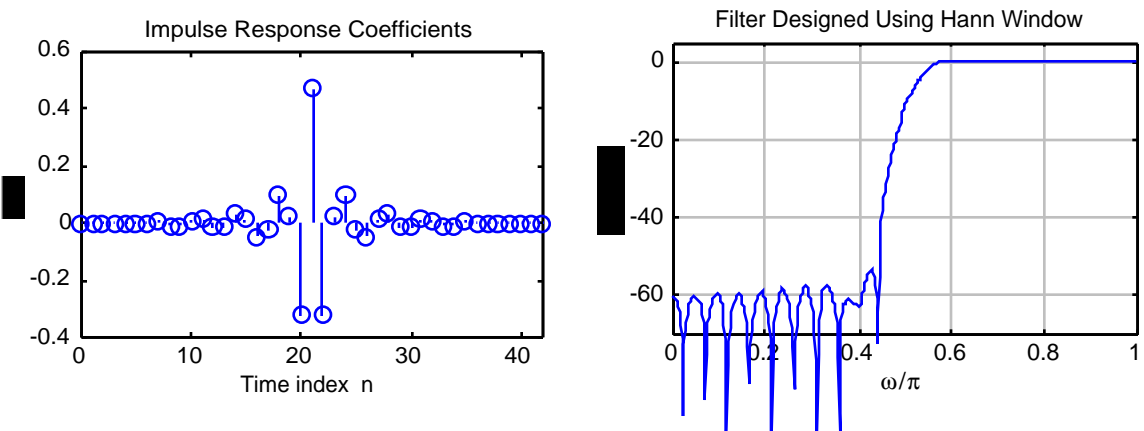
```
b = fir1(2*M,0.525,'high');
```



(b) Hann Window: From Table 7.2 and Eq. (7.77) we have  $M = \frac{3.11\pi}{\Delta\omega} = \frac{3.11\pi}{0.15\pi} = 20.733$ . We choose  $M = 21$ , i.e. filter order  $N = 2M = 42$ .

Code fragment used is:

```
b = fir1(2*M,0.525,'high',hanning(2*M+1));
```

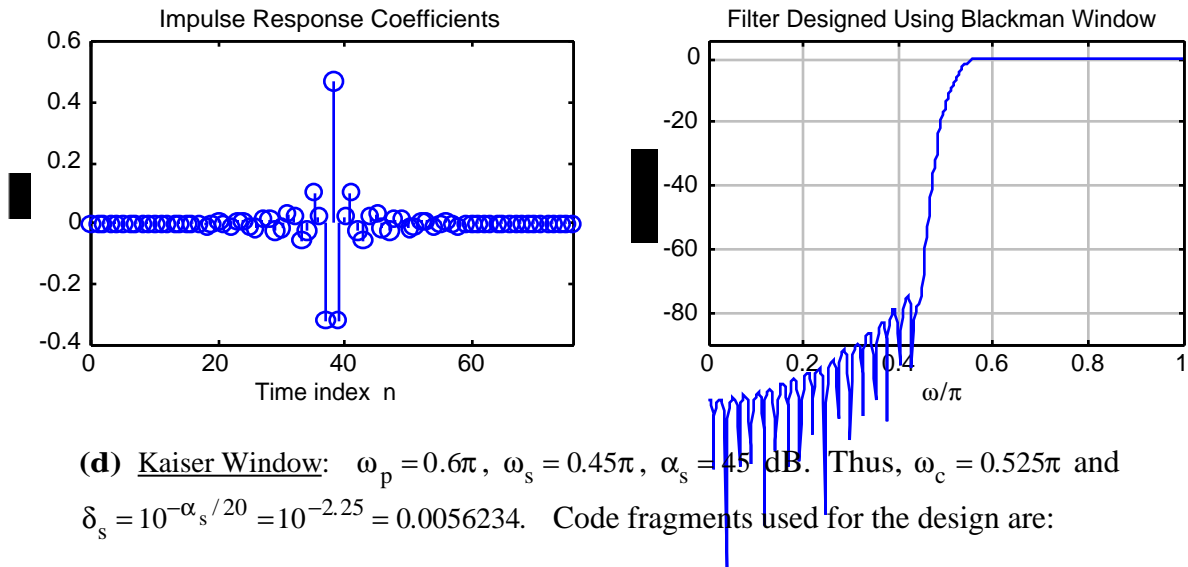


(c) Blackman Window: From Table 7.2 and Eq. (7.77) we have

$$M = \frac{5.56\pi}{\Delta\omega} = \frac{5.56\pi}{0.15\pi} = 37.067. \text{ We choose } M = 38, \text{ i.e. filter order } N = 2M = 76.$$

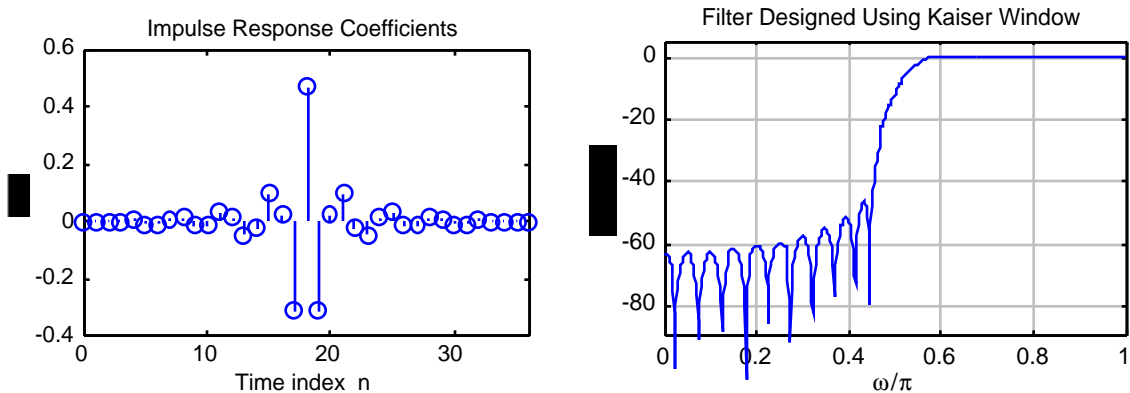
Code fragment used is:

```
b = fir1(2*M,0.525,'high',blackman(2*M+1));
```



(d) Kaiser Window:  $\omega_p = 0.6\pi$ ,  $\omega_s = 0.45\pi$ ,  $\alpha_s = 45$  dB. Thus,  $\omega_c = 0.525\pi$  and  $\delta_s = 10^{-\alpha_s/20} = 10^{-2.25} = 0.0056234$ . Code fragments used for the design are:

```
ds = 0.0056234;
[N,Wn,beta,type]=kaiserord([0.45 0.6],[1 0],[ds ds]);
N = N+1;% For highpass filter, order must be even
b = fir1(N,0.525,'high',kaiser(N+1,beta));
```



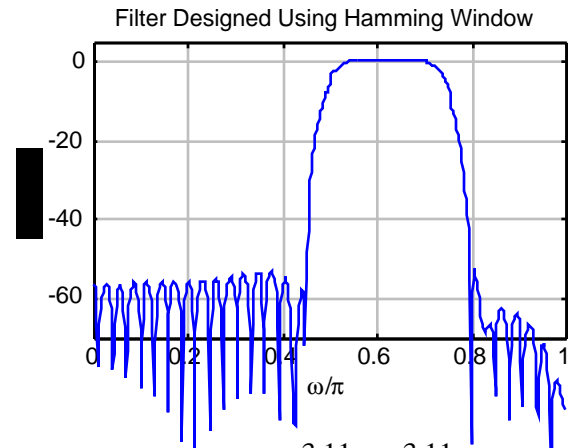
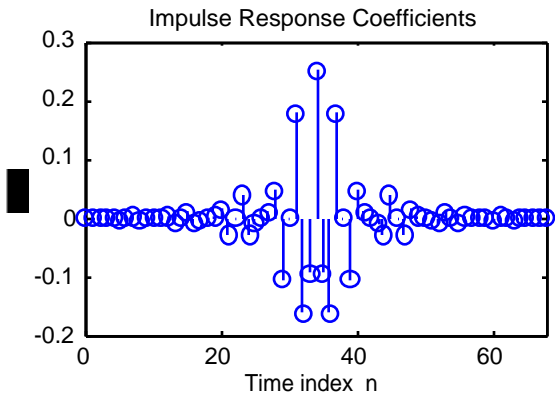
**M7.29**  $\omega_{p1} = 0.55\pi$ ,  $\omega_{p2} = 0.7\pi$ ,  $\omega_{s1} = 0.45\pi$ ,  $\omega_{s2} = 0.8\pi$ ,  $\alpha_p = 0.15$  and  $\alpha_s = 40$ . Thus,  $\Delta\omega_1 = \omega_{p1} - \omega_{s1} = 0.1\pi$  and  $\Delta\omega_2 = \omega_{s2} - \omega_{p2} = 0.1\pi = \Delta\omega_1$ .

(a) Hamming Window: From Table 7.2 and Eq. (7.77) we have

$$M = \frac{3.32\pi}{\Delta\omega_1} = \frac{3.32\pi}{0.1\pi} = 33.2 \quad \text{We choose } M = 34, \text{ i.e. filter order } N = 2M = 68.$$

Code fragment used is:

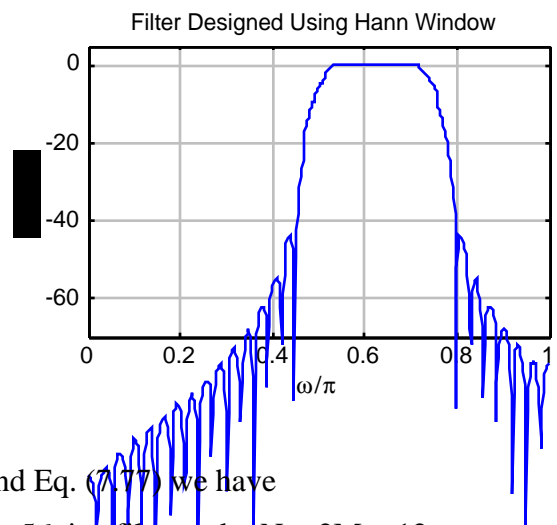
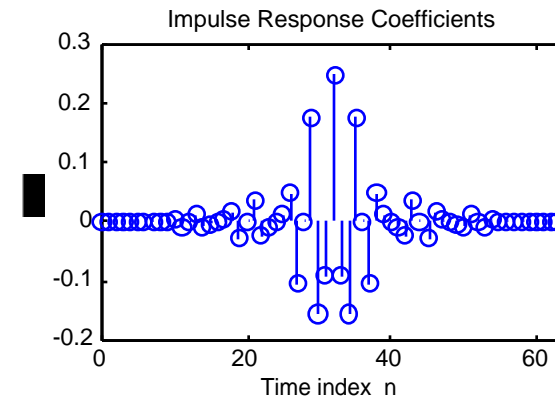
```
b = fir1(2*M,[0.5 0.75]);
```



(b) Hamming Window: From Table 7.2 and Eq. (7.77) we have  $M = \frac{3.11\pi}{\Delta\omega} = \frac{3.11\pi}{0.1\pi} = 31.1$ . We choose  $M = 32$ , i.e. filter order  $N = 2M = 64$ .

Code fragment used is:

```
b = fir1(2*M,[0.5 0.75],hanning(2*M+1));
```

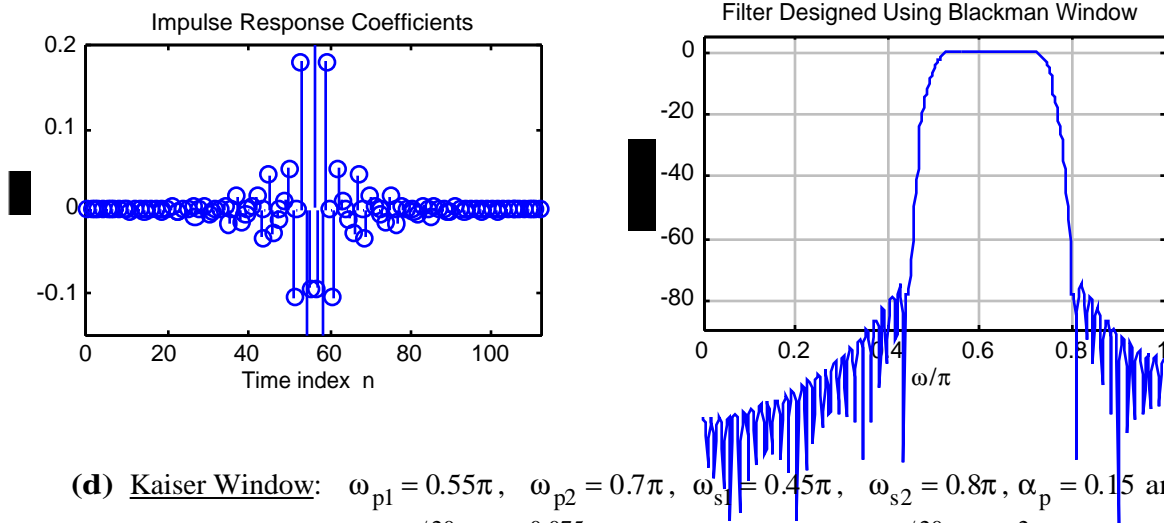


(c) Blackman Window: From Table 7.2 and Eq. (7.77) we have  $M = \frac{5.56\pi}{\Delta\omega_1} = \frac{5.56\pi}{0.1\pi} = 55.6$ . We choose  $M = 56$ , i.e. filter order  $N = 2M = 112$ .

$$\omega_{c1} = \frac{\omega_{p1} + \omega_{s1}}{2} = \frac{0.55\pi + 0.45\pi}{2} = 0.5\pi \text{ and } \omega_{c2} = \frac{\omega_{p2} + \omega_{s2}}{2} = \frac{0.7\pi + 0.8\pi}{2} = 0.75\pi.$$

Code fragment used is:

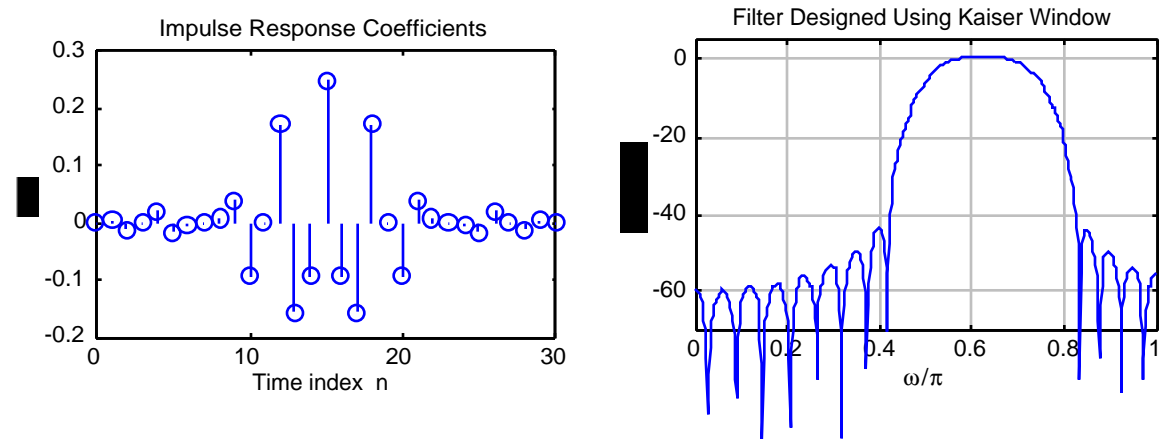
```
b = fir1(2*M,[0.55 0.7],blackman(2*M+1));
```



(d) Kaiser Window:  $\omega_{p1} = 0.55\pi$ ,  $\omega_{p2} = 0.7\pi$ ,  $\omega_{s1} = 0.45\pi$ ,  $\omega_{s2} = 0.8\pi$ ,  $\alpha_p = 0.15$  and  $\alpha_s = 40$ . Thus,  $\delta_p = 10^{-\alpha_p/20} = 10^{-0.075} = 0.98288$  and  $\delta_s = 10^{-\alpha_s/20} = 10^{-2} = 0.01$ .

Code fragments used for the design are:

```
[N,Wn,beta,type]= kaiserord([0.55 0.7],[1 0],[0.98288 0.01]);
b = fir1(N,[0.5 0.75],kaiser(N+1,beta));
```

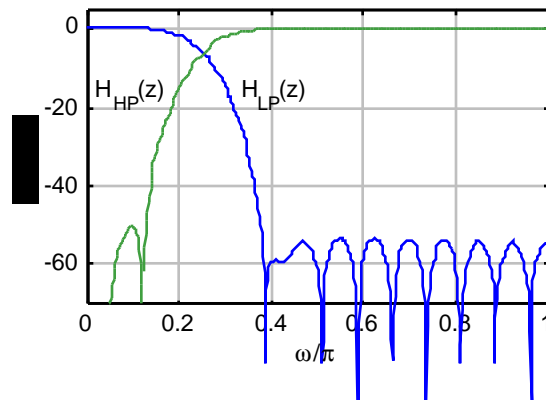


**M7.30** From Eq. (7.7), the normalized crossover frequency  $\omega_c = \frac{2\pi \times 10}{80} = 0.25\pi$ . The delay-complementary FIR lowpass and highpass filter coefficients of length 27 are then generated using the MATLAB statements:

```
d1 = fir1(26,0.7854);
d2 = -d1;d2(14) = 1 - d1(14);
```

The gain responses of the two filters are as indicated below:

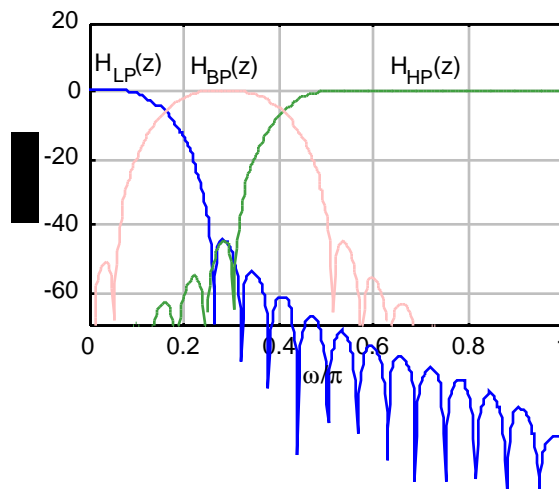




**M7.31** The filter coefficients of the FIR lowpass and the highpass, and their delay-complementary FIR bandpass filter are generated using the MATLAB program:

```
c = hanning(31);
d1 = fir1(30,0.15873,c);
d2 = fir1(30,0.40816,'high',c);
d3 = -d1-d2;d3(16) = 1 - d1(16) - d2(16);
```

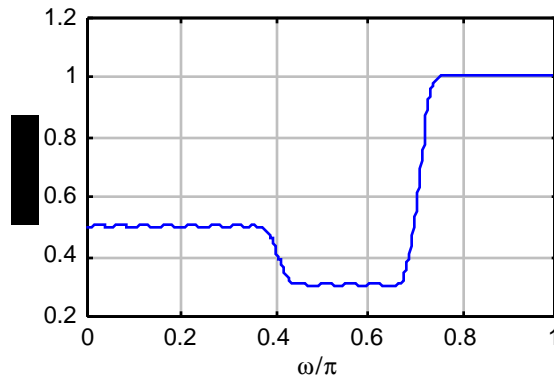
The gain responses of the three FIR filters are as given below:



**M7.32** The filter coefficients are generated using the MATLAB program:

```
fpts = [0 0.4 0.42 0.7 0.721 1];
mval = [0.5 0.5 0.3 0.3 1.0 1.0];
b = fir2(80,fpts,mval);
```

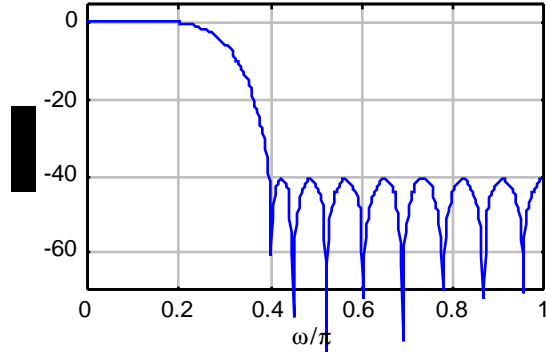
The gain response of the multiband FIR filter is as given below:



**M7.33** The filter was also designed by using `remezord` to estimate the filter order and then using `remez` to determine the filter coefficients. To this end the MATLAB program used is given below:

```
Rp = 0.1; Rs = 40; FT = 20;
f = [2 4]; m = [1 0];
dev = [(10^(Rp/20) - 1)/(10^(Rp/20) + 1) 10^(-Rs/20)];
[N, fo, mo, wo] = remezord(f, m, dev, FT);
b = remez(N, fo, mo, wo);
```

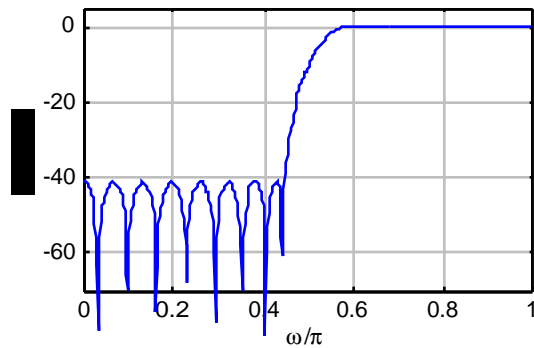
The filter obtained using the above program is of length 20, but its gain response did not meet the specifications. The specifications were met when the filter order was increased to 22.



**M7.34** The filter was also designed by using `remezord` to estimate the filter order and then using `remez` to determine the filter coefficients. To this end the MATLAB program used is given below:

```
Rp = 0.2; Rs = 45;
f = [0.45 0.65]; m = [1 0];
dev = [(10^(Rp/20) - 1)/(10^(Rp/20) + 1) 10^(-Rs/20)];
[N, fo, mo, wo] = remezord(f, m, dev);
b = remez(N, fo, mo, wo);
```

The filter obtained using the above program is of length 27, but its gain response did not meet the specifications. The specifications were met when the filter order was increased to 30.



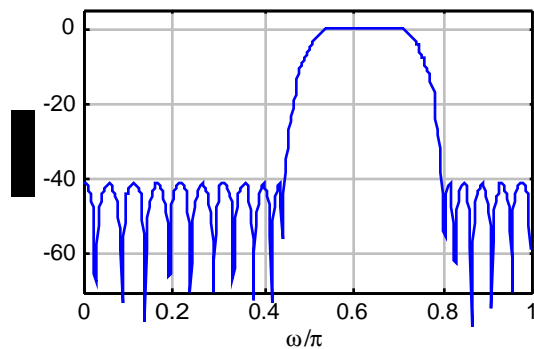
**M7.35** The code fragments used to design the FIR bandpass filter are:

```

Rp = 0.15; Rs = 40;
f = [0.45 0.55 0.7 0.8]; m = [0 1 0];
dev = [10^(-Rs/20) (10^(Rp/20) -1)/(10^(Rp/20) +1) 10^(-
Rs/20)];
[N, fo, mo, wo] = remezord(f, m, dev);
b = remez(N,fo,mo,wo);

```

The filter obtained using the above program is of length 40, but its gain response did not meet the specifications. The specifications were met when the filter order was increased to 41.



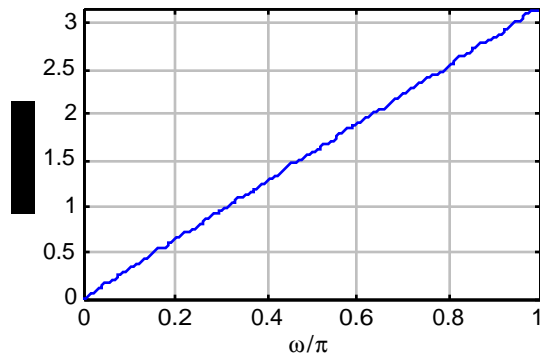
**M7.36** The code fragment used to design a length-32 differentiator is given below.

```

b = remez(31,[0 1],[0 pi],'differentiator');

```

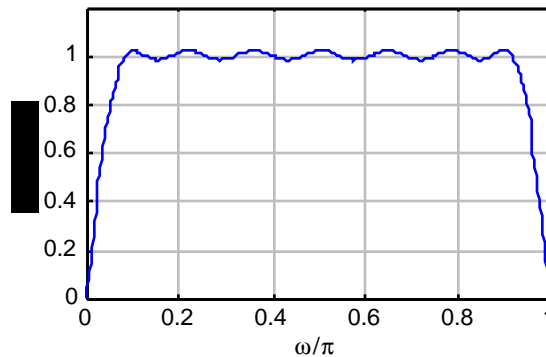
The magnitude response of the differentiator designed is shown below:



**M7.37** The code fragments used to design the 26-th order Hilbert transformer are:

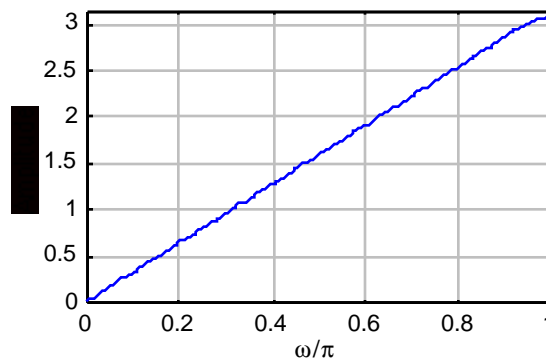
```
f = [0.01 0.078 0.08 0.92 0.94 1];
m = [0 0 1 1 0 0];
wt = [1 60 1];
b = remez(28,f,m,wt,'hilbert');
```

Note that the passband and the stopbands have been weighted to reduce the ripple in the passband. The magnitude response of the Hilbert transformer designed is shown below:



**M7.38** The code fragment used for the design is

```
b = firls(31,[0 0.9],[0 0.9*pi],'differentiator');
```



**M7.39** From the given specifications  $\frac{\omega_s}{M} = 0.2\pi$ . We also require  $\frac{\omega_s}{M} \leq \frac{2\pi - \omega_s}{M}$ , for no overlap to

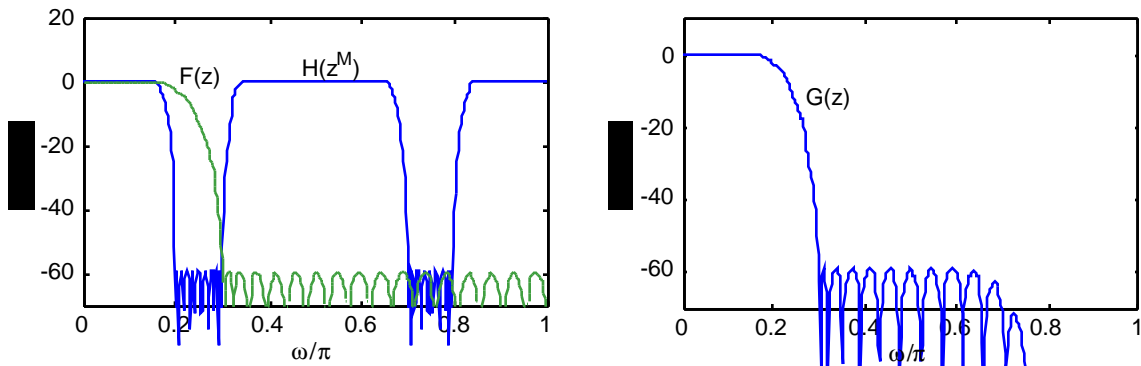
occur in  $\left|H(e^{jM\omega})\right|$  (see Figure P7.3(b)). Hence  $0.2\pi \leq \frac{2\pi}{M} - 0.2\pi$  which implies  $M \leq 5$ .

We choose  $M = 4$ . Hence specifications for  $H(z)$  are  $\omega_p = 0.6\pi$ ,  $\omega_s = 0.8\pi$ ,  $\delta_p = 0.001$  and  $\delta_s = 0.001$ . Substituting these values in Eq. (7.15), we arrive at the estimated value of the FIR filter length as  $N = 33$ .

Specifications for  $F(z)$  are  $\omega_p = 0.15\pi$ ,  $\omega_s = 0.3\pi$ ,  $\delta_p = 0.001$  and  $\delta_s = 0.001$ . Hence, from Eq. (7.15), the estimated length of  $F(z)$  is  $N = 43$ .

Therefore the total number of multiplications required per output sample is  $\left(\frac{33}{2} + 43\right) \times \frac{1}{2} = 30$ . (The division by 2 is due to the fact that  $H(z)$  and  $F(z)$  are linear phase filters.)

On the other hand for a direct single stage implementation, we note that the specifications of  $G(z)$  are:  $\omega_p = 0.15\pi$ ,  $\omega_s = 0.2\pi$ ,  $\delta_p = 0.002$ ,  $\delta_s = 0.001$ . Hence, the filter length of  $G(z)$  from Eq. (7.15) is 121. Therefore the total number of multiplications required per output sample is  $121/2 = 61$ .



**M7.40** From the specified stopband edge  $\omega_s = 0.14\pi$ , we get the estimated order  $N$  of the RRS prefilter  $H(z)$  as  $N \cong \frac{2\pi}{\omega_s} = 14.2857$ . We choose  $N = 15$ . Next using the following two MATLAB functions we design the equalizer  $F(z)$  and the overall cascaded filter  $H(z)F(z)$ .

```
% Plotting function
% plots the result of using an equalizer of length InpN
function[N] = plotfunc(InpN) ;
% Creating filters
Wfilt = ones(1,InpN);
Efilt = remezfunc(InpN, Wfilt);
% Plot running sum filter response
%figure(1)
Wfilt = Wfilt/sum(Wfilt);
[hh,w]=freqz(Wfilt,1,512);
plot(w/pi,20*log10(abs(hh)));
axis([0 1 -50 5]); grid;
xlabel('\omega/\pi'); ylabel('Gain, dB');
title('Prefilter H(z)');
```

```

% Plot equalizer filter response
%figure(2)
pause
[hw,w]=freqz(Efilt,1,512);
plot(w/pi,20*log10(abs(hw)));
axis([0 1 -50 5]); grid;
xlabel('\omega/\pi'); ylabel('Gain, dB');
title('Equalizer F(z)');
% Plot cascaded filter response
%figure(3)
pause
Cfilt = conv(Wfilt, Efilt);
[hc,w]=freqz(Cfilt,1,512);
plot(w/pi,20*log10(abs(hc)));
axis([0 1 -80 5]); grid;
title('Cascaded filter H(z)F(z)');
ylabel('G(\omega), dB'); xlabel('\omega/\pi');
% Plot filter coefficients
%figure(4)
pause
%subplot(2,1,1);
n1 = 0:length(Wfilt)-1;
stem(n1,Wfilt);
xlabel('Time index n');ylabel('Amplitude');
title('Prefilter coefficients');
pause
%subplot(2,1,2);
n2 = 0:length(Efilt)-1;
stem(n2,Efilt);
xlabel('Time index n');ylabel('Amplitude');
title('Equalizer coefficients');

% Remez function using 1/P(z) as desired amplitude
% and P(z) as weighting
function [N] = remezfunc(Nin, Wfilt);
% Nin : number of tuples in the remez equalizer filter
% Wfilt : the prefilter
a = [0:0.001:0.999]; % The accuracy of the computation
w = a.*pi;
wp = 0.042*pi; % The passband edge
ws = 0.14*pi; % The stopband edge
i = 1;
n = 1;
for t = 1 : (length(a)/2)
    if w(2*t) < wp
        pas(i) = w(2*t - 1);
        pas(i+1) = w(2*t);
        i = i + 2;
    end;
    if w(2*t-1) > ws
        sto(n) = w(2*t-1);
        sto(n+1) = w(2*t);
        n = n + 2;
    end;
end;
w = cat(2, pas, sto);
bi = length(w)/2; % bi gives the number of subintervals
for t1 = 1:bi

```

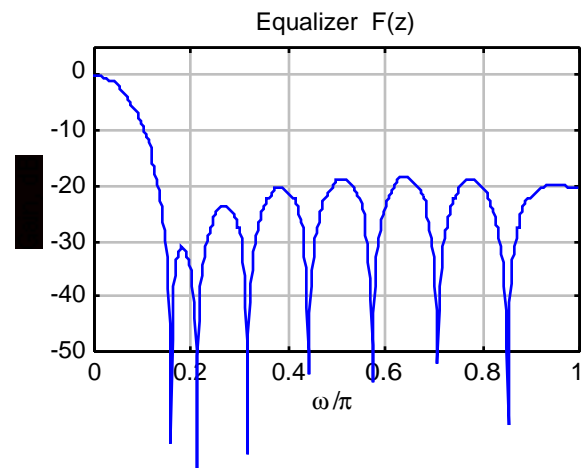
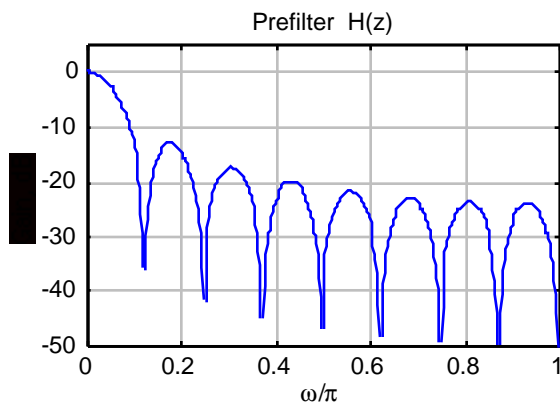
```

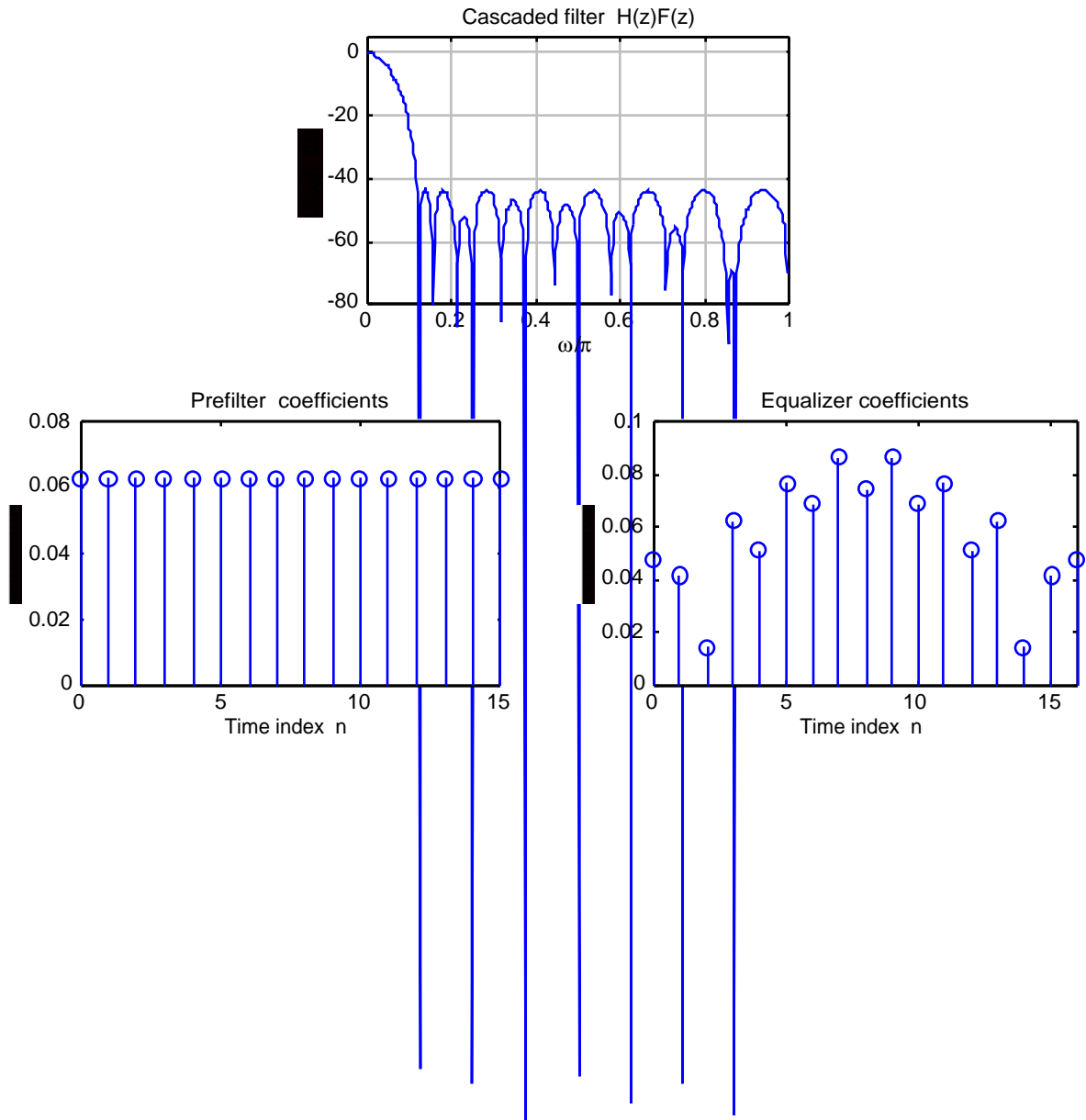
    bw(t1) = (w(2*t1) + w(2*t1-1))/2;
    W(t1) = Weight(bw(t1), Wfilt, ws);
end;
W = W/max(W);
for t2 = 1:length(w)
    G(t2) = Hdr(w(t2), Wfilt, wp);
end;
G = G/max(G);
N = remez(Nin, w/pi, G, W);

% Weighting function
function[Wout] = Weight (w, Wfilt, ws);
K = 22.8;
L = length(Wfilt);
Wtemp = 0;
Wsum = 0;
for k = 1:L
    Wtemp = Wfilt(k)*exp((k-1)*i*w);
    Wsum = Wsum + Wtemp;
end;
Wout = abs(Wsum);
if w > ws
    Wout = K*max(Wout);
end;

% Desired function
function[Wout] = Hdr (w, Wfilt, ws);
if w <= ws
    L = length(Wfilt);
    Wtemp = 0;
    Wsum = 0;
    for k = 1:L
        Wtemp = Wfilt(k)*exp(i*(k-1)*w);
        Wsum = Wsum + Wtemp;
    end;
    Wsum = abs(Wsum);
    Wout = 1/Wsum;
else Wout = 0;
end;

```

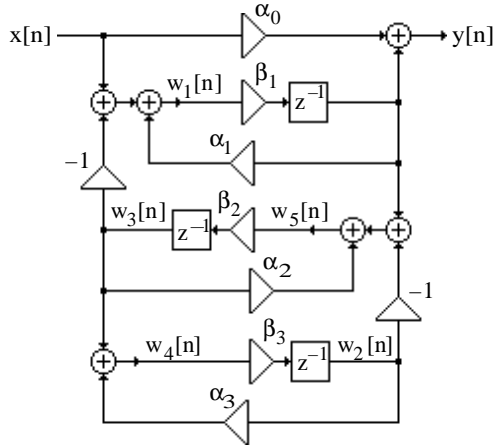






## Chapter 8 (2e)

### 8.1



Analysis yields

$$\begin{aligned}
 w_1[n] &= \alpha_1 \beta_1 w_1[n-1] + x[n] - w_3[n], \\
 w_2[n] &= \beta_3 w_4[n-1], \\
 w_3[n] &= \beta_2 w_5[n-1], \\
 w_4[n] &= \alpha_3 w_2[n] + w_3[n], \\
 w_5[n] &= \alpha_2 w_3[n] - w_2[n] + \beta_1 w_1[n-1], \\
 y[n] &= \alpha_0 x[n] + \beta_1 w_1[n-1].
 \end{aligned}$$

In matrix form the above set of equations is given by:

$$\begin{bmatrix} w_1[n] \\ w_2[n] \\ w_3[n] \\ w_4[n] \\ w_5[n] \\ y[n] \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_3 & 1 & 0 & 0 & 0 \\ 0 & -1 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n-1] \\ w_2[n-1] \\ w_3[n-1] \\ w_4[n-1] \\ w_5[n-1] \\ y[n-1] \end{bmatrix} + \begin{bmatrix} \alpha_0 \beta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n-1] \\ w_2[n-1] \\ w_3[n-1] \\ w_4[n-1] \\ w_5[n-1] \\ y[n-1] \end{bmatrix} + \begin{bmatrix} x[n] \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_0 x[n] \end{bmatrix}$$

Here the  $\mathbf{F}$  matrix is given by

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_3 & 1 & 0 & 0 & 0 \\ 0 & -1 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the  $\mathbf{F}$  matrix contains nonzero entries above the main diagonal, the above set of equations are not computable.

**8.2** A computable set of equations of the structure of Figure P8.1 is given by

$$\begin{aligned}
 w_2[n] &= \beta_3 w_4[n-1], \\
 w_3[n] &= \beta_2 w_5[n-1],
 \end{aligned}$$

$$\begin{aligned}
w_1[n] &= \alpha_1 \beta_1 w_1[n-1] - w_3[n] + x[n], \\
w_4[n] &= \alpha_3 w_2[n] + w_3[n], \\
w_5[n] &= \alpha_2 w_3[n] - w_2[n] + \beta_1 w_1[n-1], \\
y[n] &= \alpha_0 x[n] + \beta_1 w_1[n-1].
\end{aligned}$$

In matrix form the above set of equations is given by:

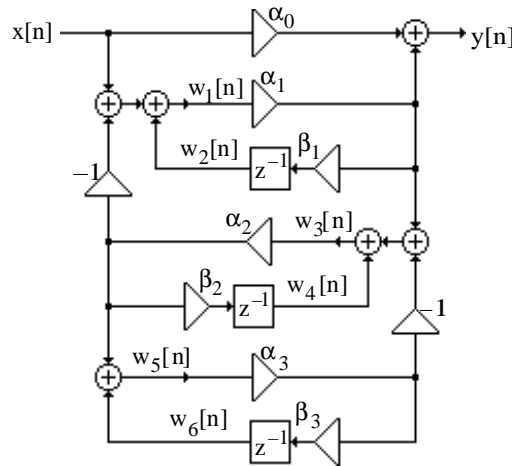
$$\begin{bmatrix} w_2[n] \\ w_3[n] \\ w_1[n] \\ w_4[n] \\ w_5[n] \\ y[n] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \alpha_3 & 1 & 0 & 0 & 0 & 0 \\ -1 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_2[n-1] \\ w_3[n-1] \\ w_1[n-1] \\ w_4[n-1] \\ w_5[n-1] \\ y[n-1] \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \beta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 & 0 \\ 0 & 0 & \alpha_1 \beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n-1] \\ w_2[n-1] \\ w_3[n-1] \\ w_4[n-1] \\ w_5[n-1] \\ y[n-1] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x[n] \\ 0 \\ 0 \\ \alpha_0 x[n] \end{bmatrix}$$

Here the  $\mathbf{F}$  matrix is given by

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ \alpha_3 & 1 & 0 & 0 & 0 & 0 \\ -1 & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the  $\mathbf{F}$  matrix does not contain nonzero entries above the main diagonal, the new set of equations are computable.

### 8.3



Analysis yields

$$\begin{aligned}
w_1[n] &= x[n] - \alpha_2 w_3[n] + w_2[n], \\
w_2[n] &= \alpha_1 \beta_1 w_1[n-1], \\
w_3[n] &= \alpha_1 w_1[n] - \alpha_3 w_5[n] + w_4[n], \\
w_4[n] &= \alpha_2 \beta_2 w_3[n-1], \\
w_5[n] &= \alpha_2 w_3[n] + w_6[n], \\
w_6[n] &= \alpha_3 \beta_3 w_5[n-1],
\end{aligned}$$

$$y[n] = \alpha_0 x[n] + \alpha_1 w_1[n].$$

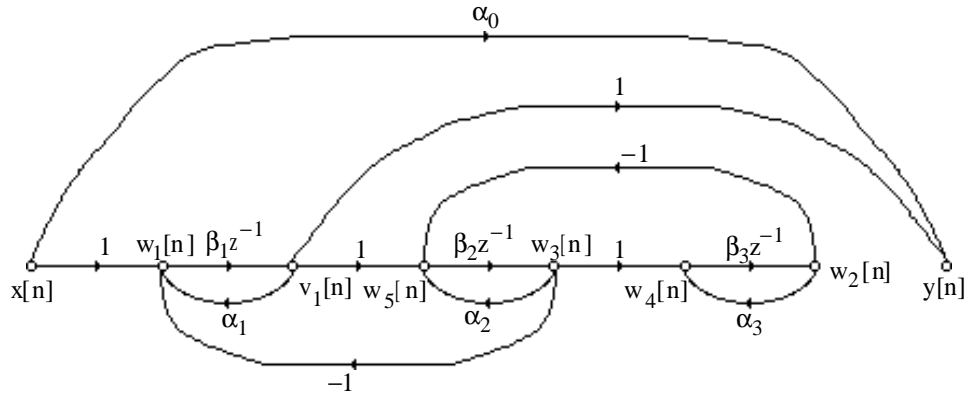
In matrix form

$$\begin{bmatrix} w_1[n] \\ w_2[n] \\ w_3[n] \\ w_4[n] \\ w_5[n] \\ w_6[n] \\ y[n] \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 1 & -\alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n] \\ w_2[n] \\ w_3[n] \\ w_4[n] \\ w_5[n] \\ w_6[n] \\ y[n] \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 \beta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 \beta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_3 \beta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1[n-1] \\ w_2[n-1] \\ w_3[n-1] \\ w_4[n-1] \\ w_5[n-1] \\ w_6[n-1] \\ y[n-1] \end{bmatrix} + \begin{bmatrix} x[n] \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_0 x[n] \end{bmatrix}$$

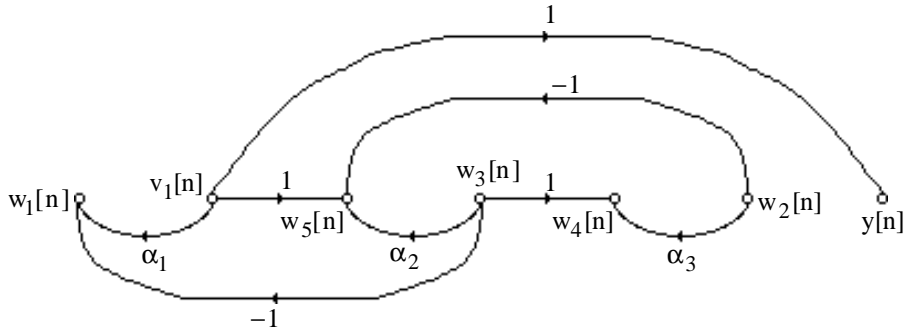
Here the  $\mathbf{F}$  matrix is given by  $\mathbf{F} = \begin{bmatrix} 0 & 1 & -\alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 1 & -\alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Since the  $\mathbf{F}$  matrix contains nonzero entries above the main diagonal, the above set of equations are not computable.

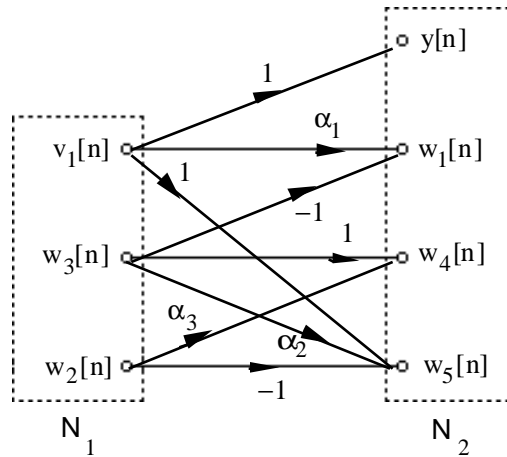
8.4 The signal-flow graph representation of the structure of Figure P8.1 is shown below:



The reduced signal-flow graph obtained by removing the branches going out of the input node and the delay branches is as indicated below:



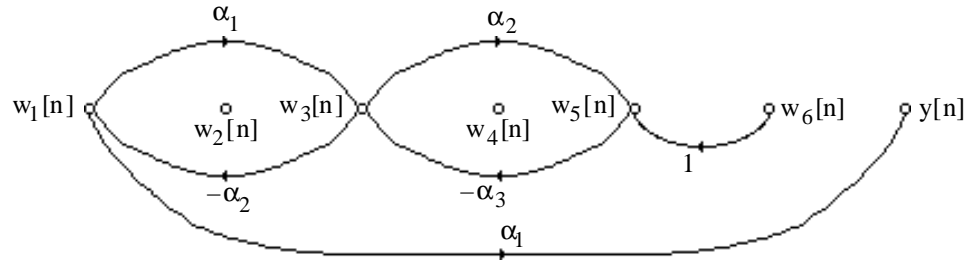
From the above signal-flow graph we arrive at its precedence graph shown below:



In the above precedence graph, the set  $N_1$  contains nodes with only outgoing branches and the final set  $N_2$  contains nodes with only incoming branches. As a result, the structure of Figure P8.1 has no delay-free loops. A valid computational algorithm by computing the node variables in set  $N_1$  first in any order followed by computing the node variables in set  $N_2$  in any order. For example, one valid computational algorithm is given by

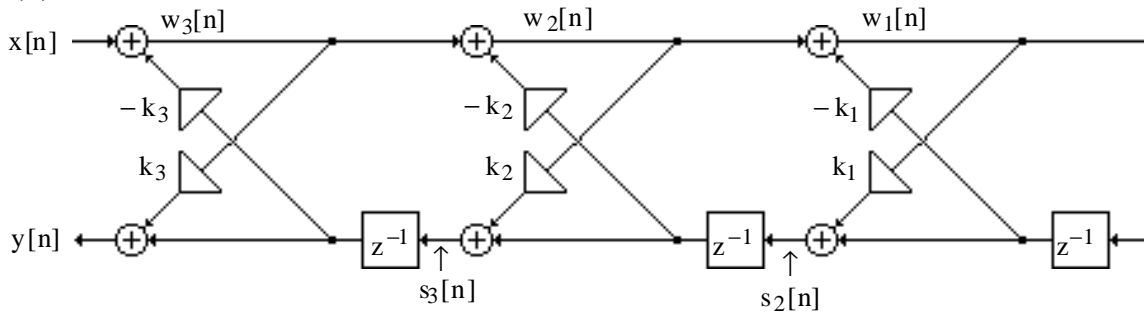
$$\begin{aligned}
 v_1[n] &= \beta_1 w_1[n-1], \\
 w_3[n] &= \beta_2 w_5[n-1], \\
 w_2[n] &= \beta_3 w_4[n-1], \\
 w_1[n] &= \alpha_1 v_1[n] - w_3[n] + x[n], \\
 w_4[n] &= \alpha_3 w_2[n] + w_3[n], \\
 w_5[n] &= \alpha_2 w_3[n] - w_2[n] + \beta_1 v_1[n], \\
 y[n] &= \alpha_0 x[n] + v_1[n].
 \end{aligned}$$

- 8.5** The reduced signal-flow graph obtained by removing the branches going out of the input node and the delay branches from the signal-flow graph representation of the structure of Figure P8.2 is as indicated below:



The only node with outgoing branch is  $w_6[n]$  and hence it is the only member of the set  $N_1$ . Since it is not possible to find a set of nodes  $N_2$  with incoming branches from  $N_1$  and all other branches being outgoing, the structure of Figure P8.2 has delay-free loops and is therefore not realizable.

**8.6 (a)**



Analysis of Figure P8.3 yields

$$\begin{aligned} w_1[n] &= w_2[n] - k_1 w_1[n-1], \\ w_2[n] &= w_3[n] - k_2 s_2[n-1], \\ w_3[n] &= x[n] - k_3 s_3[n-1], \\ s_2[n] &= k_1 w_1[n] + w_1[n-1], \\ s_3[n] &= k_2 w_2[n] + s_2[n-1], \\ y[n] &= k_3 w_3[n] + s_3[n-1]. \end{aligned}$$

In matrix form we have

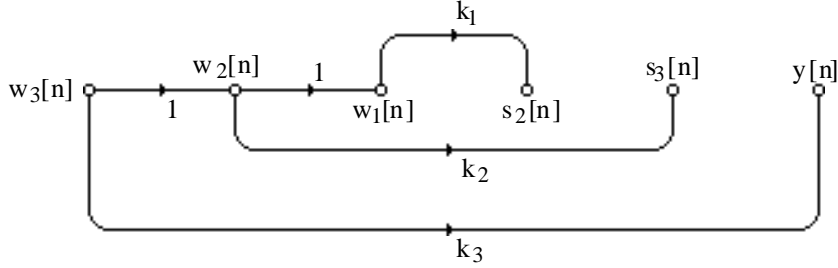
$$\begin{bmatrix} w_1[n] \\ w_2[n] \\ w_3[n] \\ s_2[n] \\ s_3[n] \\ y[n] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \mathbf{F} \begin{bmatrix} w_1[n-1] \\ w_2[n-1] \\ w_3[n-1] \\ s_2[n-1] \\ s_3[n-1] \\ y[n-1] \end{bmatrix} + \mathbf{G} \begin{bmatrix} x[n] \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\mathbf{F}$   $\mathbf{G}$

As the diagonal elements of the  $\mathbf{F}$ -matrix are all zeros, there are no delay-free loops.

However, the set of equations as ordered is not computable as there non-zero elements above the diagonal of  $\mathbf{F}$ .

**(b)** The reduced signal flow-graph representation of the structure of Figure P8.3 is shown below:



From the above flow-graph we observe that the set composed of nodes with out going branches is  $N_1 = \{w_3[n]\}$ . The set composed of nodes with out going branches and incoming branches from  $N_1$  is  $N_2 = \{w_2[n]\}$ . The set composed of nodes with out going branches and incoming branches from  $N_1$  and  $N_2$  is  $N_3 = \{w_1[n]\}$ . Finally, set composed of nodes with only incoming branches from  $N_1, N_2$  and  $N_3$  is  $N_4 = \{s_2[n], s_3[n], y[n]\}$ . Therefore, one possible ordered set of equations that is computable is given by

$$\begin{bmatrix} w_3[n] \\ w_2[n] \\ w_1[n] \\ s_2[n] \\ s_3[n] \\ y[n] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x[n] \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ k_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_3[n] \\ w_2[n] \\ w_1[n] \\ s_2[n] \\ s_3[n] \\ y[n] \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & -k_3 & 0 \\ 0 & 0 & 0 & -k_2 & 0 & 0 \\ 0 & 0 & -k_3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_3[n-1] \\ w_2[n-1] \\ w_1[n-1] \\ s_2[n-1] \\ s_3[n-1] \\ y[n-1] \end{bmatrix}$$

**F** **G**

Note that all elements on the diagonal and above diagonal are zeros.

**8.7**  $H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2}}{1 - 3z^{-1} - 5z^{-2}}$  and  $\{h[n]\} = \{2.1, 1.1, -3.2, -4.1, -28.3, \mathbf{L}\}$ ,  $n = 0, 1, \mathbf{K}$

From Eq. (8.15) we have  $\begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 2.1 & 0 & 0 \\ 1.1 & 2.1 & 0 \\ -3.2 & 1.1 & 2.1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} = \begin{bmatrix} 2.1 \\ -5.2 \\ -17 \end{bmatrix}$ . Hence,

$P(z) = 2.1 - 5.2z^{-1} - 17z^{-2}$ .

**8.8**  $H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2}}{1 + d_1 z^{-1} + d_2 z^{-2}}$  and  $\{h[n]\} = \{3, -2, -6, 12, -15\}$ ,  $0 \leq n \leq 4$ . The equation

corresponding to Eq. (8.12) is  $\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & 0 \\ -6 & -2 & 3 \\ 12 & -6 & -2 \\ -15 & 12 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ d_1 \\ d_2 \end{bmatrix}$ . Therefore, from EQ. (8.17) we

have  $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = -\begin{bmatrix} -6 & -2 \\ 12 & -6 \end{bmatrix}^{-1} \begin{bmatrix} 12 \\ -15 \end{bmatrix} = \begin{bmatrix} 1.7 \\ 0.9 \end{bmatrix}$ , and from Eq. (8.18) we have

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & 0 \\ -6 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1.7 \\ 0.9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.1 \\ -6.7 \end{bmatrix}. \text{ Hence, } H(z) = \frac{3 + 3.1z^{-1} - 6.7z^{-2}}{1 + 1.7z^{-1} + 0.9z^{-2}}.$$

**8.9**  $H(z) = \frac{p_0 + p_1z^{-1} + p_2z^{-2} + p_3z^{-3}}{1 + d_1z^{-1} + d_2z^{-2} + d_3z^{-3}}$  and

$\{h[n]\} = \{2, -5, 6, -2, -9, 18, -7, -31, 65, -30, \mathbf{L}\}$ ,  $n \geq 0$ . The equation

corresponding to Eq. (8.17) is given by  $\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -5 & 2 & 0 & 0 \\ 6 & -5 & 2 & 0 \\ -2 & 6 & -5 & 2 \\ -9 & -2 & 6 & -5 \\ 18 & -9 & -2 & 6 \\ -7 & 18 & -9 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix}$ . Therefore, from Eq.

(8.17) we have  $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = -\begin{bmatrix} -2 & 6 & -5 \\ -9 & -2 & 6 \\ 18 & -9 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -9 \\ 18 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$  and from Eq. (8.18) we have

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -5 & 2 & 0 & 0 \\ 6 & -5 & 2 & 0 \\ -2 & 6 & -5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ -3 \end{bmatrix}. \text{ Hence, } H(z) = \frac{2 - z^{-1} + 2z^{-2} - 3z^{-3}}{1 + 2z^{-1} + 3z^{-2} + z^{-3}}.$$

**8.10**  $G(z) = \frac{p_0 + p_1z^{-1} + p_2z^{-2} + p_3z^{-3}}{2 - 6z^{-1} + 8z^{-2} + 10z^{-3}}$  with  $g[n] = \{3, 7, 13, -3\}$ ,  $0 \leq n \leq 3$ . Therefore, from

Eq. (8.18) we have  $\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 13 & 7 & 3 & 0 \\ -3 & 13 & 7 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 8 \\ 2 \end{bmatrix}$ . Hence,  $G(z) = \frac{6 - 4z^{-1} + 8z^{-2} + 2z^{-3}}{2 - 6z^{-1} + 8z^{-2} + 10z^{-3}}$ .

**8.11**  $H(z) = \frac{p_0 + p_1z^{-1} + p_2z^{-2} + p_3z^{-3} + p_4z^{-4}}{1 + 2z^{-1} + 2z^{-2} + 3z^{-3} + 3z^{-4}}$  with

$\{h[n]\} = \{2, 0, -5, -10, -10, 55, -45, 40, -125, 140, -15\}$   $0 \leq n \leq 10$

The equation corresponding to Eq. (8.18) is given by

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ -5 & 0 & 2 & 0 & 0 \\ -10 & -5 & 0 & 2 & 0 \\ -10 & -10 & -5 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -1 \\ -14 \\ -34 \end{bmatrix}.$$

Therefore,  $H(z) = \frac{2 + 4z^{-1} - z^{-2} - 14z^{-3} - 34z^{-4}}{1 + 2z^{-1} + 2z^{-2} + 3z^{-3} + 3z^{-4}}$ .

**8.12** The  $k$ -th sample of an  $N$ -point DFT is given by  $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$ . Thus the computation of

$X[k]$  requires  $N$  complex multiplications and  $N-1$  complex additions. Now each complex multiplication, in turn, requires 4 real multiplications, and 2 real additions. Likewise, each complex addition requires 2 real additions. As a result, the  $N$  complex multiplications needed to compute  $X[k]$  require a total of  $4N$  real multiplications and a total of  $2N$  real additions. Similarly, the  $N-1$  complex additions needed in the computation of  $X[k]$  require a total of  $2N-2$  real additions. Hence, each sample of the  $N$ -point DFT involves  $4N$  real multiplications and  $4N-2$  real additions. The computation of all  $N$  DFT samples thus requires  $4N^2$  real multiplications and  $(4N-2)N$  real additions.

**8.13** Let the two complex numbers be  $\alpha = a + jb$  and  $\beta = c + jd$ . Then,  $\alpha\beta = (a + jb)(c + jd) = (ac - bd) + j(ad + bc)$  which requires 4 real multiplications and 2 real additions. Consider the products  $(a + b)(c + d)$ ,  $ac$  and  $bd$  which require 3 real multiplications and 2 real additions. The imaginary part of  $\alpha\beta$  can be formed from  $(a + b)(c + d) - ac - bd = ad + bc$  which now requires 2 real additions. Likewise, the real part of  $\alpha\beta$  can be formed by forming  $ac - bd$  requiring an additional real addition. Hence, the complex multiplication  $\alpha\beta$  can be computed using 3 real multiplications and 5 real additions.

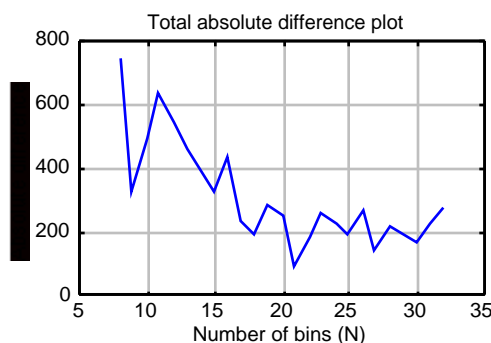
**8.14** The center frequency of bin  $k$ :  $f_c(k) = \frac{kF_T}{N}$  where  $N = \#$  of bins, and  $F_T$  is the sampling frequency. Inverting we have  $k(f) = \left\lfloor \frac{fN}{F_T} \right\rfloor$ . Therefore, the absolute difference from one of the given four tones (150 Hz, 375 Hz, 620 Hz, and 850 Hz) to the center of its bin is given by

$\text{dist}(N, f) = \left| f - \frac{F_T}{N} \left\lfloor \frac{fN}{F_T} \right\rfloor \right|$ . It follows from this equation that the distance goes to zero if

$$\frac{F_T}{N} \left\lfloor \frac{fN}{F_T} \right\rfloor = f \text{ or } \frac{fN}{F_T} \text{ is an integer.}$$

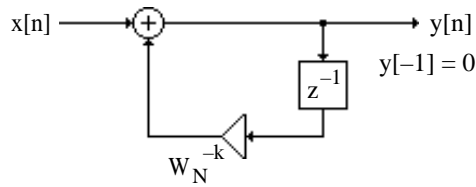
The total distance is reduced to zero if  $\frac{f_i N}{F_T}$  is an integer for  $i = 1, 2, 3, 4$ . The minimum  $N$  for which this is true is 500.

However, the total distance can be small, but nonzero, for significantly smaller values of  $N$ . The MATLAB generated plot given below shows a nice minimum at  $N = 21$ .





**8.15**



$$H_k(z) = \frac{1}{1 - W_N^{-k} z^{-1}}. \quad \text{Hence, } Y(z) = \frac{X(z)}{1 - W_N^{-k} z^{-1}} = \frac{1 + z^{-N/2}}{1 - W_N^{-k} z^{-1}}.$$

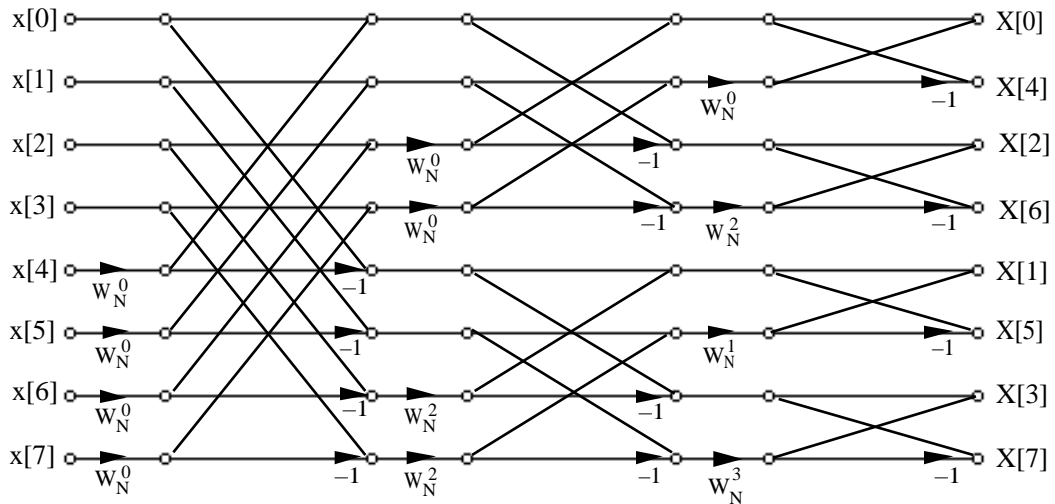
$$\text{For } k = 1, Y(z) = \frac{1 + z^{-N/2}}{1 - W_N^{-1} z^{-1}} = (1 + z^{-N/2})(1 + W_N^{-1} z^{-1} + W_N^{-2} z^{-2} + \dots + W_N^{-(N/2)} z^{-N/2} + \dots)$$

$$\text{Hence, } y[n] = \{1, W_N^{-1}, W_N^{-2}, \dots, W_N^{-(N-2)/2}, 1 + W_N^{-N/2}, \dots, 1 + W_N^{-(N-1)}, \dots\}$$

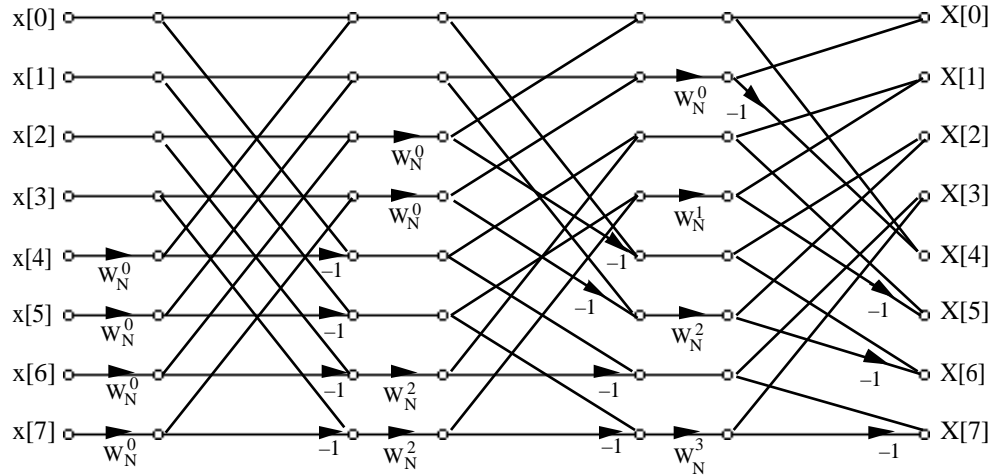
$$\text{For } k = N/2, Y(z) = \frac{1 + z^{-N/2}}{1 - W_N^{-N/2} z^{-1}}. \quad \text{Hence,}$$

$$y[n] = \{1, -1, 1, -1, \dots, -1, 2, 0, 2, 0, \dots, 0\}$$

**8.16**



**8.17**



**8.18** 
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

$$= \sum_{n=0}^{r_1-1} x[nr_1] W_N^{nkr_1} + \sum_{n=0}^{r_1-1} x[nr_1+1] W_N^{nkr_1+k} + \dots + \sum_{n=0}^{r_1-1} x[nr_1+r_1-1] W_N^{nkr_1+(r_1-1)k}$$

$$= \sum_{i=0}^{r_1-1} \sum_{n=0}^{r_1-1} x[nr_1+i] W_N^{nkr_1+i} \cdot W_N^{ki}$$

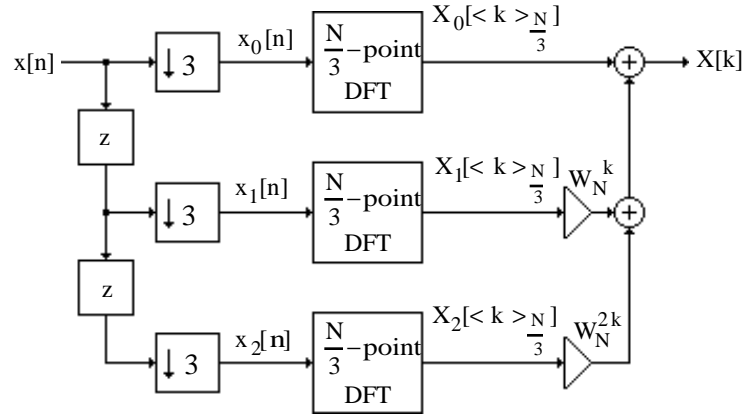
(N/r<sub>1</sub>)-point DFT

Thus, if the  $(N/r_1)$ -point DFT has been calculated, we need at the first stage an additional  $(r_1 - 1)$  multiplications to compute one sample the  $N$ -point DFT  $X[k]$  and as a result, additional  $(r_1 - 1)N$  multiplications are required to compute all  $N$  samples of the  $N$ -point DFT. Decomposing it further, it follows then that additional  $(r_2 - 1)N$  multiplications are needed at the second stage, and so on. Therefore,

Total number of multiply (add) operations =  $(r_1 - 1)N + (r_2 - 1)N + \dots + (r_v - 1)N$

$$= \left( \sum_{i=1}^v r_i - v \right) N.$$

**8.19**  $X(z) = X_0(z^3) + z^{-1}X_1(z^3) + z^{-2}X_2(z^3)$ . Thus, the  $N$ -point DFT can be expressed as  $X[k] = X_0[\langle k \rangle_{N/3}] + W_N^k X_1[\langle k \rangle_{N/3}] + W_N^{2k} X_2[\langle k \rangle_{N/3}]$ . Hence, the structural interpretation of the first stage of the radix-3 DFT is as indicated below:



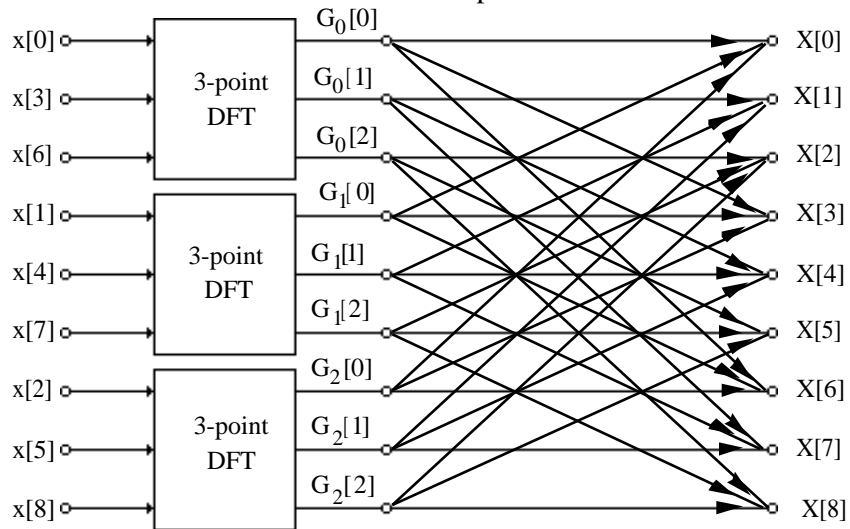
**8.20** 
$$X[k] = \sum_{n=0}^8 x[n] W_9^{nk} = (x[0]W_9^{0 \cdot k} + x[3]W_9^{3k} + x[6]W_9^{6k})$$

$$+ (x[1]W_9^k + x[4]W_9^{4k} + x[7]W_9^{7k}) + (x[2]W_9^{2k} + x[5]W_9^{5k} + x[8]W_9^{8k})$$

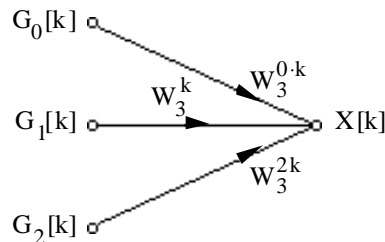
$$= (x[0]W_3^{0 \cdot k} + x[3]W_3^k + x[6]W_3^{2k}) + (x[1]W_3^{0 \cdot k} + x[4]W_3^k + x[7]W_3^{2k})W_9^k$$

$$+ (x[2]W_3^{0 \cdot k} + x[5]W_3^k + x[8]W_3^{2k})W_9^{2k} = G_0[<k >_3] + G_1[<k >_3]W_9^k + G_2[<k >_3]W_9^{2k}$$

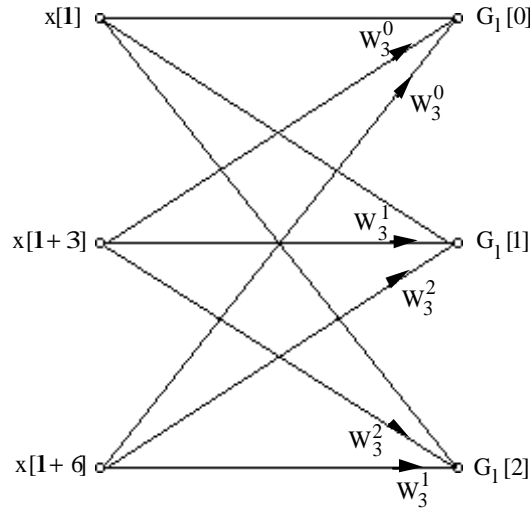
where  $G_0[<k >_3] = x[0]W_3^{0 \cdot k} + x[3]W_3^k + x[6]W_3^{2k}$ ,  $G_1[<k >_3] = x[1]W_3^{0 \cdot k} + x[4]W_3^k + x[7]W_3^{2k}$ ,  
**and**  $G_2[<k >_3] = x[2]W_3^{0 \cdot k} + x[5]W_3^k + x[8]W_3^{2k}$ , are three 3-point DFTs. A flow-graph representation of this DIT mixed-radix DFT computation scheme is shown below:



where the twiddle factors for computing the DFT samples are indicated below for a typical DFT sample:



In the above diagram, the 3-point DFT computation is carried out as indicated below:



$$\mathbf{8.21} \quad X[k] = \sum_{n=0}^{14} x[n] W_{15}^{nk} = \left( x[0] W_{15}^{0 \cdot k} + x[3] W_{15}^{3k} + x[6] W_{15}^{6k} + x[9] W_{15}^{9k} + x[12] W_{15}^{12k} \right)$$

$$+ \left( x[1] W_{15}^k + x[4] W_{15}^{4k} + x[7] W_{15}^{7k} + x[10] W_{15}^{10k} + x[13] W_{15}^{13k} \right)$$

$$+ \left( x[2] W_{15}^{2k} + x[5] W_{15}^{5k} + x[8] W_{15}^{8k} + x[11] W_{15}^{11k} + x[14] W_{15}^{14k} \right)$$

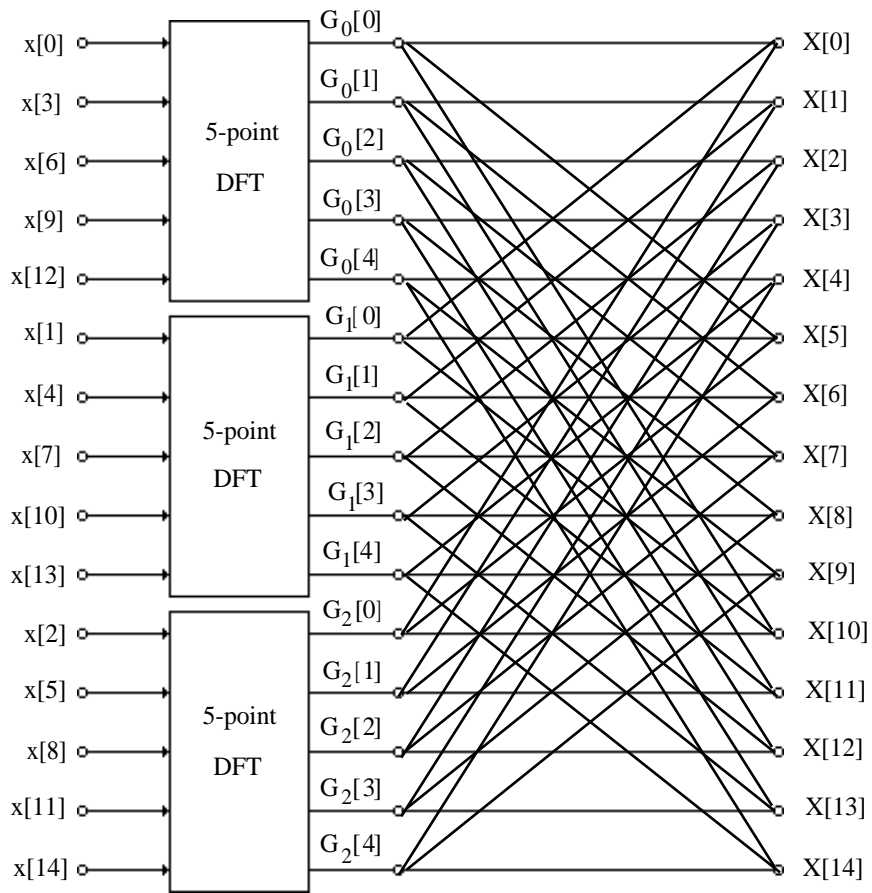
$$= G_0[\langle k \rangle_5] + G_1[\langle k \rangle_5] W_{15}^k + G_2[\langle k \rangle_5] W_{15}^{2k}, \text{ where}$$

$$G_0[\langle k \rangle_5] = x[0] W_5^{0 \cdot k} + x[3] W_5^k + x[6] W_5^{2k} + x[9] W_5^{3k} + x[12] W_5^{4k},$$

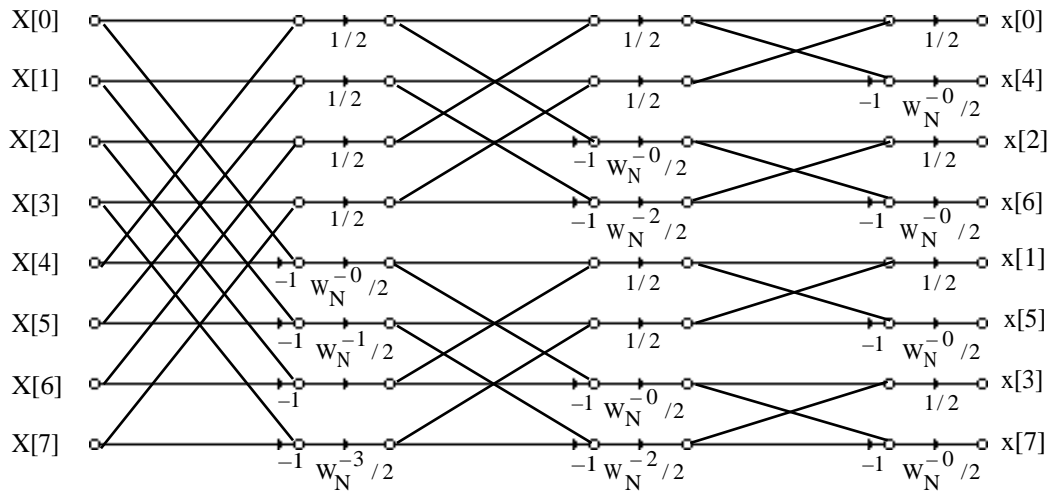
$$G_1[\langle k \rangle_5] = x[1] W_5^{0 \cdot k} + x[4] W_5^k + x[7] W_5^{2k} + x[10] W_5^{3k} + x[13] W_5^{4k}, \text{ and}$$

$$G_2[\langle k \rangle_5] = x[2] W_5^{0 \cdot k} + x[5] W_5^k + x[8] W_5^{2k} + x[11] W_5^{3k} + x[14] W_5^{4k}.$$

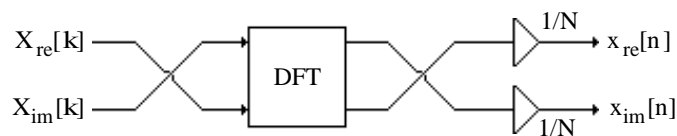
A flow-graph representation of this DIT mixed-radix DFT computation scheme is shown below:



**8.22**



**8.23**



Now, by definition,  $q[n] = \text{Im}\{X[n]\} + j\text{Re}\{X[n]\}$ . Its  $N$ -point DFT is  $Q[k] = \sum_{n=0}^{N-1} q[n] W_N^{nk}$ .

$$\text{Thus, } \text{Re}\{Q[k]\} = \sum_{m=0}^{N-1} \left( \text{Im}\{X[m]\} \cos\left(\frac{2\pi mk}{N}\right) + \text{Re}\{X[m]\} \sin\left(\frac{2\pi mk}{N}\right) \right), \quad (18)$$

$$\text{Im}\{Q[k]\} = \sum_{m=0}^{N-1} \left( -\text{Im}\{X[m]\} \sin\left(\frac{2\pi mk}{N}\right) + \text{Re}\{X[m]\} \cos\left(\frac{2\pi mk}{N}\right) \right), \quad (19)$$

From the definition of the inverse DFT we observe  $x[k] = \frac{1}{N} \sum_{m=0}^{N-1} X[m] W_N^{-mk}$ . Hence,

$$\text{Re}\{x[k]\} = \frac{1}{N} \sum_{m=0}^{N-1} \left( \text{Re}\{X[m]\} \cos\left(\frac{2\pi mk}{N}\right) - \text{Im}\{X[m]\} \sin\left(\frac{2\pi mk}{N}\right) \right), \quad (20)$$

$$\text{Im}\{x[k]\} = \frac{1}{N} \sum_{m=0}^{N-1} \left( \text{Im}\{X[m]\} \cos\left(\frac{2\pi mk}{N}\right) + \text{Re}\{X[m]\} \sin\left(\frac{2\pi mk}{N}\right) \right). \quad (21)$$

Comparing Eqs. (19) and (20), we get  $\text{Re}\{x[n]\} = \frac{1}{N} \text{Im}\{Q[k]\} \Big|_{k=n}$ ,

and comparing Eqs. (18) and (21) we get  $\text{Im}\{x[n]\} = \frac{1}{N} \text{Re}\{Q[k]\} \Big|_{k=n}$ .

**8.24**  $r[n] = X[\langle -n \rangle_N] = \begin{cases} X[0], & \text{if } n = 0, \\ X[N-n], & \text{if } n \neq 0. \end{cases}$  Therefore,

$$\begin{aligned} R[k] &= \sum_{n=0}^{N-1} r[n] W_N^{nk} = r[0] + \sum_{n=1}^{N-1} r[n] W_N^{nk} = X[0] + \sum_{n=1}^{N-1} X[N-n] W_N^{nk} \\ &= X[0] + \sum_{n=1}^{N-1} X[n] W_N^{(N-n)k} = X[0] + \sum_{n=1}^{N-1} X[n] W_N^{-nk} = \sum_{n=0}^{N-1} X[n] W_N^{-nk} = N x[k]. \end{aligned}$$

Thus,  $x[n] = \frac{1}{N} \cdot R[k] \Big|_{k=n}$ .

**8.25** Let  $y[n]$  denote the result of convolving a length- $L$  sequence  $x[n]$  with a length- $N$  sequence  $h[n]$ . The length of  $y[n]$  is then  $L + N - 1$ . Here  $L = 8$  and  $N = 5$ , hence length of  $y[n]$  is 12.

Method #1: Direct linear convolution - For a length- $L$  sequence  $x[n]$  and a length- $N$   $h[n]$ ,

$$\# \text{ of real mult.} = 2 \left( \sum_{n=1}^N n \right) + N(L - N - 1) = 2 \left( \sum_{n=1}^5 n \right) + 5(8 - 5 - 1) = 40.$$

Method #2: Linear convolution via circular convolution - Since  $y[n]$  is of length 12, to get the correct result we need to pad both sequences with zeros to increase their lengths to 12 before carrying out the circular convolution.

# of real mult. =  $12 \times 12 = 144$ .

Method #3: Linear convolution via radix-2 FFT - The process involves computing the 16-point FFT  $G[k]$  of the length-16 complex sequence  $g[n] = x_e[n] + j h_e[n]$  where  $x_e[n]$  and  $h_e[n]$  are length-16 sequences obtained by zero-padding  $x[n]$  and  $h[n]$ , respectively. Then recovering the 16-point DFTs,  $X_e[k]$  and  $H_e[k]$ , of  $x_e[n]$  and  $h_e[n]$ , respectively, from  $G[k]$ . Finally, the IDFT of the product  $Y[k] = X_e[k] \cdot H_e[k]$  yields  $y[n]$ .

Now, the first stage of the 16-point radix-2 FFT requires 0 complex multiplications, the second stage requires 0 complex multiplications, the third stage requires 4 complex multiplications, and the last stage requires 6 complex multiplications resulting in a total of 10 complex multiplications.

# of complex mult. to implement  $G[k] = 10$

# of complex mult. to recover  $X_e[k]$  and  $H_e[k]$  from  $G[k] = 0$

# of complex mult. to form  $Y[k] = X_e[k] \cdot H_e[k] = 16$

# of complex mult. to form the IDFT of  $Y[k] = 10$

Hence, the total number of complex mult. = 36

A direct implementation of a complex multiplication requires 4 real multiplications resulting in a total of  $4 \times 36 = 144$  real multiplications for Method #3. However, if a complex multiply can be implemented using 3 real multiplies (see Problem 8.13), in which case Method #3 requires a total of  $3 \times 36 = 108$  real multiplications.

**8.26** Method #1: Total # of real multiplications required

$$= 2 \left( \sum_{n=1}^N n \right) + N(L - N - 1) = 2 \left( \sum_{n=1}^6 n \right) + 6(8 - 6 - 1) = 48.$$

Method #2: Total # of real multiplications required =  $13^2 = 169$ .

Method #3: Linear convolution via radix-2 FFT - The process involves computing the 16-point FFT  $G[k]$  of the length-16 complex sequence  $g[n] = x_e[n] + j h_e[n]$  where  $x_e[n]$  and  $h_e[n]$  are length-16 sequences obtained by zero-padding  $x[n]$  and  $h[n]$ , respectively. Then recovering the 16-point DFTs,  $X_e[k]$  and  $H_e[k]$ , of  $x_e[n]$  and  $h_e[n]$ , respectively, from  $G[k]$ . Finally, the IDFT of the product  $Y[k] = X_e[k] \cdot H_e[k]$  yields  $y[n]$ .

Now, the first stage of the 16-point radix-2 FFT requires 0 complex multiplications, the second stage requires 0 complex multiplications, the third stage requires 4 complex multiplications, and the last stage requires 6 complex multiplications resulting in a total of 10 complex multiplications.

# of complex mult. to implement  $G[k] = 10$

# of complex mult. to recover  $X_e[k]$  and  $H_e[k]$  from  $G[k] = 0$

# of complex mult. to form  $Y[k] = X_e[k] \cdot H_e[k] = 16$

# of complex mult. to form the IDFT of  $Y[k] = 10$

Hence, the total number of complex mult. = 36

A direct implementation of a complex multiplication requires 4 real multiplications resulting in a total of  $4 \times 36 = 144$  real multiplications for Method #3. However, if a complex multiply can be implemented using 3 real multiplies (see Problem 8.13), in which case Method #3 requires a total of  $3 \times 36 = 108$  real multiplications.

- 8.27 (a)** Since the impulse response of the filter is of length 34, the transform length  $N$  should be greater than 34. If  $L$  denotes the number of input samples used for convolution, then  $L = N - 33$ . So for every  $L$  samples of the input sequence, an  $N$ -point DFT is computed and multiplied with an  $N$ -point DFT of the impulse response sequence  $h[n]$  (which needs to be computed only once), and finally an  $N$ -point inverse of the product sequence is evaluated. Hence, the total number  $R_M$  of complex multiplications required (assuming  $N$  is a power-of-2) is given by

$$R_M = \left\lceil \frac{1024}{N-33} \right\rceil (N \log_2 N + N) + \frac{N}{2} \log_2 N$$

It should be noted that in developing the above expression, multiplications due to twiddle factors of values  $\pm 1$  and  $\pm j$  have not been excluded. The values of  $R_M$  for different values of  $N$  are as follows:

$$\text{For } N = 64, \quad R_M = 15,424$$

$$\text{For } N = 128, \quad R_M = 11,712$$

$$\text{for } N = 256, \quad R_M = 12,544$$

$$\text{for } N = 512, \quad R_M = 17,664$$

Hence,  $N = 128$  is the appropriate choice for the transform length requiring 14,848 complex multiplications or equivalently,  $11,712 \times 3 = 35,136$  real multiplications.

Since the first stage of the FFT calculation process requires only multiplications by  $\pm 1$ , the total number of complex multiplications for  $N = 128$  is actually

$$R_M = \left\lceil \frac{1024}{N-33} \right\rceil (N \log_2 N + N) + \frac{N}{2} \log_2 N - \frac{N}{2} = 11,648$$

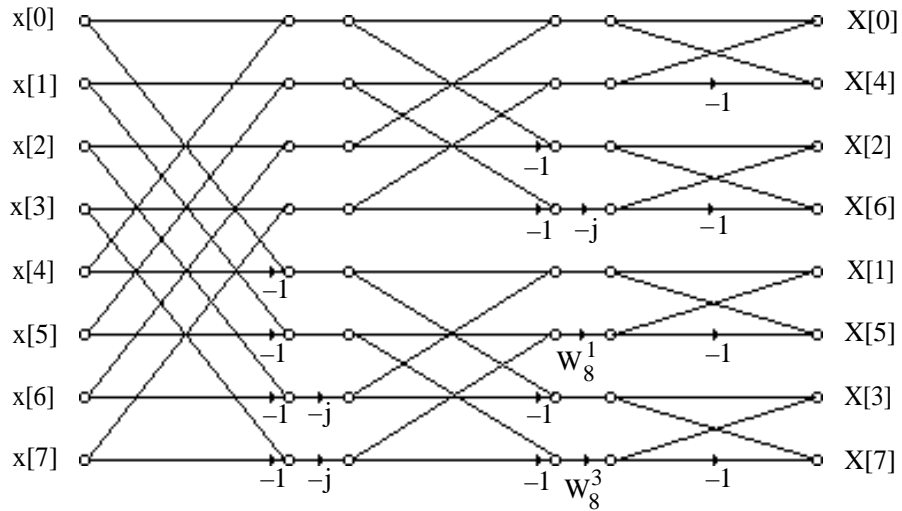
or equivalently,  $11,648 \times 3 = 34,944$  real multiplications.

- (b)** For direct convolution, # of real multiplications =

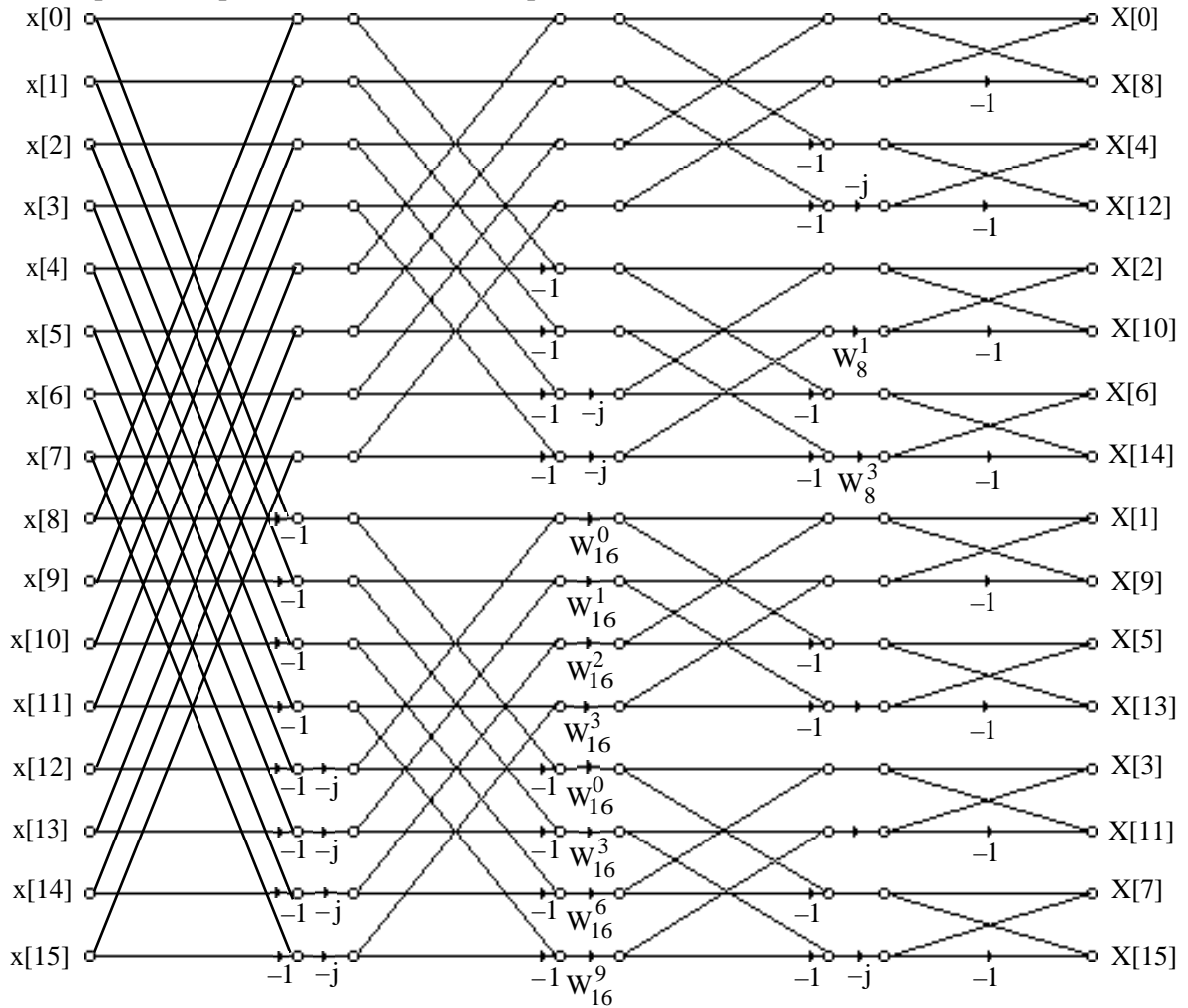
$$2 \left( \sum_{n=1}^N n \right) + N(L - N - 1) = 2 \left( \sum_{n=1}^{34} n \right) + 34(1024 - 34 - 1) = 34,816.$$

- 8.28** From the flow-graph of the 8-point split-radix FFT algorithm given below it can be seen that the total number of complex multiplications required is 2. On the other hand, the total number of complex multiplications required for a standard DIF FFT algorithm is also 2.





**8.29** If multiplications by  $\pm j$ ,  $\pm 1$  are ignored, the flow-graph shown below requires 8 complex = 24 real multiplications. A radix-2 DIF 16-point FFT algorithm, on the other hand, requires 10 complex multiplications = 30 real multiplications.



**8.30 (a)** From Eq. (8.97a),  $x[n] = x[n_1 + N_1 n_2]$  where  $0 \leq n_1 \leq N_1 - 1$ , and  $0 \leq n_2 \leq N_2 - 1$ . Now,

using Eq. (8.97b) we can rewrite  $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$ ,  $0 \leq k \leq N-1$ , as

$$\begin{aligned} X[N_2 k_1 + k_2] &= \sum_{n=0}^{N_1 N_2 - 1} x[n] W_N^{n(N_2 k_1 + k_2)} = \sum_{n_1=0}^{N_1 - 1} \sum_{n_2=0}^{N_2 - 1} x[n_1 + N_1 n_2] W_N^{(n_1 + N_1 n_2)(N_2 k_1 + k_2)} \\ &= \sum_{n_1=0}^{N_1 - 1} \sum_{n_2=0}^{N_2 - 1} x[n_1 + N_1 n_2] W_N^{n_1 N_2 k_1} W_N^{N_1 n_2 N_2 k_1} W_N^{N_1 n_2 k_2} W_N^{n_1 k_2}. \end{aligned}$$

Since,  $W_N^{n_1 N_2 k_1} = W_{N_1}^{n_1 k_1}$ ,  $W_N^{N_1 n_2 N_2 k_1} = 1$ , and  $W_N^{N_1 n_2 k_2} = W_{N_2}^{n_2 k_2}$ , we get

$$X[k] = X[N_2 k_1 + k_2] = \sum_{n_1=0}^{N_1 - 1} \left[ \left( \sum_{n_2=0}^{N_2 - 1} x[n_1 + N_1 n_2] W_{N_2}^{n_2 k_2} \right) W_N^{n_1 k_2} \right] W_{N_1}^{n_1 k_1}. \quad (22)$$

**(b)** For  $N_1 = 2$  and  $N_2 = N/2$ , the above DFT computation scheme leads to

$$\begin{aligned} X[k] &= X\left[\frac{N}{2} k_1 + k_2\right] = \sum_{n_1=0}^1 \left[ \left( \sum_{n_2=0}^{\frac{N}{2} - 1} x[n_1 + 2n_2] W_{N/2}^{n_2 k_2} \right) W_N^{n_1 k_2} \right] W_2^{n_1 k_1} \\ &= \sum_{n_2=0}^{\frac{N}{2} - 1} x[2n_2] W_{N/2}^{n_2 k_2} + W_{N_1}^{k_1} W_N^{k_2} \sum_{n_2=0}^{\frac{N}{2} - 1} x[2n_2 + 1] W_{N/2}^{n_2 k_2} \\ &= \sum_{n_2=0}^{\frac{N}{2} - 1} x[2n_2] W_{N/2}^{n_2 k_2} + W_N^k \sum_{n_2=0}^{\frac{N}{2} - 1} x[2n_2 + 1] W_{N/2}^{n_2 k_2} \end{aligned}$$

which is seen to be the first stage in the DIT DFT computation.

On the other hand, for  $N_1 = N/2$  and  $N_2 = 2$ , the above DFT computation scheme leads to

$$\begin{aligned} X[k] &= X[2k_1 + k_2] = \sum_{n_1=0}^{\frac{N}{2} - 1} \left[ \left( \sum_{n_2=0}^1 x[n_1 + \frac{N}{2} n_2] W_2^{n_2 k_2} \right) W_N^{n_1 k_2} \right] W_{N/2}^{n_1 k_1} \\ &= \sum_{n_1=0}^{\frac{N}{2} - 1} \left( x[n_1] + x[n_1 + \frac{N}{2}] (-1)^{k_2} \right) W_N^{n_1 k_2} W_{N/2}^{n_1 k_1} \\ &= \sum_{n_1=0}^{\frac{N}{2} - 1} \left\{ \left( x[n_1] + (-1)^{k_2} x[n_1 + \frac{N}{2}] \right) W_N^{n_1 k_2} \right\} W_{N/2}^{n_1 k_1} \end{aligned}$$

which represents the first stage of the DIF FFT algorithm.

(c) In the DFT computation scheme of Eq. (22), we first compute a total of  $N_2 N_1$ -point DFTs, multiply all the  $N_1 N_2 = N$  computed DFT coefficients by the twiddle factors  $W_N^{n_1 k_2}$ , and finally calculate a total of  $N_2 N_1$ -point DFTs. If  $R(N)$  denotes the total number of multiplications needed to compute an  $N$ -point DFT, then the total number of multiplications required in the DFT computation scheme of Eq. (22) is given by

- (i)  $N_2 \cdot R(N_1)$  for the first step,
- (ii)  $N_2 N_1 = N$  for multiplications by the twiddle factors, and
- (iii)  $N_1 \cdot R(N_2)$  for the last step.

Therefore,  $R(N) = N_2 \cdot R(N_1) + N + N_1 \cdot R(N_2) = N \left( \frac{1}{N_1} R(N_1) + \frac{1}{N_2} R(N_2) + 1 \right)$ .

(d) For  $N = 2^v$ , choose  $N_i = 2$ ,  $i = 1, 2, \dots, v$ . Now from Figure 8.24 for a 2-point DFT  $R(N_i) = 2$ . Hence,  $R(N) = N \left( \frac{v}{2} \right) = \frac{N}{2} \log_2 N$ .

**8.28 (a)**

$$\mathbf{V}_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & W_8^0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_8^1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_8^3 \\ 1 & 0 & 0 & 0 & W_8^4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & W_8^5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & W_8^6 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & W_8^7 \end{bmatrix}, \quad \mathbf{V}_4 = \begin{bmatrix} 1 & 0 & W_8^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & W_8^2 & 0 & 0 & 0 & 0 \\ 1 & 0 & W_8^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & W_8^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & W_8^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & W_8^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & W_8^0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & W_8^2 \end{bmatrix},$$

$$\mathbf{V}_2 = \begin{bmatrix} 1 & W_8^0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & W_8^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & W_8^0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & W_8^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & W_8^0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & W_8^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & W_8^0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & W_8^4 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

As can be seen from the above, multiplication by each matrix  $\mathbf{V}_k$ ,  $k = 1, 2, 3$ , requires at most 8 complex multiplications.

(b) The transpose of the matrices given in part (a) are as follows:

$$\mathbf{V}_8^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ W_8^0 & 0 & 0 & 0 & W_8^4 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 & 0 & W_8^5 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 & 0 & 0 & W_8^6 & 0 \\ 0 & 0 & 0 & W_8^3 & 0 & 0 & 0 & W_8^7 \end{bmatrix}, \quad \mathbf{V}_4^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ W_8^0 & 0 & W_8^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & W_8^2 & 0 & W_8^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & W_8^0 & 0 & W_8^0 & 0 \\ 0 & 0 & 0 & 0 & 0 & W_8^2 & 0 & W_8^2 \end{bmatrix},$$

$$\mathbf{V}_2^T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ W_8^0 & W_8^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & W_8^0 & W_8^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & W_8^0 & W_8^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & W_8^0 & W_8^4 \end{bmatrix}, \quad \mathbf{E}^T = \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to show that the flow-graph representation of  $\mathbf{D}_8 = \mathbf{E}^T \mathbf{V}_2^T \mathbf{V}_4^T \mathbf{V}_8^T$  is precisely the 8-point DIF FFT algorithm of Figure 8.28.

$$8.29 \quad X[2l] = \sum_{n=0}^{N-1} x[n] W_N^{2ln} = \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2ln} + \sum_{n=N/2}^{N-1} x[n] W_N^{2ln}, \quad l=0,1,\dots,\frac{N}{2}-1.$$

Replacing  $n$  by  $n + \frac{N}{2}$  in the right-most sum we get

$$X[2l] = \sum_{n=0}^{\frac{N}{2}-1} x[n] W_N^{2ln} + \sum_{n=0}^{\frac{N}{2}-1} x[n + \frac{N}{2}] W_N^{2ln} W_N^{Nl} = \sum_{n=0}^{\frac{N}{2}-1} \left\{ x[n] + x[n + \frac{N}{2}] \right\} W_N^{ln}, \quad 0 \leq l \leq \frac{N}{2} - 1.$$

$$X[4l+1] = \sum_{n=0}^{\frac{N}{4}-1} x[n] W_N^{(4l+1)n} + \sum_{n=\frac{N}{4}}^{\frac{N}{2}-1} x[n] W_N^{(4l+1)n} + \sum_{n=\frac{N}{2}}^{\frac{3N}{4}-1} x[n] W_N^{(4l+1)n} + \sum_{n=\frac{3N}{4}}^{N-1} x[n] W_N^{(4l+1)n},$$

where  $0 \leq l \leq \frac{N}{4} - 1$ . Replacing  $n$  by  $n + \frac{N}{4}$  in the second sum,  $n$  by  $n + \frac{N}{2}$  in the third sum,

and  $n$  by  $n + \frac{3N}{4}$  in the fourth sum, we get

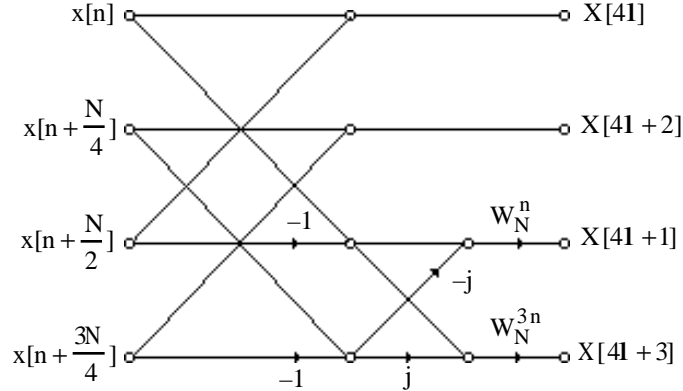
$$X[4l+1] = \sum_{n=0}^{\frac{N}{4}-1} x[n] W_N^{4ln} W_N^n + \sum_{n=0}^{\frac{N}{4}-1} x[n + \frac{N}{4}] W_N^{4ln} W_N^n W_N^{1N} W_N^{N/4} \\ + \sum_{n=0}^{\frac{N}{4}-1} x[n + \frac{N}{2}] W_N^{4ln} W_N^n W_N^{2N} W_N^{N/2} + \sum_{n=0}^{\frac{N}{4}-1} x[n + \frac{3N}{4}] W_N^{4ln} W_N^n W_N^{3N} W_N^{3N/4}$$

Now,  $W_N^{1N} = W_N^{2N} = W_N^{3N} = 1$ ,  $W_N^{N/4} = -j$ ,  $W_N^{N/2} = -1$ , and  $W_N^{3N/4} = +j$ . Therefore,

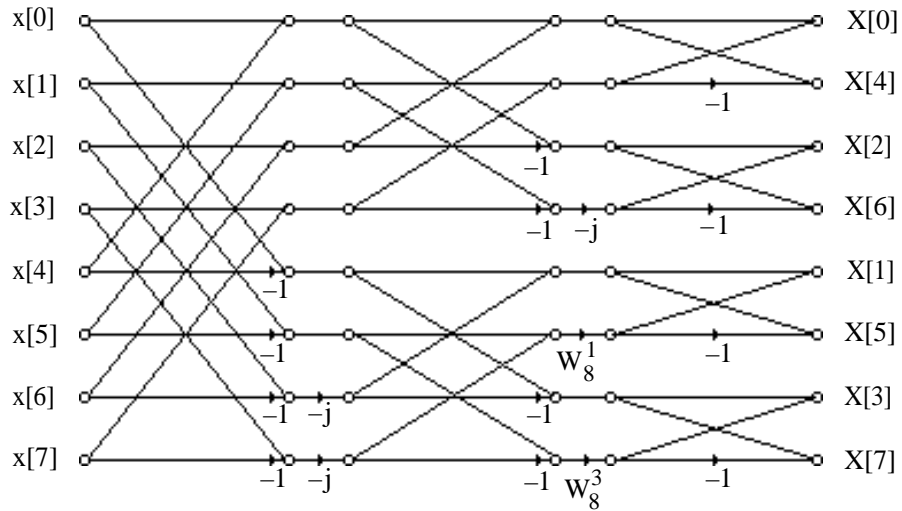
$$X[4l + 1] = \sum_{n=0}^{\frac{N}{4}-1} \left\{ \left( x[n] - x\left[n + \frac{N}{2}\right] \right) - j \left( x\left[n + \frac{N}{4}\right] - x\left[n + \frac{3N}{4}\right] \right) \right\} W_N^n W_{N/4}^{1n}, \quad 0 \leq l \leq \frac{N}{4} - 1.$$

$$\begin{aligned} \text{Similarly, } X[4l + 3] &= \sum_{n=0}^{\frac{N}{4}-1} x[n] W_N^{(4l+3)n} + \sum_{n=0}^{\frac{N}{4}-1} x\left[n + \frac{N}{4}\right] W_N^{(4l+3)n} W_N^{(4l+3)N/4} \\ &\quad + \sum_{n=0}^{\frac{N}{4}-1} x\left[n + \frac{N}{2}\right] W_N^{(4l+3)n} W_N^{(4l+3)N/2} + \sum_{n=0}^{\frac{N}{4}-1} x\left[n + \frac{3N}{4}\right] W_N^{(4l+3)n} W_N^{(4l+3)3N/4} \\ &= \sum_{n=0}^{\frac{N}{4}-1} x[n] W_N^{4l n} W_N^{3n} + \sum_{n=0}^{\frac{N}{4}-1} x\left[n + \frac{N}{4}\right] W_N^{4l n} W_N^{3n} W_N^{1N} W_N^{3N/4} \\ &= \sum_{n=0}^{\frac{N}{4}-1} x\left[n + \frac{N}{2}\right] W_N^{4l n} W_N^{3n} W_N^{2N} W_N^{6N/4} + \sum_{n=0}^{\frac{N}{4}-1} x\left[n + \frac{3N}{4}\right] W_N^{4l n} W_N^{3n} W_N^{3N} W_N^{9N/4} \\ &= \sum_{n=0}^{\frac{N}{4}-1} \left\{ \left( x[n] - x\left[n + \frac{N}{2}\right] \right) + j \left( x\left[n + \frac{N}{4}\right] - x\left[n + \frac{3N}{4}\right] \right) \right\} W_N^{3n} W_{N/4}^{1n}, \quad 0 \leq l \leq \frac{N}{4} - 1. \end{aligned}$$

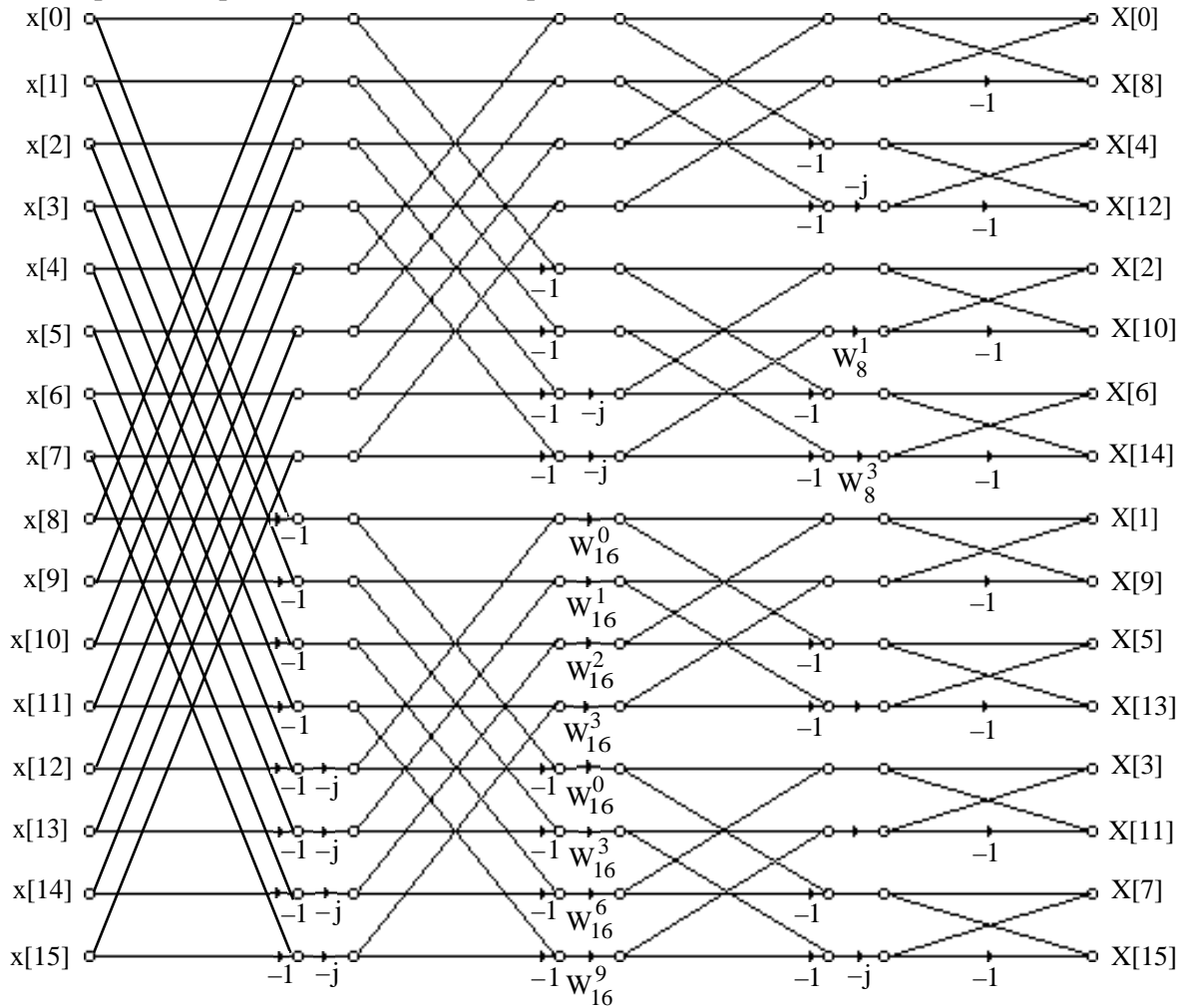
The butterfly here is as shown below which is seen to require two complex multiplications.



**8.30** From the flow-graph of the 8-point split-radix FFT algorithm given below it can be seen that the total number of complex multiplications required is 2. On the other hand, the total number of complex multiplications required for a standard DIF FFT algorithm is also 2.



**8.31** If multiplications by  $\pm j$ ,  $\pm 1$  are ignored, the flow-graph shown below requires 8 complex = 24 real multiplications. A radix-2 DIF 16-point FFT algorithm, on the other hand, requires 10 complex multiplications = 30 real multiplications.



**8.32 (a)** From Eq. (8.97a),  $x[n] = x[n_1 + N_1 n_2]$  where  $0 \leq n_1 \leq N_1 - 1$ , and  $0 \leq n_2 \leq N_2 - 1$ . Now,

using Eq. (8.97b) we can rewrite  $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$ ,  $0 \leq k \leq N-1$ , as

$$\begin{aligned} X[N_2 k_1 + k_2] &= \sum_{n=0}^{N_1 N_2 - 1} x[n] W_N^{n(N_2 k_1 + k_2)} = \sum_{n_1=0}^{N_1 - 1} \sum_{n_2=0}^{N_2 - 1} x[n_1 + N_1 n_2] W_N^{(n_1 + N_1 n_2)(N_2 k_1 + k_2)} \\ &= \sum_{n_1=0}^{N_1 - 1} \sum_{n_2=0}^{N_2 - 1} x[n_1 + N_1 n_2] W_N^{n_1 N_2 k_1} W_N^{N_1 n_2 N_2 k_1} W_N^{N_1 n_2 k_2} W_N^{n_1 k_2}. \end{aligned}$$

Since,  $W_N^{n_1 N_2 k_1} = W_{N_1}^{n_1 k_1}$ ,  $W_N^{N_1 n_2 N_2 k_1} = 1$ , and  $W_N^{N_1 n_2 k_2} = W_{N_2}^{n_2 k_2}$ , we get

$$X[k] = X[N_2 k_1 + k_2] = \sum_{n_1=0}^{N_1 - 1} \left[ \left( \sum_{n_2=0}^{N_2 - 1} x[n_1 + N_1 n_2] W_{N_2}^{n_2 k_2} \right) W_N^{n_1 k_2} \right] W_{N_1}^{n_1 k_1}. \quad (22)$$

**(b)** For  $N_1 = 2$  and  $N_2 = N/2$ , the above DFT computation scheme leads to

$$\begin{aligned} X[k] &= X\left[\frac{N}{2} k_1 + k_2\right] = \sum_{n_1=0}^1 \left[ \left( \sum_{n_2=0}^{\frac{N}{2} - 1} x[n_1 + 2n_2] W_{N/2}^{n_2 k_2} \right) W_N^{n_1 k_2} \right] W_2^{n_1 k_1} \\ &= \sum_{n_2=0}^{\frac{N}{2} - 1} x[2n_2] W_{N/2}^{n_2 k_2} + W_{N_1}^{k_1} W_N^{k_2} \sum_{n_2=0}^{\frac{N}{2} - 1} x[2n_2 + 1] W_{N/2}^{n_2 k_2} \\ &= \sum_{n_2=0}^{\frac{N}{2} - 1} x[2n_2] W_{N/2}^{n_2 k_2} + W_N^k \sum_{n_2=0}^{\frac{N}{2} - 1} x[2n_2 + 1] W_{N/2}^{n_2 k_2} \end{aligned}$$

which is seen to be the first stage in the DIT DFT computation.

On the other hand, for  $N_1 = N/2$  and  $N_2 = 2$ , the above DFT computation scheme leads to

$$\begin{aligned} X[k] &= X[2k_1 + k_2] = \sum_{n_1=0}^{\frac{N}{2} - 1} \left[ \left( \sum_{n_2=0}^1 x[n_1 + \frac{N}{2} n_2] W_2^{n_2 k_2} \right) W_N^{n_1 k_2} \right] W_{N/2}^{n_1 k_1} \\ &= \sum_{n_1=0}^{\frac{N}{2} - 1} \left( x[n_1] + x[n_1 + \frac{N}{2}] (-1)^{k_2} \right) W_N^{n_1 k_2} W_{N/2}^{n_1 k_1} \\ &= \sum_{n_1=0}^{\frac{N}{2} - 1} \left\{ \left( x[n_1] + (-1)^{k_2} x[n_1 + \frac{N}{2}] \right) W_N^{n_1 k_2} \right\} W_{N/2}^{n_1 k_1} \end{aligned}$$

which represents the first stage of the DIF FFT algorithm.

(c) In the DFT computation scheme of Eq. (22), we first compute a total of  $N_2 N_1$ -point DFTs, multiply all the  $N_1 N_2 = N$  computed DFT coefficients by the twiddle factors  $W_N^{n_1 k_2}$ , and finally calculate a total of  $N_2 N_1$ -point DFTs. If  $R(N)$  denotes the total number of multiplications needed to compute an  $N$ -point DFT, then the total number of multiplications required in the DFT computation scheme of Eq. (22) is given by

- (i)  $N_2 \cdot R(N_1)$  for the first step,
- (ii)  $N_2 N_1 = N$  for multiplications by the twiddle factors, and
- (iii)  $N_1 \cdot R(N_2)$  for the last step.

Therefore,  $R(N) = N_2 \cdot R(N_1) + N + N_1 \cdot R(N_2) = N \left( \frac{1}{N_1} R(N_1) + \frac{1}{N_2} R(N_2) + 1 \right)$ .

(d) For  $N = 2^v$ , choose  $N_i = 2$ ,  $i = 1, 2, \dots, v$ . Now from Figure 8.24 for a 2-point DFT  $R(N_i) = 2$ . Hence,  $R(N) = N \left( \frac{v}{2} \right) = \frac{N}{2} \log_2 N$ .

**8.33 (a)**  $N = 12$ . Choose  $N_1 = 4$  and  $N_2 = 3$ . Thus,

$$n = n_1 + 4n_2, \begin{cases} 0 \leq n_1 \leq 3 \\ 0 \leq n_2 \leq 2 \end{cases}, \text{ and } k = 3k_1 + k_2, \begin{cases} 0 \leq k_1 \leq 3 \\ 0 \leq k_2 \leq 2 \end{cases}.$$

$n_2 \backslash n_1$	0	1	2	3
0	x[0]	x[1]	x[2]	x[3]
1	x[4]	x[5]	x[6]	x[7]
2	x[8]	x[9]	x[10]	x[11]

$k_2 \backslash k_1$	0	1	2	3
0	X[0]	X[3]	X[6]	X[9]
1	X[1]	X[4]	X[7]	X[10]
2	X[2]	X[5]	X[8]	X[11]

(b)  $N = 15$ . Choose  $N_1 = 3$  and  $N_2 = 5$ . Thus,

$$n = n_1 + 3n_2, \begin{cases} 0 \leq n_1 \leq 2 \\ 0 \leq n_2 \leq 4 \end{cases}, \text{ and } k = 5k_1 + k_2, \begin{cases} 0 \leq k_1 \leq 2 \\ 0 \leq k_2 \leq 4 \end{cases}.$$

$n_2 \backslash n_1$	0	1	2
0	x[0]	x[1]	x[2]
1	x[3]	x[4]	x[5]
2	x[6]	x[7]	x[8]
3	x[9]	x[10]	x[11]
4	x[12]	x[13]	x[14]

$k_2 \backslash k_1$	0	1	2
0	X[0]	X[5]	X[10]
1	X[1]	X[6]	X[11]
2	X[2]	X[7]	X[12]
3	X[3]	X[8]	X[13]
4	X[4]	X[9]	X[14]

(c)  $N = 21$ . Choose  $N_1 = 7$ , and  $N_2 = 3$ . Thus,

$$n = n_1 + 7n_2, \begin{cases} 0 \leq n_1 \leq 6 \\ 0 \leq n_2 \leq 2 \end{cases}, \text{ and } k = 3k_1 + k_2, \begin{cases} 0 \leq k_1 \leq 6 \\ 0 \leq k_2 \leq 2 \end{cases}.$$



	$n_1$	0	1	2	3	4	5	6
$n_2$	0	x[0]	x[1]	x[2]	x[3]	x[4]	x[5]	x[6]
1	x[7]	x[8]	x[9]	x[10]	x[11]	x[12]	x[13]	
2	x[14]	x[15]	x[16]	x[17]	x[18]	x[19]	x[20]	

	$k_1$	0	1	2	3	4	5	6
$k_2$	0	X[0]	X[3]	X[6]	X[9]	X[12]	X[15]	X[18]
1	X[1]	X[4]	X[7]	X[10]	X[13]	X[16]	X[19]	
2	X[2]	X[5]	X[8]	X[11]	X[14]	X[17]	X[20]	

(d)  $N = 35$ . Choose  $N_1 = 7$ , and  $N_2 = 5$ . Thus,

$$n = n_1 + 7n_2, \quad \begin{cases} 0 \leq n_1 \leq 6 \\ 0 \leq n_2 \leq 4 \end{cases}, \quad \text{and} \quad k = 5k_1 + k_2, \quad \begin{cases} 0 \leq k_1 \leq 6 \\ 0 \leq k_2 \leq 4 \end{cases}$$

	$n_1$	0	1	2	3	4	5	6
$n_2$	0	x[0]	x[1]	x[2]	x[3]	x[4]	x[5]	x[6]
1	x[7]	x[8]	x[9]	x[10]	x[11]	x[12]	x[13]	
2	x[14]	x[15]	x[16]	x[17]	x[18]	x[19]	x[20]	
3	x[21]	x[22]	x[23]	x[24]	x[25]	x[26]	x[27]	
4	x[28]	x[29]	x[30]	x[31]	x[32]	x[33]	x[34]	

	$k_1$	0	1	2	3	4	5	6
$k_2$	0	X[0]	X[5]	X[10]	X[15]	X[20]	X[25]	X[30]
1	X[1]	X[6]	X[11]	X[16]	X[21]	X[26]	X[31]	
2	X[2]	X[7]	X[12]	X[17]	X[22]	X[27]	X[32]	
3	X[3]	X[8]	X[13]	X[18]	X[23]	X[28]	X[33]	
4	X[4]	X[9]	X[14]	X[19]	X[24]	X[29]	X[34]	

8.34 (a)  $n = \langle A n_1 + B n_2 \rangle_N$ ,  $k = \langle C k_1 + D k_2 \rangle_N$ , where  $N = N_1 N_2$ .

$$\begin{aligned} X[k] &= X[\langle C k_1 + D k_2 \rangle_N] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[\langle A n_1 + B n_2 \rangle_N] W_N^{(A n_1 + B n_2)(C k_1 + D k_2)} \\ &= \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[\langle A n_1 + B n_2 \rangle_N] W_N^{A C n_1 k_1} W_N^{A D n_1 k_2} W_N^{B D n_2 k_2} W_N^{B C n_2 k_1}. \end{aligned}$$

To completely eliminate the twiddle factors we require

$$W_N^{A C n_1 k_1} W_N^{A D n_1 k_2} W_N^{B D n_2 k_2} W_N^{B C n_2 k_1} = W_{N_1}^{n_1 k_1} W_{N_2}^{n_2 k_2}.$$

To achieve this we need to choose the constants A, B, C, and D, such that

$$\langle A D \rangle_N = 0, \quad \langle B C \rangle_N = 0, \quad \langle A C \rangle_N = N_2, \quad \text{and} \quad \langle B D \rangle_N = N_1. \quad (23)$$

Then, we can write  $X[\langle Ck_1 + Dk_2 \rangle_N] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[\langle An_1 + Bn_2 \rangle_N] W_{N_1}^{n_1 k_1} W_{N_2}^{n_2 k_2}$ .

(b) We shall show that the following choice of the constants satisfy the constraints of Eq. (23):

$$A = N_2, \quad B = N_1, \quad C = N_2 \langle N_2^{-1} \rangle_{N_1}, \quad \text{and} \quad D = N_1 \langle N_1^{-1} \rangle_{N_2},$$

where  $\langle N_1^{-1} \rangle_{N_2}$  is the multiplicative inverse of  $N_1$  evaluated modulo  $N_2$ , i.e. if  $\langle N_1^{-1} \rangle_{N_2} = \alpha$ , then  $\langle N_1 \alpha \rangle_{N_2} = 1$ . Hence,  $N_1 \alpha$  must be expressible as  $N_2 \beta + 1$ , where  $\beta$  is any integer.

Likewise,  $\langle N_2^{-1} \rangle_{N_1}$  is the multiplicative inverse of  $N_2$  evaluated modulo  $N_1$ , and if  $\langle N_2^{-1} \rangle_{N_1} = \gamma$ , then  $N_2 \gamma = N_1 \delta + 1$ , where  $\delta$  is any integer. Now, from Eq. (23),

$$\langle AC \rangle_N = \langle N_2 \cdot N_2 \langle N_2^{-1} \rangle_{N_1} \rangle_N = \langle N_2 (N_1 \delta + 1) \rangle_N = \langle N_2 N_1 \delta + N_2 \rangle_N = N_2. \quad \text{Similarly,}$$

$$\langle BD \rangle_N = \langle N_1 \cdot N_1 \langle N_1^{-1} \rangle_{N_2} \rangle_N = \langle N_1 (N_2 \beta + 1) \rangle_N = \langle N_1 N_2 \beta + N_1 \rangle_N = N_1. \quad \text{Next, we}$$

observe that  $\langle AD \rangle_N = \langle N_2 \cdot N_1 \langle N_1^{-1} \rangle_{N_2} \rangle_N = \langle N \langle N_1^{-1} \rangle_{N_2} \rangle_N = \langle N \alpha \rangle_N = 0$ , and

$$\langle BC \rangle_N = \langle N_1 \cdot N_2 \langle N_2^{-1} \rangle_{N_1} \rangle_N = \langle N \langle N_2^{-1} \rangle_{N_1} \rangle_N = \langle N \gamma \rangle_N = 0.$$

$$\text{Hence, } X[k] = X[\langle Ck_1 + Dk_2 \rangle_N] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[\langle An_1 + Bn_2 \rangle_N] W_N^{N_2 n_1 k_1} W_N^{N_1 n_2 k_2}$$

$$= \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[\langle An_1 + Bn_2 \rangle_N] W_{N_1}^{n_1 k_1} W_{N_2}^{n_2 k_2}$$

$$= \sum_{n_2=0}^{N_2-1} \left( \sum_{n_1=0}^{N_1-1} x[\langle An_1 + Bn_2 \rangle_N] W_{N_1}^{n_1 k_1} \right) W_{N_2}^{n_2 k_2}.$$

**8.35 (a)**  $N = 12$ . Choose  $N_1 = 4$  and  $N_2 = 3$ .

$$A = 3, \quad B = 4, \quad C = 3 \langle 3^{-1} \rangle_4 = 9, \quad D = 4 \langle 4^{-1} \rangle_3 = 4.$$

$$n = \langle 3n_1 + 4n_2 \rangle_{12}, \quad \begin{cases} 0 \leq n_1 \leq 3, \\ 0 \leq n_2 \leq 2, \end{cases} \quad k = \langle 9k_1 + 4k_2 \rangle_{12}, \quad \begin{cases} 0 \leq k_1 \leq 3, \\ 0 \leq k_2 \leq 2. \end{cases}$$

$n_2 \backslash n_1$	0	1	2	3
0	x[0]	x[3]	x[6]	x[9]
1	x[4]	x[7]	x[10]	x[1]
2	x[8]	x[11]	x[2]	x[5]

$k_2 \backslash k_1$	0	1	2	3
0	X[0]	X[9]	X[6]	X[3]
1	X[4]	X[1]	X[10]	X[7]
2	X[8]	X[5]	X[2]	X[11]

(b)  $N = 15$ . Choose  $N_1 = 3$  and  $N_2 = 5$ .

$$A = 5, \quad B = 3, \quad C = 5 \langle 5^{-1} \rangle_3 = 10, \quad D = 3 \langle 3^{-1} \rangle_5 = 6.$$

$$n = \langle 5n_1 + 3n_2 \rangle_{15}, \quad \begin{cases} 0 \leq n_1 \leq 2, \\ 0 \leq n_2 \leq 4, \end{cases} \quad k = \langle 10k_1 + 6k_2 \rangle_{15}, \quad \begin{cases} 0 \leq k_1 \leq 2, \\ 0 \leq k_2 \leq 4. \end{cases}$$

$n_2 \backslash n_1$	0	1	2
0	x[0]	x[5]	x[10]
1	x[3]	x[8]	x[13]
2	x[6]	x[11]	x[1]
3	x[9]	x[14]	x[4]
4	x[12]	x[2]	x[7]

$k_2 \backslash k_1$	0	1	2
0	X[0]	X[10]	X[5]
1	X[6]	X[1]	X[11]
2	X[12]	X[7]	X[2]
3	X[3]	X[13]	X[8]
4	X[9]	X[4]	X[14]

(c)  $N = 21$ . Choose  $N_1 = 7$  and  $N_2 = 3$ .

$$A = 3, B = 7, C = 3 < 3^{-1} >_7 = 3 \times 5 = 15, D = 7 < 7^{-1} >_3 = 7 \times 1 = 7.$$

$$n = < 3n_1 + 7n_2 >_{21}, \begin{cases} 0 \leq n_1 \leq 6, \\ 0 \leq n_2 \leq 2, \end{cases} \quad k = < 15k_1 + 7k_2 >_{21}, \begin{cases} 0 \leq k_1 \leq 6, \\ 0 \leq k_2 \leq 2. \end{cases}$$

$n_2 \backslash n_1$	0	1	2	3	4	5	6
0	x[0]	x[3]	x[6]	x[9]	x[12]	x[15]	x[18]
1	x[7]	x[10]	x[13]	x[16]	x[19]	x[1]	x[4]
2	x[14]	x[17]	x[20]	x[2]	x[5]	x[8]	x[11]

$k_2 \backslash k_1$	0	1	2	3	4	5	6
0	X[0]	X[15]	X[9]	X[3]	X[18]	X[12]	X[6]
1	X[7]	X[1]	X[16]	X[10]	X[4]	X[19]	X[13]
2	X[14]	X[8]	X[2]	X[17]	X[11]	X[5]	X[20]

(d)  $N = 35$ . Choose  $N_1 = 7$  and  $N_2 = 5$ .

$$A = 5, B = 7, C = 5 < 5^{-1} >_7 = 5 \times 3 = 15, D = 7 < 7^{-1} >_5 = 7 \times 3 = 21.$$

$$n = < 5n_1 + 7n_2 >_{35}, \begin{cases} 0 \leq n_1 \leq 6, \\ 0 \leq n_2 \leq 4, \end{cases} \quad k = < 15k_1 + 21k_2 >_{35}, \begin{cases} 0 \leq k_1 \leq 6, \\ 0 \leq k_2 \leq 4. \end{cases}$$

$n_2 \backslash n_1$	0	1	2	3	4	5	6
0	x[0]	x[5]	x[10]	x[15]	x[20]	x[25]	x[30]
1	x[7]	x[12]	x[17]	x[22]	x[27]	x[32]	x[2]
2	x[14]	x[19]	x[24]	x[29]	x[34]	x[4]	x[9]
3	x[21]	x[26]	x[31]	x[1]	x[6]	x[11]	x[16]
4	x[28]	x[33]	x[3]	x[8]	x[13]	x[18]	x[23]

$k_2 \backslash k_1$	0	1	2	3	4	5	6
0	X[0]	X[15]	X[30]	X[10]	X[25]	X[5]	X[20]
1	X[21]	X[1]	X[16]	X[31]	X[11]	X[26]	X[6]
2	X[7]	X[22]	X[2]	X[17]	X[32]	X[12]	X[27]
3	X[28]	X[8]	X[23]	X[3]	X[18]	X[33]	X[13]
4	X[14]	X[29]	X[9]	X[24]	X[4]	X[19]	X[34]

**8.36**  $N = 12$ .  $N_1 = 4$  and  $N_2 = 3$ .

$$A = 3, B = 4, C = 3 \langle 3^{-1} \rangle_4 = 9, D = 4 \langle 4^{-1} \rangle_3 = 4.$$

$$n = \langle 3n_1 + 4n_2 \rangle_{12}, \begin{cases} 0 \leq n_1 \leq 3, \\ 0 \leq n_2 \leq 4, \end{cases} \quad k = \langle 9k_1 + 4k_2 \rangle_{12}, \begin{cases} 0 \leq k_1 \leq 3, \\ 0 \leq k_2 \leq 2. \end{cases}$$

$n_2 \backslash n_1$	0	1	2	3
0	x[0]	x[3]	x[6]	x[9]
1	x[4]	x[7]	x[10]	x[1]
2	x[8]	x[11]	x[2]	x[5]

$k_2 \backslash k_1$	0	1	2	3
0	X[0]	X[9]	X[6]	X[3]
1	X[4]	X[1]	X[10]	X[7]
2	X[8]	X[5]	X[2]	X[11]

$$n = \langle 9n_1 + 4n_2 \rangle_{12}$$

$n_2 \backslash n_1$	0	1	2	3
0	x[0]	x[9]	x[6]	x[3]
1	x[4]	x[1]	x[10]	x[7]
2	x[8]	x[5]	x[2]	x[11]

$$k = \langle 3k_1 + 4k_2 \rangle_{12}$$

$k_2 \backslash k_1$	0	1	2	3
0	Y[0]	Y[3]	Y[6]	Y[9]
1	Y[4]	Y[7]	Y[10]	Y[1]
2	Y[8]	Y[11]	Y[2]	Y[5]

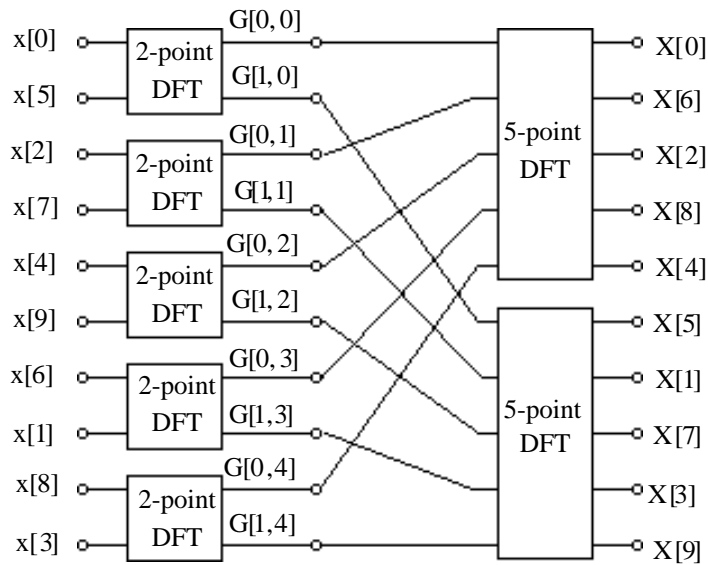
Hence,  $X[2k] = Y[2k]$ , and  $X[2k + 1] = X[2k + 1] = Y[\langle 6 + (2k + 1) \rangle_{12}]$ ,  $k = 0, 1, \dots, 5$ ,

**8.37**  $N = 10$ ,  $N_1 = 2$ , and  $N_2 = 5$ . Choose  $A = 5$ ,  $B = 2$ ,  $C = 5 \langle 5^{-1} \rangle_2 = 5$ ,  $D = 2 \langle 2^{-1} \rangle_5 = 6$ .

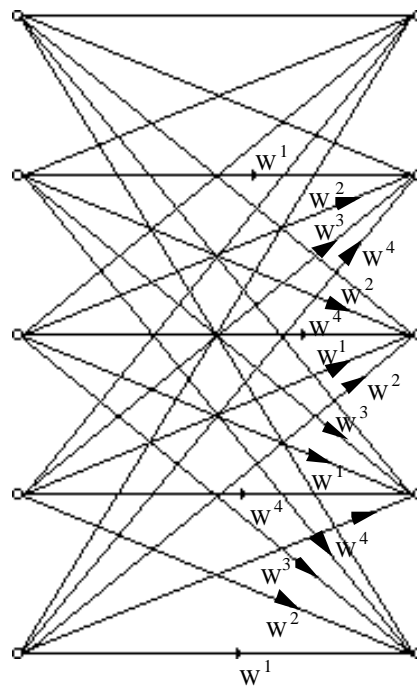
$n_2 \backslash n_1$	0	1
0	x[0]	x[5]
1	x[2]	x[7]
2	x[4]	x[9]
3	x[6]	x[1]
4	x[8]	x[3]

$n_2 \backslash k_1$	0	1
0	G[0,0]	G[1,0]
1	G[0,1]	G[1,1]
2	G[0,2]	G[1,2]
3	G[0,3]	G[1,3]
4	G[0,4]	G[1,4]

$k_2 \backslash k_1$	0	1
0	X[0]	X[5]
1	X[6]	X[1]
2	X[2]	X[7]
3	X[8]	X[3]
4	X[4]	X[9]



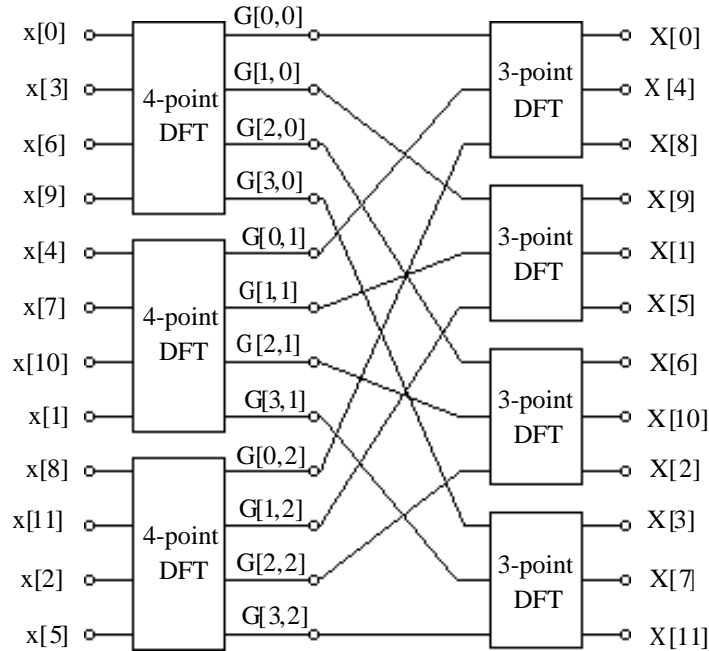
The flow-graph of the 2-point DFT is given in Figure 8.21. The flow-graph of the 5-point DFT is shown below



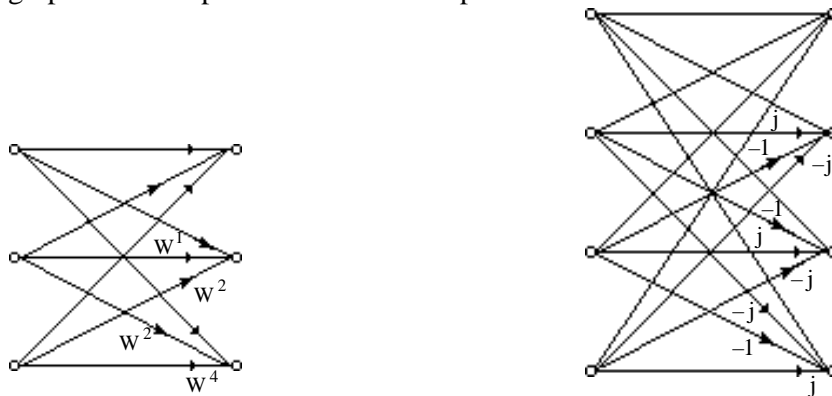
**8.38**  $N = 12$ ,  $N_1 = 4$ , and  $N_2 = 3$ . Choose  $A = 3$ ,  $B = 4$ ,  $C = 3 < 3^{-1} >_4 = 9$ , and  $D = 4 < 4^{-1} >_3 = 4$ .

$n_2 \backslash n_1$	0	1	2	3	$n_2 \backslash k_1$	0	1	2	3
0	x[0]	x[3]	x[6]	x[9]	0	G[0, 0]	G[1,0]	G[2,0]	G[3,0]
1	x[4]	x[7]	x[10]	x[1]	1	G[0, 1]	G[1,1]	G[2,1]	G[3,1]
2	x[8]	x[11]	x[2]	x[5]	2	G[0, 2]	G[1,2]	G[2,2]	G[3,2]

$k_2 \backslash k_1$	0	1	2	3
0	X[0]	X[9]	X[6]	X[3]
1	X[4]	X[1]	X[10]	X[7]
2	X[8]	X[5]	X[2]	X[11]



The flow-graphs of the 3-point DFT and the 4-point DFT are shown below:



**8.39** Note that  $3072 = 512 \times 6$ . Now an N-point DFT, with N divisible by 6, can be computed as

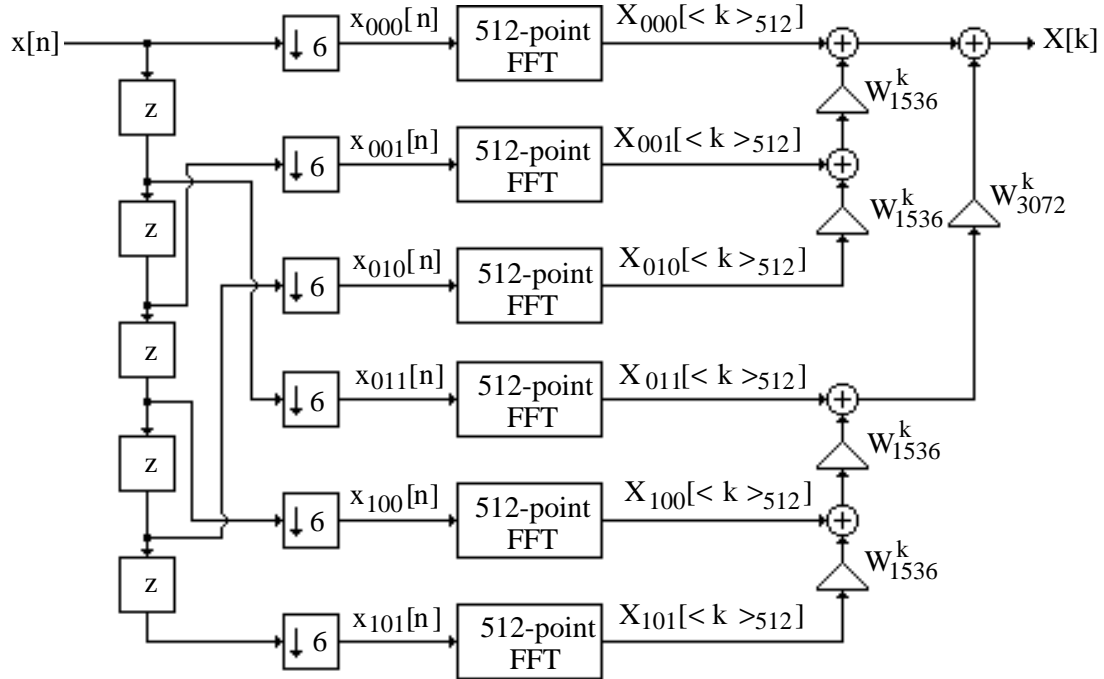
$$\text{follows: } X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} = X_0[\langle k \rangle_{N/6}] + W_N^k \cdot X_1[\langle k \rangle_{N/6}] + W_N^{2k} \cdot X_2[\langle k \rangle_{N/6}] \\ + W_N^{3k} \cdot X_3[\langle k \rangle_{N/6}] + W_N^{4k} \cdot X_4[\langle k \rangle_{N/6}] + W_N^{5k} \cdot X_5[\langle k \rangle_{N/6}], \text{ where}$$

$$X_1[\langle k \rangle_{N/6}] = \sum_{r=0}^{\frac{N}{6}-1} x[6r+1] W_{N/6}^{rk}, \quad 0 \leq 1 \leq 5. \text{ For } N = 3072, \text{ we thus get}$$

$$X[k] = X_0[\langle k \rangle_{512}] + W_{3072}^k \cdot X_1[\langle k \rangle_{512}] + W_{3072}^{2k} \cdot X_2[\langle k \rangle_{512}]$$

$$+ W_{3072}^{3k} \cdot X_3[\langle k \rangle_{512}] + W_{3072}^{4k} \cdot X_4[\langle k \rangle_{512}] + W_{3072}^{5k} \cdot X_5[\langle k \rangle_{512}], \text{ where}$$

$$X_1[\langle k \rangle_{512}] = \sum_{r=0}^{511} x[6r+1]W_{512}^{rk}, \quad 0 \leq k \leq 511.$$



Now an  $N$ -point FFT algorithm requires  $\frac{N}{2} \log_2 N$  complex multiplications and  $N \log_2 N$  complex additions. Hence, an  $\frac{N}{6}$ -point FFT algorithm requires  $\frac{N}{12} \log_2 \left(\frac{N}{6}\right)$  complex multiplications and  $\frac{N}{6} \log_2 \left(\frac{N}{6}\right)$  complex additions. In addition, we need  $5 \times N$  complex multiplications and  $5 \times N$  complex additions to compute the  $N$ -point DFT  $X[k]$ . Hence, for  $N = 3072$ , the evaluation of  $X[k]$  using 6 (512)-point FFT modules requires  $\frac{N}{12} \log_2 \left(\frac{N}{6}\right) + 5 \times N = 256 \times \log_2(512) + 5 \times 3072 = 17,664$  complex multiplications and  $\frac{N}{6} \log_2 \left(\frac{N}{6}\right) + 5 \times N = 512 \times \log_2(512) + 5 \times 3072 = 19,968$  complex additions.

It should be noted that a direct computation of the 3072-point DFT would require 9,437,184 complex multiplications and 9,434,112 complex additions.

**8.40 (a)** # of zero-valued samples to be added is  $1024 - 1000 = 24$ .

**(b)** Direct computation of a 1024-point DFT of a length-1000 sequence requires  $(1000)^2 = 1,000,000$  complex multiplications and  $999 \times 1000 = 999,000$  complex additions.

(c) A 1024-point Cooley-Tukey type FFT algorithm requires  $512 \times \log_2(1024) = 5,120$  complex multiplications and  $1024 \times \log_2(1024) = 10,240$  complex additions.

**8.41 (a)**  $Y(z) = H(z)X(z)$  or  $y[0] + y[1]z^{-1} + y[2]z^{-2} = (h[0] + h[1]z^{-1})(x[0] + x[1]z^{-1})$ .

Now,  $Y(z_0) = Y(-1) = y[0] - y[1] + y[2] = H(-1)X(-1) = (h[0] - h[1])(x[0] - x[1])$ ,  
 $Y(z_1) = Y(\infty) = y[0] = H(\infty)X(\infty) = h[0]x[0]$ ,  
 $Y(z_2) = Y(1) = y[0] + y[1] + y[2] = H(1)X(1) = (h[0] + h[1])(x[0] + x[1])$ .

From Eqs. (3.158) and (3.159), we can write

$$Y(z) = \frac{I_0(z)}{I_0(z_0)} Y(z_0) + \frac{I_1(z)}{I_1(z_1)} Y(z_1) + \frac{I_2(z)}{I_2(z_2)} Y(z_2),$$

where  $I_0(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1}) = (1 - z_1 z^{-1})(1 - z^{-1}) \Big|_{z_1 = \infty}$ ,

$$I_1(z) = (1 - z_0 z^{-1})(1 - z_2 z^{-1}) = (1 + z^{-1})(1 - z^{-1}) = 1 - z^{-2},$$

$$I_2(z) = (1 - z_0 z^{-1})(1 - z_1 z^{-1}) = (1 + z^{-1})(1 - z_1 z^{-1}) \Big|_{z_1 = \infty}.$$

Therefore,  $\frac{I_0(z)}{I_0(z_0)} = -\frac{1}{2}z^{-1}(1 - z^{-1})$ ,  $\frac{I_1(z)}{I_1(z_1)} = (1 - z^{-2})$ , and  $\frac{I_2(z)}{I_2(z_2)} = \frac{1}{2}z^{-1}(1 + z^{-1})$ . Hence,

$$\begin{aligned} Y(z) &= -\frac{1}{2}z^{-1}(1 - z^{-1})Y(z_0) + (1 - z^{-2})Y(z_1) + \frac{1}{2}z^{-1}(1 + z^{-1})Y(z_2) \\ &= Y(z_1) + \left(-\frac{1}{2}Y(z_0) + \frac{1}{2}Y(z_2)\right)z^{-1} + \left(\frac{1}{2}Y(z_0) - Y(z_1) + \frac{1}{2}Y(z_2)\right)z^{-2} \\ &= h[0]x[0] + \left(-\frac{1}{2}(h[0] - h[1])(x[0] - x[1]) + \frac{1}{2}(h[0] + h[1])(x[0] + x[1])\right)z^{-1} \\ &\quad + \left(\frac{1}{2}(h[0] - h[1])(x[0] - x[1]) - h[0]x[0] + \frac{1}{2}(h[0] + h[1])(x[0] + x[1])\right)z^{-2} \\ &= h[0]x[0] + (h[0]x[1] + h[1]x[0])z^{-1} + h[1]x[1]z^{-2}. \end{aligned}$$

Ignoring the multiplications by  $\frac{1}{2}$ , computation of the coefficients of  $Y(z)$  require the values of  $Y(z_0)$ ,  $Y(z_1)$ , and  $Y(z_2)$  which can be evaluated using only 3 multiplications.

**(b)**  $Y(z) = H(z)X(z)$  or

$$y[0] + y[1]z^{-1} + y[2]z^{-2} + y[3]z^{-3} + y[4]z^{-4} = (h[0] + h[1]z^{-1} + h[2]z^{-2})(x[0] + x[1]z^{-1} + x[2]z^{-2}).$$

Now,  $Y(z_0) = Y(-\frac{1}{2}) = (h[0] - 2h[1] + 4h[2])(x[0] - 2x[1] + 4x[2])$ ,

$$Y(z_1) = Y(-1) = (h[0] - h[1] + h[2])(x[0] - x[1] + x[2]),$$

$$Y(z_2) = Y(\infty) = h[0]x[0],$$

$$Y(z_3) = Y(1) = (h[0] + h[1] + h[2])(x[0] + x[1] + x[2]),$$

$$Y(z_4) = Y(\frac{1}{2}) = (h[0] + 2h[1] + 4h[2])(x[0] + 2x[1] + 4x[2]).$$

From Eqs. (3.158) and (3.159), we can write



$$Y(z) = \frac{I_0(z)}{I_0(z_0)} Y(z_0) + \frac{I_1(z)}{I_1(z_1)} Y(z_1) + \frac{I_2(z)}{I_2(z_2)} Y(z_2) + \frac{I_3(z)}{I_3(z_3)} Y(z_3) + \frac{I_4(z)}{I_4(z_4)} Y(z_4),$$

where

$$I_0(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1})(1 - z_3 z^{-1})(1 - z_4 z^{-1}) = (1 - z^{-2})(1 - \frac{1}{2} z^{-1})(1 - z_2 z^{-1}) \Big|_{z_2=\infty},$$

$$I_1(z) = (1 - z_0 z^{-1})(1 - z_2 z^{-1})(1 - z_3 z^{-1})(1 - z_4 z^{-1}) = (1 - \frac{1}{4} z^{-2})(1 - z^{-1})(1 - z_2 z^{-1}) \Big|_{z_2=\infty},$$

$$I_2(z) = (1 - z_0 z^{-1})(1 - z_1 z^{-1})(1 - z_3 z^{-1})(1 - z_4 z^{-1}) = (1 - \frac{1}{4} z^{-2})(1 - z^{-2}),$$

$$I_3(z) = (1 - z_0 z^{-1})(1 - z_1 z^{-1})(1 - z_2 z^{-1})(1 - z_4 z^{-1}) = (1 - \frac{1}{4} z^{-2})(1 + z^{-1})(1 - z_2 z^{-1}) \Big|_{z_2=\infty},$$

$$I_4(z) = (1 - z_0 z^{-1})(1 - z_1 z^{-1})(1 - z_2 z^{-1})(1 - z_3 z^{-1}) = (1 - z^{-2})(1 + \frac{1}{2} z^{-1})(1 - z_2 z^{-1}) \Big|_{z_2=\infty}.$$

Therefore,

$$\frac{I_0(z)}{I_0(z_0)} = \frac{1}{12} z^{-1} (1 - \frac{1}{2} z^{-1})(1 - z^{-2}), \quad \frac{I_1(z)}{I_1(z_1)} = -\frac{2}{3} z^{-1} (1 - z^{-1})(1 - \frac{1}{4} z^{-2}),$$

$$\frac{I_2(z)}{I_2(z_2)} = (1 - z^{-2})(1 - \frac{1}{4} z^{-2}), \quad \frac{I_3(z)}{I_3(z_3)} = \frac{2}{3} z^{-1} (1 + z^{-1})(1 - \frac{1}{4} z^{-2}), \text{ and}$$

$$\frac{I_4(z)}{I_4(z_4)} = -\frac{1}{12} z^{-1} (1 + \frac{1}{2} z^{-1})(1 - z^{-2}).$$

Hence,

$$Y(z) = \frac{1}{12} (z^{-1} - \frac{1}{2} z^{-2} - z^{-3} + \frac{1}{2} z^{-4}) Y(z_0) - \frac{2}{3} (z^{-1} - z^{-2} - \frac{1}{4} z^{-3} + \frac{1}{4} z^{-4}) Y(z_1)$$

$$+ \frac{2}{3} (z^{-1} + z^{-2} - \frac{1}{4} z^{-3} - \frac{1}{4} z^{-4}) Y(z_3) - \frac{1}{12} (z^{-1} + \frac{1}{2} z^{-2} - z^{-3} - \frac{1}{2} z^{-4}) Y(z_4)$$

$$= Y(z_2) + \left( \frac{1}{12} Y(z_0) - \frac{2}{3} Y(z_1) + \frac{2}{3} Y(z_3) - \frac{1}{12} Y(z_4) \right) z^{-1}$$

$$+ \left( -\frac{1}{24} Y(z_0) + \frac{2}{3} Y(z_1) - \frac{5}{4} Y(z_2) + \frac{2}{3} Y(z_3) - \frac{1}{24} Y(z_4) \right) z^{-2}$$

$$+ \left( -\frac{1}{12} Y(z_0) + \frac{1}{6} Y(z_1) - \frac{1}{6} Y(z_3) + \frac{1}{12} Y(z_4) \right) z^{-3}$$

$$+ \left( \frac{1}{24} Y(z_0) - \frac{1}{6} Y(z_1) + \frac{1}{4} Y(z_2) - \frac{1}{6} Y(z_3) + \frac{1}{24} Y(z_4) \right) z^{-4}.$$

Substituting the expressions for  $Y(z_0)$ ,  $Y(z_1)$ ,  $Y(z_2)$ ,  $Y(z_3)$ , and  $Y(z_4)$ , in the above equation, we then arrive at the expressions for the coefficients  $\{y[n]\}$  in terms of the coefficients  $\{h[n]\}$  and  $\{x[n]\}$ . For example,  $y[0] = Y(z_2) = h[0]x[0]$ .

$$y[1] = \frac{1}{12} (Y(z_0) - Y(z_4)) + \frac{2}{3} (Y(z_3) - Y(z_1)) =$$

$$= \frac{1}{12} \left( [(h[0] - 2h[1] + 4h[2])(x[0] - 2x[1] + 4x[2])] - [(h[0] + 2h[1] + 4h[2])(x[0] + 2x[1] + 4x[2])] \right)$$

$$+ \frac{2}{3} \left( [(h[0] + h[1] + h[2])(x[0] + x[1] + x[2])] - [(h[0] - h[1] + h[2])(x[0] - x[1] + x[2])] \right)$$

$$= h[0]x[1] + h[1]x[0]. \text{ In a similar manner we can show,}$$

$$y[2] = -\frac{1}{24} (Y(z_0) + Y(z_4)) + \frac{2}{3} (Y(z_1) + Y(z_3)) - \frac{5}{4} Y(z_2) = h[0]x[2] + h[1]x[1] + h[2]x[0],$$

$$y[3] = \frac{1}{12} (Y(z_4) - Y(z_0)) + \frac{1}{2} (Y(z_1) - Y(z_3)) = h[1]x[2] + h[2]x[1], \text{ and}$$

$$y[4] = \frac{1}{24} (Y(z_0) + Y(z_4)) - \frac{1}{6} (Y(z_1) + Y(z_3)) + \frac{1}{4} Y(z_2) = h[2]x[2].$$

Hence, ignoring the multiplications by  $\frac{1}{12}$ ,  $\frac{2}{3}$ ,  $\frac{5}{4}$ ,  $\frac{1}{4}$ ,  $\frac{1}{6}$ , and  $\frac{1}{24}$ , computation of the coefficients of  $Y(z)$  require the values of  $Y(z_0)$ ,  $Y(z_1)$ ,  $Y(z_2)$ ,  $Y(z_3)$ , and  $Y(z_4)$  which can be evaluated using only 5 multiplications.

**8.42**  $Y(z) = H(z)X(z)$  or  $y[0] + y[1]z^{-1} + y[2]z^{-2} = (h[0] + h[1]z^{-1})(x[0] + x[1]z^{-1})$   
 $h[0]x[0] + (h[0]x[1] + h[1]x[0])z^{-1} + h[1]x[1]z^{-2}.$

Hence,  $y[0] = h[0]x[0]$ ,  $y[1] = h[0]x[1] + h[1]x[0]$ , and  $y[2] = h[1]x[1]$ .

Now,  $(h[0] + h[1])(x[0] + x[1]) - h[0]x[0] - h[1]x[1] = h[0]x[1] + h[1]x[0] = y[1]$ . As a result, evaluation of  $H(z)X(z)$  requires the computation of 3 products,  $h[0]x[0]$ ,  $h[1]x[1]$ , and  $(h[0] + h[1])(x[0] + x[1])$ . In addition, it requires 4 additions,  $h[0] + h[1]$ ,  $x[0] + x[1]$ , and  $(h[0] + h[1])(x[0] + x[1]) - h[0]x[0] - h[1]x[1]$ .

**8.43** Let the two length- $N$  sequences be denoted by  $\{h[n]\}$  and  $\{x[n]\}$ . Denote the sequence generated by the linear convolution of  $h[n]$  and  $x[n]$  as  $y[n]$ , i.e.  $y[n] = \sum_{l=0}^{2N-1} h[l]x[n-l]$ . Computation of  $\{y[n]\}$  thus requires  $2N$  multiplications. Let  $H(z)$  and  $X(z)$  denote the  $z$ -transforms of  $\{h[n]\}$  and  $\{x[n]\}$ , i.e.  $H(z) = \sum_{n=0}^{N-1} h[n]z^{-n}$ , and  $X(z) = \sum_{n=0}^{N-1} x[n]z^{-n}$ . Rewrite  $H(z)$  and  $X(z)$  in the form  $H(z) = H_0(z) + z^{-N/2}H_1(z)$ , and  $X(z) = X_0(z) + z^{-N/2}X_1(z)$ , where  $H_0(z) = \sum_{n=0}^{(N/2)-1} h[n]z^{-n}$ ,  $H_1(z) = \sum_{n=0}^{(N/2)-1} h[n + \frac{N}{2}]z^{-n}$ ,  $X_0(z) = \sum_{n=0}^{(N/2)-1} x[n]z^{-n}$ , and  $X_1(z) = \sum_{n=0}^{(N/2)-1} x[n + \frac{N}{2}]z^{-n}$ . Therefore, we can write

$$Y(z) = (H_0(z) + z^{-N/2}H_1(z))(X_0(z) + z^{-N/2}X_1(z))$$

$$= H_0(z)X_0(z) + z^{-N/2}(H_0(z)X_1(z) + H_1(z)X_0(z)) + z^{-N}H_1(z)X_1(z)$$

$$= Y_0(z) + z^{-N/2}Y_1(z) + z^{-N}Y_2(z),$$

where  $Y_0(z) = H_0(z)X_0(z)$ ,  $Y_1(z) = H_0(z)X_1(z) + H_1(z)X_0(z)$ ,  $Y_2(z) = H_1(z)X_1(z)$ .

Now  $Y_0(z)$  and  $Y_1(z)$  are products of two polynomials of degree  $\frac{N}{2}$ , and hence, require  $\left(\frac{N}{2}\right)^2$  multiplications each. Now, we can write  $Y_1(z) = (H_0(z) + H_1(z))(X_0(z) + X_1(z)) - Y_0(z) - Y_2(z)$ .

Since  $(H_0(z) + H_1(z))(X_0(z) + X_1(z))$  is a product of two polynomials of degree  $\frac{N}{2}$ , and hence,

it can be computed using  $\left(\frac{N}{2}\right)^2$  multiplications. As a result,  $Y(z) = H(z)X(z)$  can be computed using  $3\left(\frac{N}{2}\right)^2$  multiplications instead of  $N^2$  multiplications.

If  $N$  is a power-of-2,  $\frac{N}{2}$  is even, and the same procedure can be applied to compute  $Y_0(z)$ ,  $Y_1(z)$ , and  $Y_2(z)$  reducing further the number of multiplications. This process can be continued until, the sequences being convolved are of length 1 each.

Let  $R(N)$  denote the total number of multiplications required to compute the linear convolution of two length- $N$  sequences. Then, in the method outlined above, we have  $R(N) = 3 \cdot R(N/2)$  with  $R(1) = 1$ . A solution of this equation is given by  $R(N) = 3^{\log_2 N}$ .

**8.44** The dynamic range of a signed  $B$ -bit integer  $\eta$  is given by  $-(2^{(B-1)} - 1) \leq \eta < (2^{(B-1)} - 1)$  which for  $B = 32$  is given by  $-(2^{31} - 1) \leq \eta < (2^{31} - 1)$ .

(a) For  $E = 6$  and  $M = 25$ , the value of a 32-bit floating point number is given by  $\eta = (-1)^s 2^{E-31}(M)$ . Hence, the value of the largest number is  $\approx 2^{32}$ , and the value of the smallest number is  $\approx -2^{32}$ . The dynamic range is therefore  $\approx 2 \times 2^{32}$ .

(b) For  $E = 7$  and  $M = 24$ , the value of a 32-bit floating point number is given by  $\eta = (-1)^s 2^{E-63}(M)$ . Hence, the value of the largest number is  $\approx 2^{64}$ , and the value of the smallest number is  $\approx -2^{64}$ . The dynamic range is therefore  $\approx 2 \times 2^{64}$ .

(c) For  $E = 8$  and  $M = 23$ , the value of a 32-bit floating point number is given by  $\eta = (-1)^s 2^{E-127}(M)$ . Hence, the value of the largest number is  $\approx 2^{128}$ , and the value of the smallest number is  $\approx -2^{128}$ . The dynamic range is therefore  $\approx 2 \times 2^{128}$ .

Hence, the dynamic range in a floating-point representation is much larger than that in a fixed-point representation with the same wordlength.

**8.45** A 32-bit floating-point number in the IEEE Format has  $E = 8$  and  $M = 23$ . Also the exponent  $E$  is coded in a biased form as  $E - 127$  with certain conventions for special cases such as  $E = 0$ , 255, and  $M = 0$  (See text pages 540 and 541).

Now a positive 32-bit floating point number  $\eta$  represented in the "normalized" form have an exponent in the range  $0 < E < 255$ , and is of the form  $\eta = (-1)^s 2^{E-127}(1_{\Delta}M)$ . Hence, the smallest positive number that can be represented will have  $E = 1$ , and  $M = \underset{22 \text{ bits}}{0 \ 0 \ 1 \ 0 \ 0 \ 0}$ , and has

therefore a value given by  $2^{-126} \approx 1.18 \times 10^{-38}$ . For the largest positive number,  $E = 254$ , and  $M = \underset{22 \text{ bits}}{1 \ 1 \ 1 \ 1 \ 1 \ 1}$ . Thus here  $\eta = (-1)^0 2^{127}(1_{\Delta} \underset{22 \text{ bits}}{1 \ 1 \ 1 \ 1 \ 1 \ 1}) \approx 2^{127} \times 2 \approx 3.4 \times 10^{38}$ .

Note: For representing numbers less than  $2^{-126}$ , IEEE format uses the "de-normalized" form where  $E = 0$ , and  $\eta = (-1)^s 2^{-126}(0_{\Delta}M)$ . In this case, the smallest positive number that can be represented is given by  $\eta = (-1)^0 2^{-126}(0_{\Delta} \underset{22 \text{ bits}}{0 \ 0 \ 0 \ 1 \ 0 \ 0}) \approx 2^{-149} \approx 1.4013 \times 10^{-45}$ .

**8.46** For a two's-complement binary fraction  $s_{\Delta} a_{-1} a_{-2} \dots a_{-b}$  the decimal equivalent for  $s = 0$  is

simply  $\sum_{i=1}^b a_{-i} 2^{-i}$ . For  $s = 1$ , the decimal equivalent is given by  $-\left(\sum_{i=1}^b (1 - a_{-i}) 2^{-i} + 2^{-b}\right)$

$$= -\sum_{i=1}^b 2^{-i} + \sum_{i=1}^b a_{-i} 2^{-i} - 2^{-b} = -(1-2^{-b}) + \sum_{i=1}^b a_{-i} 2^{-i} - 2^{-b} = -1 + \sum_{i=1}^b a_{-i} 2^{-i}. \text{ Hence, the}$$

decimal equivalent of  $s_{\Delta} a_{-1} a_{-2} \mathbf{K} a_{-b}$  is given by  $-s + \sum_{i=1}^b a_{-i} 2^{-i}$ .

**8.47** For a two's-complement binary fraction  $s_{\Delta} a_{-1} a_{-2} \mathbf{K} a_{-b}$  the decimal equivalent for  $s = 0$  is

simply  $\sum_{i=1}^b a_{-i} 2^{-i}$ . For  $s = 1$ , the decimal equivalent is given by  $-\sum_{i=1}^b (1-a_{-i}) 2^{-i}$

$$= -\sum_{i=1}^b 2^{-i} + \sum_{i=1}^b a_{-i} 2^{-i} = -(1-2^{-b}) + \sum_{i=1}^b a_{-i} 2^{-i}. \text{ Hence, the decimal equivalent of}$$

$s_{\Delta} a_{-1} a_{-2} \mathbf{K} a_{-b}$  is given by  $-s(1-2^{-b}) + \sum_{i=1}^b a_{-i} 2^{-i}$ .

**8.48 (a)**  $\eta = -0.625_{10}$ .

- (i) Signed-magnitude representation =  $1_{\Delta} 10100000$
- (ii) Ones'-complement representation =  $1_{\Delta} 01011111$
- (iii) Two's-complement representation =  $1_{\Delta} 01100000$

**(b)**  $\eta = -0.7734375_{10}$ .

- (i) Signed-magnitude representation =  $1_{\Delta} 11000110$
- (ii) Ones'-complement representation =  $1_{\Delta} 00111001$
- (iii) Two's-complement representation =  $1_{\Delta} 00111010$

**(c)**  $\eta = -0.36328125_{10}$ .

- (i) Signed-magnitude representation =  $1_{\Delta} 01011101$
- (ii) Ones'-complement representation =  $1_{\Delta} 10100010$
- (iii) Two's-complement representation =  $1_{\Delta} 10100011$

**(d)**  $\eta = -0.94921875_{10}$ .

- (i) Signed-magnitude representation =  $1_{\Delta} 11110011$
- (ii) Ones'-complement representation =  $1_{\Delta} 00001100$
- (iii) Two's-complement representation =  $1_{\Delta} 00001101$

**8.49 (a)**  $\eta = 0.625_{10} = 1_{\Delta} 101000$ , **(b)**  $\eta = -0.625_{10} = 0_{\Delta} 011000$ ,

**(c)**  $\eta = 0.359375_{10} = 1_{\Delta} 010111$ , **(d)**  $\eta = -0.359375_{10} = 0_{\Delta} 101001$ ,

**(e)**  $\eta = 0.90625_{10} = 1_{\Delta} 111010$ , **(f)**  $\eta = -0.90625_{10} = 0_{\Delta} 000110$ .

**8.50 (a)** SD-representation =  $0_{\Delta} 00 \bar{1} 10 \bar{1} 1 \bar{1}$ , **(b)** SD-representation =  $0_{\Delta} 10000 \bar{1} 1 \bar{1}$ ,

**(c)** SD-representation =  $0_{\Delta} \bar{1} 1 \bar{1} 1000 \bar{1}$ .

**8.51 (a)**  $\underset{D}{1} \underset{6}{0} \underset{C}{1} \underset{7}{0} 101100011 = D6C7$ , **(b)**  $0101 \ 1111 \ 1010 \ 1001 = 5FA9$ ,

**(c)**  $1011 \ 0100 \ 0010 \ 1110 = B42E$ .

**8.52 (a)** The addition of the positive binary fractions  $0_{\Delta} 10101$  and  $0_{\Delta} 01111$  is given below:

$$\begin{array}{r}
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \leftarrow \text{carry} \\
0 \quad \Delta \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \\
+ 0 \quad \Delta \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \\
\hline
1 \quad \Delta \quad 0 \quad 0 \quad 1 \quad 0 \quad 0
\end{array}$$

As the sign bit is a 1 there has been an overflow and the sum is not correct.

**(b)** The addition of the positive binary fractions  $0_{\Delta}01011$  and  $0_{\Delta}10001$  is given below:

$$\begin{array}{r}
\quad \quad \quad 1 \quad 1 \quad \leftarrow \text{carry} \\
0 \quad \Delta \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \\
+ 0 \quad \Delta \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \\
\hline
0 \quad \Delta \quad 1 \quad 1 \quad 1 \quad 0 \quad 0
\end{array}$$

**8.53 (a)** The difference of the two positive binary fractions  $0_{\Delta}10101 - 0_{\Delta}01111$  can be carried out as an addition of the positive binary fraction  $0_{\Delta}10101$  and the two's-complement representation of  $-0_{\Delta}01111$  which is given by  $1_{\Delta}10001$ . The process is illustrated below:

$$\begin{array}{r}
\quad \quad \quad 1 \quad \quad \quad \quad \quad 1 \quad \leftarrow \text{carry} \\
0 \quad \Delta \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \\
+ 1 \quad \Delta \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \\
\hline
1 \quad 0 \quad \Delta \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \\
\uparrow \\
\text{drop}
\end{array}$$

The extra bit 1 on the left of the sign bit is dropped resulting in  $0_{\Delta}00110$  which is the correct difference.

**(b)** The difference of the two positive binary fractions  $0_{\Delta}10001 - 0_{\Delta}01011$  can be carried out as an addition of the positive binary fraction  $0_{\Delta}10001$  and the two's-complement representation of  $-0_{\Delta}01011$  which is given by  $1_{\Delta}10101$ . The process is illustrated below:

$$\begin{array}{r}
\quad \quad \quad 1 \quad \quad \quad \quad \quad 1 \quad \leftarrow \text{carry} \\
0 \quad \Delta \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \\
+ 1 \quad \Delta \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \\
\hline
1 \quad 0 \quad \Delta \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \\
\uparrow \\
\text{drop}
\end{array}$$

The extra bit 1 on the left of the sign bit is dropped resulting in  $0_{\Delta}00110$  which is the correct difference.

**8.54 (a)** The difference of the two positive binary fractions  $0_{\Delta}10101 - 0_{\Delta}01111$  can be carried out as an addition of the positive binary fraction  $0_{\Delta}10101$  and the ones'-complement representation of  $-0_{\Delta}01111$  which is given by  $1_{\Delta}10000$ . The process is illustrated below:

$$\begin{array}{r}
\phantom{+} \phantom{0} \phantom{\Delta} \phantom{1} \phantom{0} \phantom{1} \phantom{0} \phantom{1} \phantom{1} \phantom{\leftarrow} \phantom{\text{carry}} \\
\phantom{+} \phantom{0} \phantom{\Delta} \phantom{1} \phantom{0} \phantom{1} \phantom{0} \phantom{1} \\
+ \phantom{0} \phantom{\Delta} \phantom{1} \phantom{0} \phantom{0} \phantom{0} \phantom{0} \phantom{0} \\
\hline
1 \phantom{0} \phantom{\Delta} \phantom{0} \phantom{0} \phantom{1} \phantom{0} \phantom{1} \\
\phantom{1} \phantom{0} \phantom{\Delta} \phantom{0} \phantom{0} \phantom{1} \phantom{0} \phantom{1} \phantom{1} \phantom{\leftarrow} \phantom{\text{end around carry}} \\
\hline
\phantom{0} \phantom{\Delta} \phantom{0} \phantom{0} \phantom{1} \phantom{1} \phantom{0}
\end{array}$$

The extra bit 1 on the left side of the sign bit is brought around and added to the LSB resulting in the correct difference given by  $0_{\Delta}00110$ .

**(b)** The difference of the two positive binary fractions  $0_{\Delta}10001 - 0_{\Delta}01011$  can be carried out as an addition of the positive binary fraction  $0_{\Delta}10001$  and the ones'-complement representation of  $-0_{\Delta}01011$  which is given by  $1_{\Delta}10100$ . The process is illustrated below:

$$\begin{array}{r}
\phantom{+} \phantom{0} \phantom{\Delta} \phantom{1} \phantom{0} \phantom{0} \phantom{0} \phantom{1} \phantom{1} \phantom{\leftarrow} \phantom{\text{carry}} \\
\phantom{+} \phantom{0} \phantom{\Delta} \phantom{1} \phantom{0} \phantom{0} \phantom{0} \phantom{1} \\
+ \phantom{0} \phantom{\Delta} \phantom{1} \phantom{0} \phantom{1} \phantom{0} \phantom{0} \\
\hline
1 \phantom{0} \phantom{\Delta} \phantom{0} \phantom{0} \phantom{1} \phantom{0} \phantom{1} \\
\phantom{1} \phantom{0} \phantom{\Delta} \phantom{0} \phantom{0} \phantom{1} \phantom{0} \phantom{1} \phantom{1} \phantom{\leftarrow} \phantom{\text{end around carry}} \\
\hline
\phantom{0} \phantom{\Delta} \phantom{0} \phantom{0} \phantom{1} \phantom{1} \phantom{0}
\end{array}$$

The extra bit 1 on the left side of the sign bit is brought around and added to the LSB resulting in the correct difference given by  $0_{\Delta}00110$ .

**8.52**  $\eta_1 = 0.6875_{10} = 0_{\Delta}1011$ ,  $\eta_2 = 0.8125_{10} = 0_{\Delta}1101$ ,  $\eta_3 = -0.5625_{10} = 1_{\Delta}0111$ . Now,  $\eta_1 + \eta_2 + \eta_3 = 0.6875_{10} + 0.8125_{10} - 0.5625_{10} = 0.9375_{10}$ . We first form the binary addition  $\eta_1 + \eta_2 = 0_{\Delta}1011 + 0_{\Delta}1101 = 1_{\Delta}1000$  indicating an overflow. If we ignore the overflow and add to the partial sum  $\eta_3$  we arrive at  $(\eta_1 + \eta_2) + \eta_3 = 1_{\Delta}1000 + 1_{\Delta}0111 = 10_{\Delta}1111$ . Dropping the leading 1 from the sum we get  $\eta_1 + \eta_2 + \eta_3 = 0_{\Delta}1111$  whose decimal equivalent is  $0.9375_{10}$ . As a result, the correct sum is obtained in spite of intermediate overflow that occurred.

**8.53 (a)**  $0_{\Delta}11101 \times 1_{\Delta}10111$ . In this case the multiplier is a negative number. Now a multiplier  $D = d_s d_{-1} d_{-2} \dots d_{-b}$  has the value  $-d_s + \sum_{i=1}^b d_{-i} 2^{-i}$ . So the product is given by  $x = A \cdot D = -A d_s + A \cdot (0_{\Delta} d_{-1} d_{-2} \dots d_{-b})$ . Hence we first ignore the sign bit of the multiplier and in the end if the sign bit is a 1, the value of the multiplicand is subtracted from the partial product.

$$\begin{array}{r}
\begin{array}{r}
0 \ \Delta \ 1 \ 1 \ 1 \ 0 \ 1 \\
\times 0 \ \Delta \ 1 \ 0 \ 1 \ 1 \ 1 \\
\hline
P^{(0)} \ 0 \ \Delta \ 0 \ 0 \ 0 \ 0 \ 0 \\
+ 0 \ \Delta \ 1 \ 1 \ 1 \ 0 \ 1 \\
\hline
0 \ \Delta \ 1 \ 1 \ 1 \ 0 \ 1 \\
\hline
P^{(1)} \ 0 \ \Delta \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \\
+ 0 \ \Delta \ 1 \ 1 \ 1 \ 0 \ 1 \\
\hline
1 \ \Delta \ 0 \ 1 \ 0 \ 1 \ 1 \ 1 \\
\hline
P^{(2)} \ 0 \ \Delta \ 1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 1 \\
+ 0 \ \Delta \ 1 \ 1 \ 1 \ 0 \ 1 \\
\hline
1 \ \Delta \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \\
\hline
P^{(3)} \ 0 \ \Delta \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \\
\hline
P^{(4)} \ 0 \ \Delta \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \\
+ 0 \ \Delta \ 1 \ 1 \ 1 \ 0 \ 1 \\
\hline
1 \ \Delta \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \\
\hline
P^{(5)} \ 0 \ \Delta \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1
\end{array}
\end{array}$$

As  $d_s = 1$ , we need to subtract  $0_{\Delta}11101$  from  $P^{(5)}$  or equivalently, adding its two's-complement  $1_{\Delta}00011$  to  $P^{(5)}$  as indicated below:

$$\begin{array}{r}
0 \ \Delta \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \\
+ 1 \ \Delta \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \\
\hline
1 \ \Delta \ 1 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1
\end{array}$$

Hence the final product is  $1_{\Delta}1011111011$ . Its decimal equivalent is  $-0.254882813_{10}$  which is the correct result of the product  $0.90625_{10} \times (-0.28125_{10})$ .

**(b)** In this case the multiplicand is negative. Hence we follow the same steps as given on Page 547 of Text except the addition is now two's-complement addition.

$$\begin{array}{r}
\begin{array}{r}
1 \quad \Delta \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \\
\times 0 \quad \Delta \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \\
\hline
P^{(0)} \quad 0 \quad \Delta \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
+ 1 \quad \Delta \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \\
\hline
P^{(1)} \quad 1 \quad \Delta \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \\
+ 1 \quad \Delta \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \\
\hline
1 \quad 1 \quad \Delta \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\
\hline
P^{(2)} \quad 1 \quad \Delta \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\
+ 1 \quad \Delta \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \\
\hline
1 \quad 1 \quad \Delta \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \\
\hline
P^{(3)} \quad 1 \quad \Delta \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \\
\hline
P^{(4)} \quad 1 \quad \Delta \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \\
+ 1 \quad \Delta \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \\
\hline
1 \quad 1 \quad \Delta \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \\
\hline
P^{(5)} \quad 1 \quad \Delta \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1
\end{array}
\end{array}$$

Hence the result of the multiplication  $1_{\Delta}1 \ 0 \ 1 \ 0 \ 1 \times 0_{\Delta}1 \ 0 \ 1 \ 1 \ 1$  is  $1_{\Delta}1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1$ ,

The decimal equivalent is  $(-0.34375_{10}) \times 0.71875 = -0.247070313_{10}$ .

**8.54 (a)** Again in this case the multiplier is negative. Hence we can write

$$x = A \cdot D = A \cdot \left( -d_s(1 - 2^{-b}) + \sum_{i=1}^b d_{-i} 2^{-i} \right) = -Ad_s + Ad_s 2^{-b} + \tilde{x},$$

where  $\tilde{x}$  is the product of A and D with its sign bit  $d_s$  ignored. Hence, we first form  $\tilde{x}$  and if  $d_s = 1$ , we subtract A from  $\tilde{x}$ , and add to the result a shifted version (b bits to the right) of A.

Now  $\tilde{x}$  is same as the product obtained in Problem 8.52(a) and is given by

$$\tilde{x} = 0_{\Delta}1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1.$$

Subtracting A from  $\tilde{x}$  is equivalent to adding the ones'-complement of A to  $\tilde{x}$  as indicated below:

$$\begin{array}{r}
0 \quad \Delta \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \\
+ 1 \quad \Delta \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\
\hline
1 \quad \Delta \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0
\end{array}$$

Next we apply the final correction of adding  $0_{\Delta}0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1$  to  $\tilde{x} - A$ :

$$\begin{array}{r}
1 \quad \Delta \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \\
+ 0 \quad \Delta \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 1 \\
\hline
1 \quad \Delta \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1
\end{array}$$

Hence the result of the product  $0_{\Delta}1 \ 1 \ 1 \ 0 \ 1 \times 1_{\Delta}1 \ 0 \ 1 \ 1 \ 1$  is given by  $1_{\Delta}1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 1$ .

The decimal equivalent is  $0.90625_{10} \times (-0.25_{10}) = -0.22265625_{10}$ .

**(b)** In this case the multiplicand is negative. Hence we follow the same steps as on Page 547 of Text except the addition is now ones'-complement addition:



$$\begin{array}{r}
\begin{array}{cccccccc}
& 1 & \Delta & 1 & 0 & 1 & 0 & 1 \\
\times & 0 & \Delta & 1 & 0 & 1 & 1 & 1 \\
\hline
P^{(0)} & 0 & \Delta & 0 & 0 & 0 & 0 & 0 \\
+ & 1 & \Delta & 1 & 0 & 1 & 0 & 1 \\
\hline
& 1 & \Delta & 1 & 0 & 1 & 0 & 1 \\
\hline
P^{(1)} & 1 & \Delta & 1 & 1 & 0 & 1 & 0 & 1 \\
+ & 1 & \Delta & 1 & 0 & 1 & 0 & 1 \\
\hline
& 1 & 1 & \Delta & 0 & 1 & 1 & 1 & 1 & 1 \\
\vdots & & & & & & & & & \rightarrow 1 \\
\hline
& 1 & \Delta & 1 & 0 & 0 & 0 & 0 & 1 \\
\hline
P^{(2)} & 1 & \Delta & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
+ & 1 & \Delta & 1 & 0 & 1 & 0 & 1 \\
\hline
& 1 & 1 & \Delta & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
\vdots & & & & & & & & & & \rightarrow 1 \\
\hline
& 1 & \Delta & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
\hline
P^{(3)} & 1 & \Delta & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
\hline
P^{(4)} & 1 & \Delta & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
+ & 1 & \Delta & 1 & 0 & 1 & 0 & 1 \\
\hline
& 1 & 1 & \Delta & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\vdots & & & & & & & & & & & & \rightarrow 1 \\
\hline
& 1 & \Delta & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\hline
P^{(5)} & 1 & \Delta & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{array}
\end{array}$$

Hence the result of the product  $1_{\Delta}10101 \times 0_{\Delta}10111$  is given by  $1_{\Delta}1100011001$ .

The decimal equivalent is  $(-0.3125_{10}) \times 0.71875_{10} = -0.224609375_{10}$ .

**8.58** The transformation  $\cos \omega = \alpha + \beta \cos \hat{\omega}$  is equivalent to  $\frac{e^{j\omega} + e^{-j\omega}}{2} = \alpha + \beta \left( \frac{e^{j\hat{\omega}} + e^{-j\hat{\omega}}}{2} \right)$ , which

by analytic continuation can be expressed as  $\frac{z+z^{-1}}{2} = \alpha + \beta \left( \frac{\hat{z} + \hat{z}^{-1}}{2} \right)$ . Now, let  $H(z)$  be a

Type I linear-phase FIR transfer function of degree  $2M$ . As indicated in Eq. (6.143),  $H(z)$  can

be expressed as  $H(z) = z^{-M} \sum_{n=0}^M a[n] \left( \frac{z+z^{-1}}{2} \right)^n$  with a frequency response given by

$H(e^{j\omega}) = e^{-jM\omega} \sum_{n=0}^M a[n] (\cos \omega)^n$ , with  $\hat{H}(\omega) = \sum_{n=0}^M a[n] (\cos \omega)^n$  denoting the amplitude function

or the zero-phase frequency response. The amplitude function or the zero-phase frequency response of the transformed filter obtained by applying the mapping  $\cos \omega = \alpha + \beta \cos \hat{\omega}$  is

therefore given by  $\hat{H}(\hat{\omega}) = \sum_{n=0}^M a[n](\alpha + \beta \cos \hat{\omega})^n$ . Or equivalently, the transfer function of the

transformed filter is given by  $H(\hat{z}) = \hat{z}^{-M} \sum_{n=0}^M a[n] \left( \alpha + \beta \frac{\hat{z} + \hat{z}^{-1}}{2} \right)^n$ . A convenient way to realize

$H(\hat{z})$  is to consider the realization of the parent transfer function  $H(z)$  in the form of a Taylor structure as outlined in Problem 6.23 which is obtained by expressing  $H(z)$  in the form

$H(z) = \sum_{n=0}^M a[n] z^{-M+n} \left( \frac{1+z^{-2}}{2} \right)^n$ . Similarly, the transfer function of the transformed filter can

be expressed as  $H(\hat{z}) = \sum_{n=0}^M a[n] \hat{z}^{-M+n} \left( \alpha \hat{z}^{-1} + \beta \frac{1+\hat{z}^{-2}}{2} \right)^n$ . As a result, the transformed filter

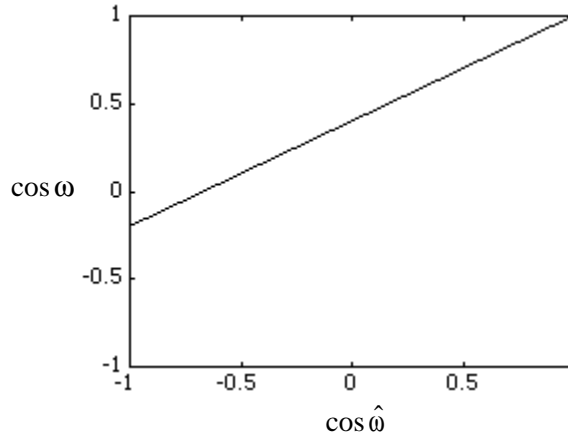
can be realized by replacing each block  $\frac{1+z^{-2}}{2}$  in the Taylor structure realization of  $H(z)$  by

the block  $\alpha \hat{z}^{-1} + \beta \frac{1+\hat{z}^{-2}}{2}$ .

Now, for a lowpass-to-lowpass transformation, we can impose the condition

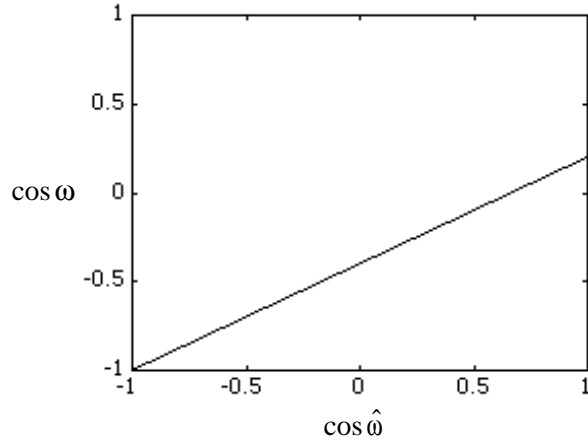
$\hat{H}(\hat{\omega})|_{\hat{\omega}=0} = H(\omega)|_{\omega=0}$ . This condition is met if  $\alpha + \beta = 1$  and  $0 \leq \alpha < 1$ . In this case, the

transformation reduces to  $\cos \omega = \alpha + (1-\alpha) \cos \hat{\omega}$ . From the plot of the mapping given below it follows that as  $\alpha$  is varied between 0 and 1,  $\hat{\omega}_c < \omega_c$ .

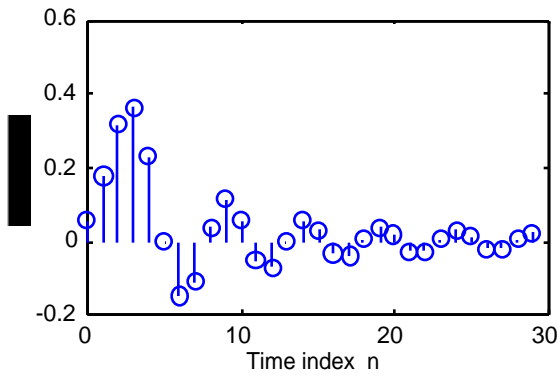


On the other hand if  $\hat{\omega}_c > \omega_c$  is desired along with a lowpass-to-lowpass transformation, we can impose the condition  $\hat{H}(\hat{\omega})|_{\hat{\omega}=\pi} = H(\omega)|_{\omega=\pi}$ . This condition is met if  $\beta = 1 + \alpha$  and  $-1 < \alpha \leq 0$ .

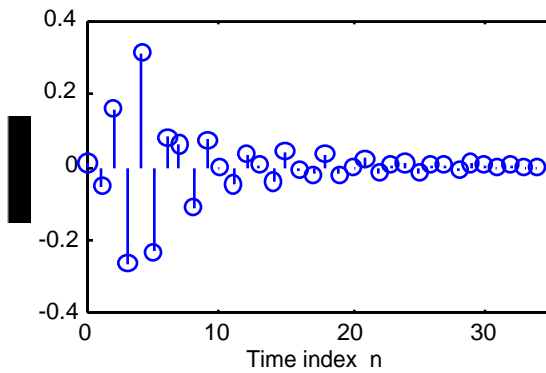
The corresponding transformation is now given by  $\cos \omega = \alpha + (1+\alpha) \cos \hat{\omega}$ . From the plot of the mapping given below it follows that as  $\alpha$  is varied between  $-1$  and  $0$ ,  $\hat{\omega}_c > \omega_c$ .



**M8.1** 
$$H(z) = \frac{0.052817 + 0.079708z^{-1} + 0.1295z^{-2} + 0.1295z^{-3} + 0.079708z^{-4} + 0.052817z^{-5}}{1 - 1.8107z^{-1} + 2.4947z^{-2} - 1.8801z^{-3} + 0.95374z^{-4} - 0.23359z^{-5}}$$



**M8.2** 
$$H(z) = \frac{0.0083632 - 0.033453z^{-1} + 0.050179z^{-2} - 0.033453z^{-3} + 0.0083632z^{-4}}{1 + 2.3741z^{-1} + 2.7057z^{-2} + 1.5917z^{-3} + 0.41032z^{-4}}$$



**M8.3** Numerator coefficients

Columns 1 through 7  
 2.7107e-04   0   -1.8975e-03   0   5.6924e-03   0   -9.4874e-03

Columns 8 through 14

0 9.4874e-03 0 -5.6924e-03 0 1.8975e-03 0

Column 15

-2.7107e-04

Denominator coefficients

Columns 1 through 5

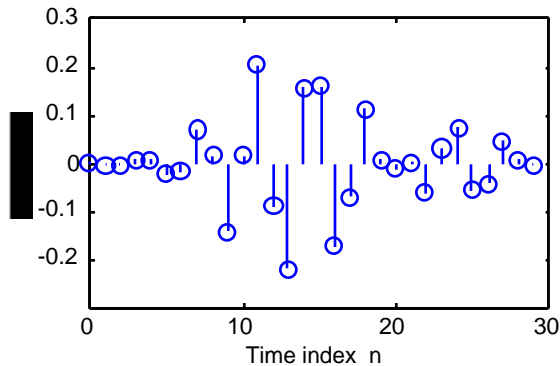
1.0000e+00 1.7451e+00 4.9282e+00 6.1195e+00 9.8134e+00

Columns 6 through 10

9.2245e+00 1.0432e+01 7.5154e+00 6.4091e+00 3.4595e+00

Columns 11 through 15

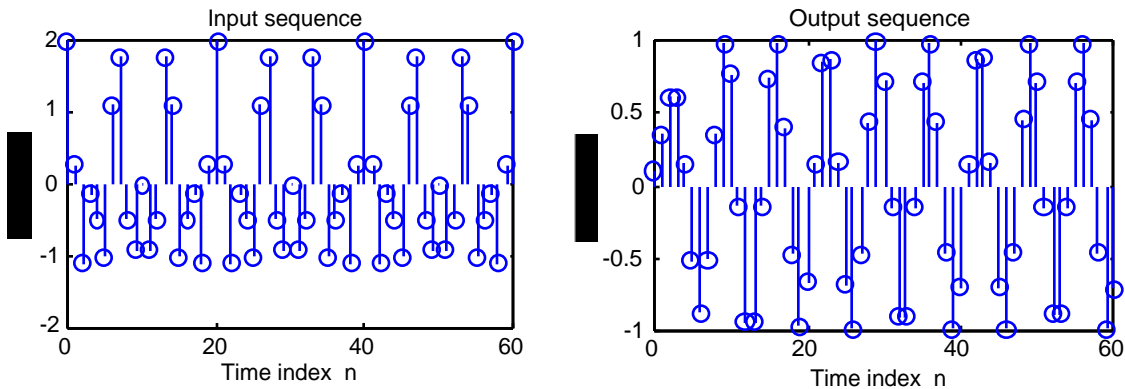
2.2601e+00 8.4696e-01 4.1671e-01 8.5581e-02 2.9919e-02



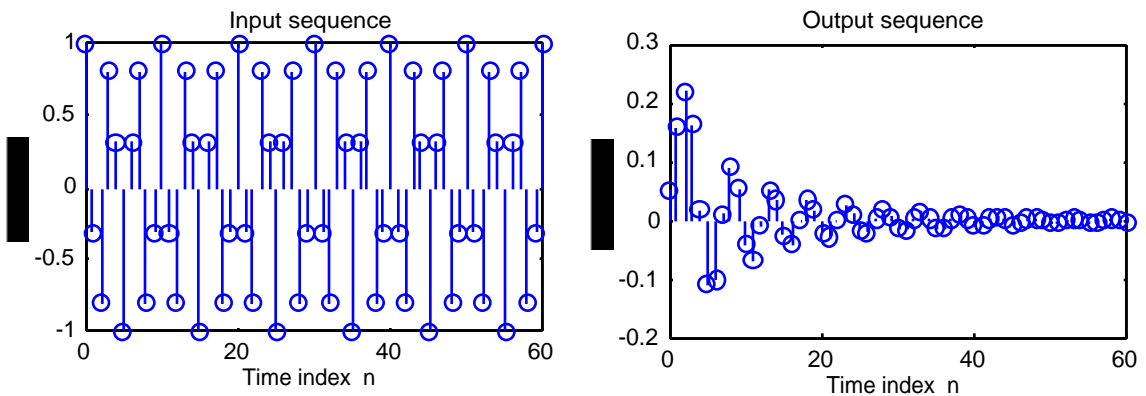
**M8.4** Modified Program 8\_2 is given below:

```
n = 0:60;
w = input('Angular frequency vector = ');
num = input('Numerator coefficients = ');
den = input('Denominator coefficients = ');
w = input('Normalized angular frequency vector = ');
x1 = cos(w(1)*pi*n); x2 = cos(w(2)*pi*n);
x = x1+x2;
subplot(2,1,1);
stem(n,x);
title('Input sequence');
xlabel('Time index n');ylabel('Amplitude');
[N,Wn] = ellipord(0.4,0.5,0.5,40);
[num,den] = ellip(N,0.5,40,Wn);
y = filter(num,den,x);
subplot(2,1,2);
stem(n,y);
title('Output sequence');
xlabel('Time index n'); ylabel('Amplitude');
```

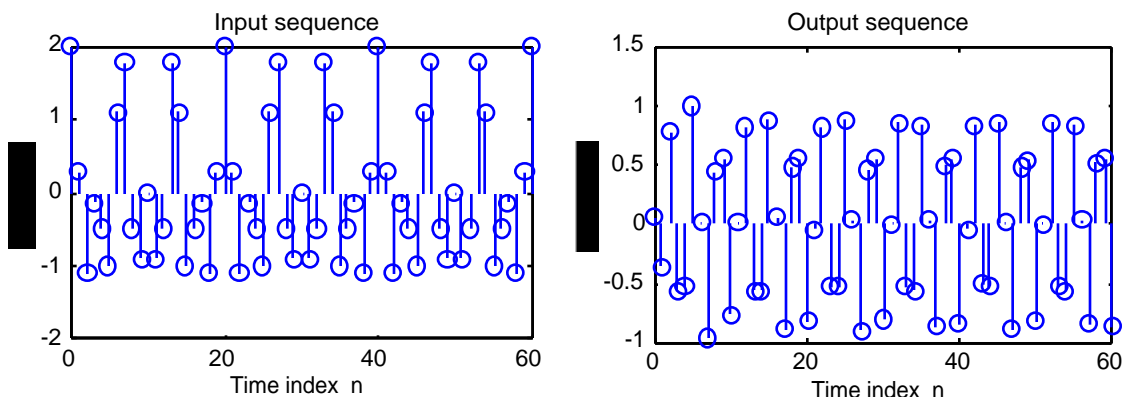
The plots generated by this program for the filter of Example 7.19 for an input composed of a sum of two sinusoidal sequences of angular frequencies,  $0.3\pi$  and  $0.6\pi$ , are given below:



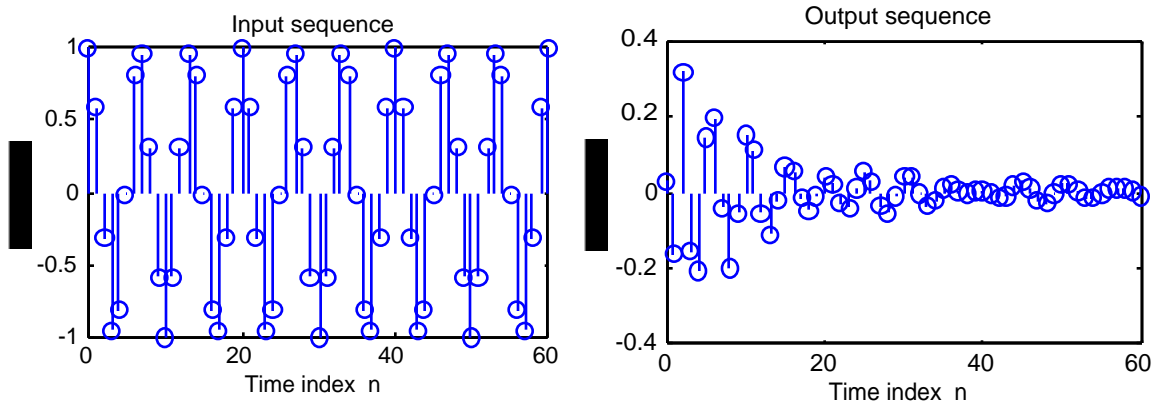
The blocking of the high-frequency signal by the lowpass filter can be demonstrated by replacing the statement `stem(k, x);` with `stem(k, x2);` and the statement `y = filter(num, den, x);` in the above program with the following:  
`y = filter(num, den, x2);`. The plots of the input high-frequency signal component and the corresponding output are indicated below:



**M8.5** The plots generated by using the modified program of Problem M8.4 and using the data of this problem are shown below:



The blocking of the low-frequency signal by the highpass filter can be demonstrated by replacing the statement `stem(k, x);` with `stem(k, x1);` and the statement `y = filter(num, den, x);` in the above program with the following:  
`y = filter(num, den, x1);`. The plots of the input high-frequency signal component and the corresponding output are indicated below:



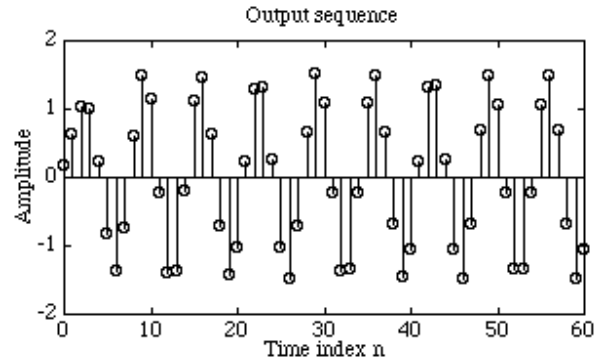
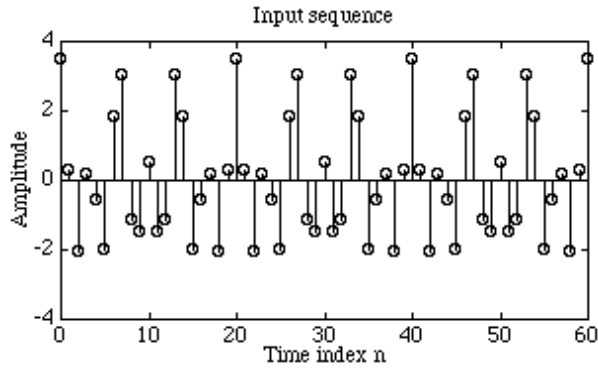
**M8.6** % The factors for the transfer of example 7.16 are

```
% num1 = [0.2546    0.2546    0]
% den1 = [1.0000   -0.4909    0]
% num2 = [0.2982    0.1801    0.2982]
% den2 = [1.0000   -0.7624    0.5390]
% num3 = [0.6957   -0.0660    0.6957]
% den3 = [1.0000   -0.5574    0.8828]
N = input('The total number of sections = ');
for k = 1:N;
    num(k,:) = input('The numerator = ');
    den(k,:) = input('The denominator = ');
end
n = 0:60;
w = input('Normalized angular frequency vector = ');
x1 = cos(w(1)*pi*n); x2 = cos(w(2)*pi*n);
x = x1+x2;
subplot(2,1,1);
stem(n,x);
xlabel('Time index n'); ylabel('Amplitude');
title('Input sequence')
si = [0 0];
for k = 1:N
    y(k,:) = filter(num(k,:),den(k,:),x,si);
    x = y(k,:);
end
subplot(2,1,2);
stem(n,x);axis([0 50 -4 4]);
xlabel('Time index n'); ylabel('Amplitude');
title('Output sequence')
```

**M8.7** %The factors for the highpass filter are

```
% num1 = [0.0495   -0.1006    0.0511]
% den1 = [1.0000    1.3101    0.5151]
% num2 = [0.1688   -0.3323    0.1636]
% den2 = [1.0000    1.0640    0.7966]
```

**M8.8** To apply the function `direct2` to filter a sum of two sinusoidal sequences, we replace the statement `y = filter(num,den,x,si)` in the MATLAB program given in the solution of Problem M8.4 with the statement `y = direct2(num,den,x,si)`. The plots generated by the modified program for the data given in this problem are given below:



**M8.9** The MATLAB program that can be used to compute all DFT samples using the function `gfft` and the function `fft` is as follows:

```
clear
N = input('Desired DFT length = ');
x = input('Input sequence = ');
for j = 1:N
    Y(j) = gfft(x,N,j-1);
end
disp('DFT samples computed using gfft are ');
disp(Y);
disp('DFT samples computed using fft are ');
X = fft(x,N); disp(X);
```

Results obtained for the computation of two input sequences  $\{x[n]\}$  of lengths 8, and 12, respectively, are given below:

```
Desired DFT length = 8
Input sequence = [1 2 3 4 4 3 2 1]

FFT values computed using gfft are
Columns 1 through 4
20.0000    -5.8284 + 2.4142i    0    -0.1716 + 0.4142i
Columns 5 through 8
0    -0.1716 - 0.4142i    0    -5.8284 - 2.4142i
```

```
FFT values using fft are
Columns 1 through 4
20.0000    -5.8284 - 2.4142i    0    -0.1716 - 0.4142i
Columns 5 through 8
0    -0.1716 + 0.4142i    0    -5.8284 + 2.4142i
```

```
Desired DFT length = 12
Input sequence = [2 4 8 12 1 3 5 7 9 6 0 1]
```

```
FFT values computed using gfft are
Columns 1 through 4
58.0000    -8.3301 + 5.5000i    -12.5000 +19.9186i
-1.0000 - 7.0000i
Columns 5 through 8
8.5000 - 7.7942i    0.3301 + 5.5000i    -8.0000
0.3301 - 5.5000i
Columns 9 through 12
8.5000 + 7.7942i    -1.0000 + 7.0000i    -12.5000
```

```
-19.9186i -8.3301 - 5.5000i
```

FFT values using fft are

Columns 1 through 4

```
58.0000 -8.3301 - 5.5000i -12.5000 -19.9186i  
-1.0000 + 7.0000i
```

Columns 5 through 8

```
8.5000 + 7.7942i 0.3301 - 5.5000i -8.0000 + 0.0000i  
0.3301 + 5.5000i
```

Columns 9 through 12

```
8.5000 - 7.7942i -1.0000 - 7.0000i -12.5000  
+19.9186i -8.3301 + 5.5000i
```

**M8.10** The MATLAB program that can be used to verify the plots of Figure 8.37 is given below:

```
[z,p,k] = ellip(5,0.5,40,0.4);  
a = conv([1 -p(1)],[1 -p(2)]);b = [1 -p(5)];  
c = conv([1 -p(3)],[1 -p(4)]);  
w = 0:pi/255:pi;  
alpha = 0;  
an1 = a(2) + (a(2)*a(2) - 2*(1 + a(3)))*alpha;  
an2 = a(3) + (a(3) -1)*a(2)*alpha;  
g = b(2) - (1 - b(2)*b(2))*alpha;  
cn1 = c(2) + (c(2)*c(2) - 2*(1 + c(3)))*alpha;  
cn2 = c(3) + (c(3) -1)*c(2)*alpha;  
a = [1 an1 an2];b = [1 g]; c = [1 cn1 cn2];  
h1 = freqz(fliplr(a),a,w); h2 = freqz(fliplr(b),b,w);  
h3 = freqz(fliplr(c),c,w);  
ha = 0.5*(h1.*h2 + h3);ma = 20*log10(abs(ha));  
alpha = 0.1;  
an1 = a(2) + (a(2)*a(2) - 2*(1 + a(3)))*alpha;  
an2 = a(3) + (a(3) -1)*a(2)*alpha;  
g = b(2) - (1 - b(2)*b(2))*alpha;  
cn1 = c(2) + (c(2)*c(2) - 2*(1 + c(3)))*alpha;  
cn2 = c(3) + (c(3) -1)*c(2)*alpha;  
a = [1 an1 an2];b = [1 g]; c = [1 cn1 cn2];  
h1 = freqz(fliplr(a),a,w); h2 = freqz(fliplr(b),b,w);  
h3 = freqz(fliplr(c),c,w);  
hb = 0.5*(h1.*h2 + h3);mb = 20*log10(abs(hb));  
alpha = -0.25;  
an1 = a(2) + (a(2)*a(2) - 2*(1 + a(3)))*alpha;  
an2 = a(3) + (a(3) -1)*a(2)*alpha;  
g = b(2) - (1 - b(2)*b(2))*alpha;  
cn1 = c(2) + (c(2)*c(2) - 2*(1 + c(3)))*alpha;  
cn2 = c(3) + (c(3) -1)*c(2)*alpha;  
a = [1 an1 an2];b = [1 g]; c = [1 cn1 cn2];  
h1 = freqz(fliplr(a),a,w); h2 = freqz(fliplr(b),b,w);  
h3 = freqz(fliplr(c),c,w);  
hc = 0.5*(h1.*h2 + h3);mc = 20*log10(abs(hc));  
plot(w/pi,ma,'r-',w/pi,mb,'b--',w/pi,mc,'g-.');axis([0 1 -80  
5]);  
xlabel('Normalized frequency');ylabel('Gain, dB');  
legend('b--','alpha = 0.1 ','w',' ','r-','alpha = 0  
,','w',' ','g-.','alpha = -0.25 ');
```

**M8.11** The MATLAB program that can be used to verify the plots of Figure 8.39 is given below:

```
w = 0:pi/255:pi;
```



```

wc2 = 0.31*pi;
f = [0 0.36 0.46 1];m = [1 1 0 0];
b1 = remez(50, f, m);
h1 = freqz(b1,1,w);
m1 = 20*log10(abs(h1));
n = -25:-1;
c = b1(1:25)./sin(0.41*pi*n);
d = c.*sin(wc2*n);q = (b1(26)*wc2)/(0.4*pi);
b2 = [d q fliplr(d)];
h2 = freqz(b2,1,w);
m2 = 20*log10(abs(h2));
wc3 = 0.51*pi;
d = c.*sin(wc3*n);q = (b1(26)*wc3)/(0.4*pi);
b3 = [d q fliplr(d)];
h3 = freqz(b3,1,w);
m3 = 20*log10(abs(h3));
plot(w/pi,m1,'r-',w/pi,m2,'b--',w/pi,m3,'g-.');
axis([0 1 -80 5]);
xlabel('Normalized frequency');ylabel('Gain, dB');
legend('b--','wc = 0.31π','w',' ','r-','wc = 0.41π','w',' ','g-','wc = 0.51π')

```

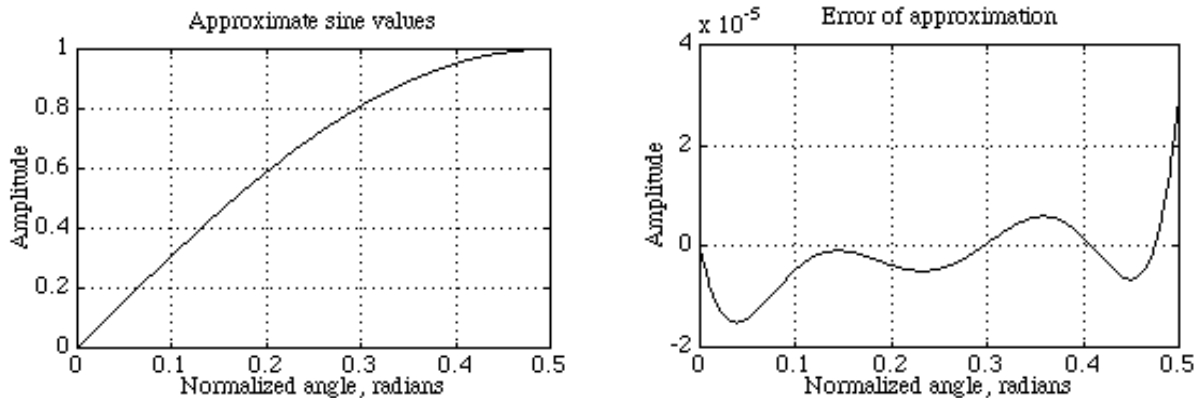
**M8.12** The MATLAB program to evaluate Eq. (8.104) is given below:

```

x = 0:0.001:0.5;
y = 3.140625*x + 0.0202636*x.^2 - 5.325196*x.^3 +
0.5446778*x.^4 + 1.800293*x.^5;
x1 = pi*x;
z = sin(x1);
plot(x,y);xlabel('Normalized angle,
radians');ylabel('Amplitude');
title('Approximate sine values');grid;axis([0 0.5 0 1]);
pause
plot(x,y-z);xlabel('Normalized angle,
radians');ylabel('Amplitude');
title('Error of approximation');grid;

```

The plots generated by the above program are as indicated below:



**M8.13** The MATLAB program to evaluate Eq. (8.110) is given below:

```

k = 1;

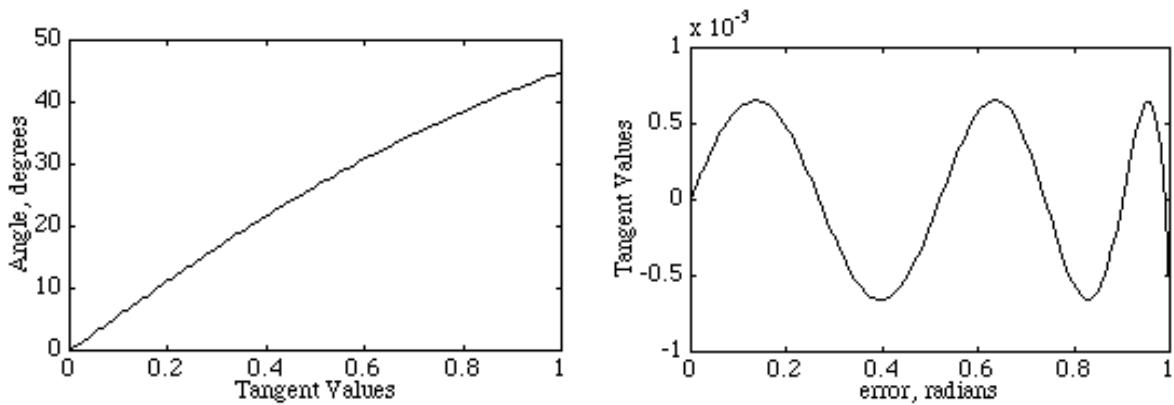
```

```

for x = 0:.01:1
    op1 = 0.318253*x+0.00331*x^2-0.130908*x^3+0.068524*x^4-
0.009159*x^5;
    op2 = 0.999866*x-0.3302995*x^3+0.180141*x^5-
0.085133*x^7+0.0208351*x^9;
    arctan1(k) = op1*180/pi;
    arctan2(k) = 180*op2/pi;
    actual(k) = atan(x)*180/pi;
    k = k+1;
end
subplot(211)
x = 0:.01:1;
plot(x,arctan2);
ylabel('Angle, degrees');
xlabel('Tangent Values');
subplot(212)
plot(x,actual-arctan2,'--');
ylabel('Tangent Values');
xlabel('error, radians');

```

The plots generated by the above program are as indicated below:



Note: Expansion for  $\arctan(x)$  given by Eq. (8.110) gives the result in normalized radians, i.e. the actual value in radians divided by  $\pi$ .

## Chapter 9 (2e)

**9.1 Two's Complement Truncation** - Assume  $x > 0$ . The relative error  $e_t$  is given by

$$e_t = \frac{Q(x) - x}{x} = \frac{Q(M) - M}{M} = \frac{\sum_{i=1}^b a_{-i} 2^{-i} - \sum_{i=1}^{\beta} a_{-i} 2^{-i} - \sum_{i=b+1}^{\beta} a_{-i} 2^{-i}}{M}.$$

Now  $e_t$  will be a minimum if all  $a_{-i}$ 's are 1 and will be a maximum if all  $a_{-i}$ 's are 0 for  $b+1 \leq i \leq \beta$ , Thus  $-\frac{\delta}{M} \leq e_t \leq \frac{0}{M}$ . Since  $0.5 \leq M \leq 1$  hence  $-2\delta \leq e_t \leq 0$ .

Now consider  $x < 0$ . Here, the relative error  $e_t$  is given by

$$e_t = \frac{Q(x) - x}{x} = \frac{Q(M) - M}{M} = \frac{-1 + \sum_{i=1}^b a_{-i} 2^{-i} + 1 - \sum_{i=1}^{\beta} a_{-i} 2^{-i}}{M}. \text{ As before, } -\frac{\delta}{M} \leq e_t \leq \frac{0}{M}. \text{ In this case } -1 < M \leq -0.5, \text{ and, hence } 0 \leq e_t \leq 2\delta.$$

**Ones' Complement Truncation** - Assume again  $x > 0$ . The relative error  $e_t$  is given by

$$e_t = \frac{Q(x) - x}{x} = \frac{Q(M) - M}{M} = \frac{\sum_{i=1}^b a_{-i} 2^{-i} - \sum_{i=1}^{\beta} a_{-i} 2^{-i} - \sum_{i=b+1}^{\beta} a_{-i} 2^{-i}}{M}.$$

Now  $e_t$  will be a minimum if all  $a_{-i}$ 's are 1 and will be a maximum if all  $a_{-i}$ 's are 0 for  $b+1 \leq i \leq \beta$ , Thus  $-\frac{\delta}{M} \leq e_t \leq \frac{0}{M}$ . Since  $0.5 \leq M \leq 1$  hence  $-2\delta \leq e_t \leq 0$ .

Now consider  $x < 0$ . Here, the relative error  $e_t$  is given by

$$e_t = \frac{Q(x) - x}{x} = \frac{Q(M) - M}{M} = \frac{-(1 - 2^{-b}) + \sum_{i=1}^b a_{-i} 2^{-i} + (1 - 2^{-\beta}) - \sum_{i=1}^{\beta} a_{-i} 2^{-i}}{M} \\ = \frac{(2^{-b} - 2^{-\beta}) + \sum_{i=b+1}^{\beta} a_{-i} 2^{-i}}{M}.$$

Now  $e_t$  will be a maximum if all  $a_{-i}$ 's are 0 and will be a minimum if all  $a_{-i}$ 's are 1, In this case,

$$\frac{0}{M} < e_t \leq \frac{\delta}{M}. \text{ Since, } -0.5 \leq M < -1, \text{ hence } -2\delta < e_t \leq 0.$$

**Sign-Magnitude Truncation** - Assume  $x > 0$ . Here,

$$e_t = \frac{Q(M) - M}{M} = \frac{-\sum_{i=1}^b a_{-i} 2^{-i} + \sum_{i=1}^{\beta} a_{-i} 2^{-i} - \sum_{i=b+1}^{\beta} a_{-i} 2^{-i}}{M}. \text{ Since, } -1 \leq M \leq -0.5, \text{ hence, } -2\delta \leq e_t < 0.$$

Rounding -  $\varepsilon = Q(M) - M$ . Hence  $\frac{-\delta}{2} \leq \varepsilon \leq \frac{\delta}{2}$ . This implies  $\frac{-\delta}{2M} \leq \mathbf{e}_r \leq \frac{\delta}{2M}$ .  
 Since  $-1 < M \leq -0.5$ , and, hence  $-\delta \leq \mathbf{e}_r \leq \delta$ .

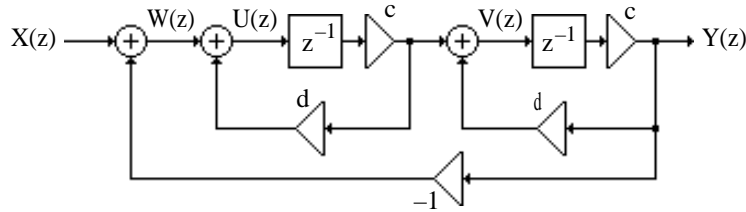
**9.2** The denominator is given by  $D(z) = z - \alpha$ . Hence, the pole is at  $z = r e^{j\theta} = \alpha$  where  $r = \alpha$  and  $\theta = 0$ . The components of the pole sensitivities are therefore given by  $\frac{\partial r}{\partial \alpha} = 1$  and  $\frac{\partial \theta}{\partial \alpha} = 0$ .

**9.3** The denominator is given by  $D(z) = z^2 - \beta(1 + \alpha)z + \alpha = (z - r e^{j\theta})(z - r e^{-j\theta}) = z^2 - 2r \cos \theta z + r^2$ . Comparing we get,  $2r \cos \theta = \beta(1 + \alpha)$  and  $r^2 = \alpha$ . Hence, taking the partials of both sides of the two equations we get  $\begin{bmatrix} 2r & 0 \\ 2 \cos \theta & -2r \sin \theta \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \beta & 1 + \alpha \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta \beta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2r \cos \theta}{1+r^2} & 1+r^2 \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta \beta \end{bmatrix}$  or  $\begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} 2r & 0 \\ 2 \cos \theta & -2r \sin \theta \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ \frac{2r \cos \theta}{1+r^2} & 1+r^2 \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta \beta \end{bmatrix} = \begin{bmatrix} \frac{1}{2r} & 0 \\ \frac{(1-r^2) \cos \theta}{2r^2(1+r^2) \sin \theta} & -\frac{1+r^2}{2r \sin \theta} \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta \beta \end{bmatrix}$ .

The components of the pole sensitivities are therefore given by,

$$\frac{\partial r}{\partial \alpha} = \frac{1}{2r}, \quad \frac{\partial r}{\partial \beta} = 0, \quad \frac{\partial \theta}{\partial \alpha} = \frac{(1-r^2) \cot \theta}{2r^2(1+r^2)}, \quad \text{and} \quad \frac{\partial \theta}{\partial \beta} = -\frac{1+r^2}{2r \sin \theta}.$$

**9.4** Analysis of the digital filter structure yields



$$W(z) = X(z) - Y(z),$$

$$U(z) = W(z) + c d z^{-1} U(z) \text{ which implies } U(z) = \frac{W(z)}{1 - c d z^{-1}} = \frac{X(z) - Y(z)}{1 - c d z^{-1}},$$

$$c z^{-1} V(z) = Y(z) \text{ implies } V(z) = \frac{z Y(z)}{c}. \text{ Also } V(z) = c z^{-1} U(z) + d Y(z).$$

Substituting for  $U(z)$  and  $V(z)$  we get  $\frac{z Y(z)}{c} = c z^{-1} \frac{X(z) - Y(z)}{1 - c d z^{-1}} + d Y(z)$ . Thus,

$$Y(z) \{ 1 - c d z^{-1} - c d z^{-1} (1 - c d z^{-1}) + c^2 z^{-2} \} = c^2 z^{-2} X(z), \text{ hence}$$

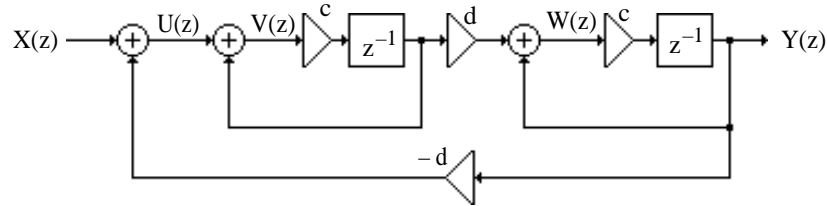
$$H(z) = \frac{Y(z)}{X(z)} = \frac{c^2}{z^2 - 2c d z + c^2(1 + d^2)}.$$

$$\text{Let } c^2(1 + d^2) = r^2 \text{ and } c d = r \cos \theta, \text{ Solving we get } \cos \theta = \frac{d}{\sqrt{1 + d^2}}.$$

Thus,  $d = \cot \theta$  and  $c = r \sin \theta$ . Hence,  $\Delta d = -\operatorname{cosec}^2 \theta (\Delta \theta)$ , and  $\Delta c = \sin \theta (\Delta r) + r \cos \theta (\Delta \theta)$ .

Or,  $\begin{bmatrix} \Delta c \\ \Delta d \end{bmatrix} = \begin{bmatrix} \sin \theta & r \cos \theta \\ 0 & -\operatorname{cosec}^2 \theta \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix}$ , i.e.

$$\begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} = -\sin \theta \begin{bmatrix} -\operatorname{cosec}^2 \theta & -r \cos \theta \\ 0 & \sin \theta \end{bmatrix} \begin{bmatrix} \Delta c \\ \Delta d \end{bmatrix} = \begin{bmatrix} \frac{1}{\sin \theta} & r \sin \theta \cdot \cos \theta \\ 0 & -\sin^2 \theta \end{bmatrix} \begin{bmatrix} \Delta c \\ \Delta d \end{bmatrix}$$



Analysis of the above structure yields,  $U(z) = X(z) - dY(z)$

$$V(z) = U(z) + cz^{-1}V(z) \text{ which implies } V(z) = \frac{U(z)}{1 - cz^{-1}}.$$

$$W(z) = cdz^{-1}V(z) + cz^{-1}W(z) \text{ or } W(z) = \frac{cdz^{-1}}{1 - cz^{-1}} V(z) = \frac{cdz^{-1}}{(1 - cz^{-1})^2} U(z).$$

$$Y(z) = cz^{-1}W(z) \text{ or } \frac{Y(z)}{cz^{-1}} = \frac{cdz^{-1}(X(z) - dY(z))}{(1 - cz^{-1})^2}.$$

Thus  $Y(z)\{(1 - cz^{-1})^2 + c^2dz^{-2}\} = c^2dz^{-2} X(z)$ . Hence,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{c^2d}{z^2 - 2cz + c^2(1 + d^2)}.$$

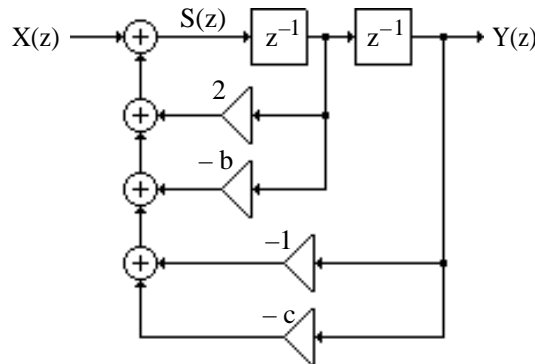
Again let  $c = r \cos \theta$  and  $c^2(1 + d^2) = r^2$ .

Therefore,  $1 + d^2 = \sec^2 \theta$  which implies  $d = \tan \theta$ . Taking partials we then get

$$\Delta c = \cos \theta (\Delta r) - r \sin \theta (\Delta \theta), \text{ and } \Delta d = \sec^2 \theta (\Delta \theta). \text{ Thus,}$$

$$\begin{bmatrix} \Delta c \\ \Delta d \end{bmatrix} = \begin{bmatrix} \cos \theta & -r \sin \theta \\ 0 & \sec^2 \theta \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix}, \text{ which yields } \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} \frac{1}{\cos \theta} & r \sin \theta \cdot \cos \theta \\ 0 & \cos^2 \theta \end{bmatrix} \begin{bmatrix} \Delta c \\ \Delta d \end{bmatrix}.$$

## 9.5



Analysis yields  $X(z) + 2z^{-1}S(z) - bz^{-1}S(z) - Y(z) - cY(z) = S(z)$  and  $Y(z) = z^{-2} S(z)$ . Thus

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2}}{1 + (b-2)z^{-1} + (1+c)z^{-2}} = \frac{1}{z^2 + (b-2)z + (1+c)}.$$

Let  $1 + c = r^2$  and  $2 - b = 2r \cos \theta$ . Taking partials we get  $\Delta c = 2r(\Delta r)$ , and  $-\Delta b = 2 \cos \theta(\Delta r) - 2r \sin \theta(\Delta \theta)$ . Equivalently,

$$\begin{bmatrix} \Delta c \\ \Delta b \end{bmatrix} = \begin{bmatrix} 2r & 0 \\ 2 \cos \theta & -2r \sin \theta \end{bmatrix} \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} \text{ or } \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} \frac{1}{2r} & 0 \\ \frac{1}{2r^2 \tan \theta} & -\frac{1}{2r \sin \theta} \end{bmatrix} \begin{bmatrix} \Delta c \\ \Delta b \end{bmatrix}.$$

From Example 9.2 the pole sensitivities for Figure 9.9 are given by

$$\begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\frac{1}{r} \sin \theta & \frac{1}{r} \cos \theta \end{bmatrix} \begin{bmatrix} \Delta \alpha \\ \Delta \beta \end{bmatrix}.$$

- 9.6 (a)** For direct form implementation  $B(z) = z^3 + b_2 z^2 + b_1 z + b_0 = (z - z_1)(z - z_2)(z - z_3)$  where  $z_1 = r_1 e^{j\theta_1}$ ,  $z_2 = r_1 e^{-j\theta_1}$ , and  $z_3 = r_3 e^{j\theta_3}$ . Thus,  $B(z) = (z^2 - 2r_1 \cos \theta_1 z + r_1^2)(z - r_3) = (z^2 + 0.5111z + 0.7363)(z + 0.3628)$ . Thus,  $2r_1 \cos \theta_1 = -0.5111$ ,  $r_1^2 = 0.7363$ ,  $r_3 = 0.3628$ , and  $\theta_3 = \pi$ .

From the above  $r_1 = \sqrt{0.7363} = 0.8581$  and  $\cos \theta_1 = \frac{-0.5111}{2\sqrt{0.7363}} = -0.2978$ .

$$\begin{aligned} \frac{1}{B(z)} &= \frac{1}{(z^2 + 0.5111z + 0.7363)(z + 0.3628)} \\ &= \frac{-0.7326 - j0.0959}{z + 0.25555 - j0.81914} + \frac{-0.7326 + j0.0959}{z + 0.25555 + j0.81914} + \frac{1.4652}{z + 0.3628}. \end{aligned}$$

$$\mathbf{P}_1 = [\cos \theta_1 \quad r_1 \quad r_1^2 \cos \theta_1] = [-0.2978 \quad 0.8581 \quad -0.2193],$$

$$\mathbf{Q}_1 = [-\sin \theta_1 \quad 0 \quad r_1^2 \sin \theta_1] = [-0.9546 \quad 0 \quad 0.7029],$$

$$R_1 = 0.40357, \quad X_1 = -0.36481,$$

$$\mathbf{P}_3 = [\cos \theta_3 \quad r_3 \quad r_3^2 \cos \theta_3] = [-1 \quad 0.3628 \quad -0.1316], \quad \mathbf{Q}_3 = [-\sin \theta_3 \quad 0 \quad r_3^2 \sin \theta_3] = [0 \quad 0 \quad 0],$$

$$R_3 = 1.4652, \text{ and } X_3 = 0.$$

Thus,  $\Delta \eta_1 = (-R_1 \mathbf{P}_1 + X_1 \mathbf{Q}_1) \cdot \Delta \mathbf{B} = -0.1266 \Delta b_0 + 0.6286 \Delta b_1 - 0.2281 \Delta b_2$ ,

$$\Delta \theta_1 = -\frac{1}{r_1} (\mathbf{X}_1 \mathbf{P}_1 + \mathbf{R}_1 \mathbf{Q}_1) \cdot \Delta \mathbf{B} = -0.8483 \Delta b_0 + 0.0959 \Delta b_1 + 0.5756 \Delta b_2,$$

$$\Delta r_3 = (-R_3 \mathbf{P}_3 + X_3 \mathbf{Q}_3) \cdot \Delta \mathbf{B} = -R_3 \mathbf{P}_3 \cdot \Delta \mathbf{B} = 1.4652 \Delta b_0 - 0.5316 \Delta b_1 + 0.1929 \Delta b_2,$$

$$\Delta \theta_3 = -\frac{1}{r_3} (\mathbf{X}_3 \mathbf{P}_3 + \mathbf{R}_3 \mathbf{Q}_3) \cdot \Delta \mathbf{B} = 0.$$

- (b) Cascade Form:**  $B(z) = z^3 + b_2 z^2 + b_1 z + b_0 = (z^2 + c_1 z + c_0)(z + d_0) = B_1(z)B_2(z)$  where  $B_1(z) = z^2 + c_1 z + c_0 = (z - z_1)(z - z_2) = (z - r_1 e^{j\theta_1})(z - r_1 e^{-j\theta_1}) = z^2 - 2r_1 \cos \theta_1 z + r_1^2$  and  $B_2(z) = z + d_0 = z - r_3 e^{j\theta_3}$ . Comparing with the denominator of the given transfer function we get  $2r_1 \cos \theta_1 = -0.5111$ ,  $r_1^2 = 0.7363$ ,  $r_3 = 0.3628$ , and  $\theta_3 = \pi$ . Hence,  $r_1 = 0.8581$  and  $\cos \theta_1 = -0.2978$ .

$$\text{Now, } \frac{1}{B_1(z)} = \frac{-j0.6104}{z + 0.2556 - j0.8191} + \frac{j0.6104}{z + 0.2556 + j0.8191}.$$

Hence,  $R_1 = 0$  and  $X_1 = -0.6104$ .

$\mathbf{P}_1 = [\cos\theta_1 \ r_1] = [-0.2978 \ 0.8581]$ , and  $\mathbf{Q}_1 = [-\sin\theta_1 \ 0] = [-0.9546 \ 0]$ , Next, we observe

$\frac{1}{B_2(z)} = \frac{1}{z + 0.3628}$ . This implies,  $R_3 = 1$  and  $X_3 = 0$ .  $\mathbf{P}_3 = \cos\theta_3 = -1$ ,  $\mathbf{Q}_3 = -\sin\theta_3 = 0$ .

Thus,

$$\Delta r_1 = (-R_1 \mathbf{P}_1 + X_1 \mathbf{Q}_1) \cdot [\Delta c_0 \ \Delta c_1]^T = X_1 \mathbf{Q}_1 \cdot [\Delta c_0 \ \Delta c_1]^T = 0.5827 \Delta c_0,$$

$$\Delta \theta_1 = -\frac{1}{r_1} (X_1 \mathbf{P}_1 + R_1 \mathbf{Q}_1) \cdot [\Delta c_0 \ \Delta c_1]^T = -0.2118 \Delta c_0 + 0.6104 \Delta c_1$$

$$\Delta r_3 = (-R_3 \mathbf{P}_3 + X_3 \mathbf{Q}_3) \cdot \Delta d_0 = -R_3 \mathbf{P}_3 \cdot \Delta d_0 = -\Delta d_0$$

$$\Delta \theta_3 = -\frac{1}{r_3} (X_3 \mathbf{P}_3 + R_3 \mathbf{Q}_3) \cdot \Delta d_0 = -\frac{1}{r_3} R_3 \mathbf{Q}_3 \cdot \Delta d_0 = 0$$

**9.7** In terms of transfer parameters, the input-output relation of the two-pair of Figure P9.3 is given by

$Y_1 = t_{11}X_1 + t_{12}X_2$ , and  $Y_2 = t_{21}X_1 + t_{22}X_2$ , where  $X_2 = \alpha Y_2$ . From Eq. (4.156b) we get

$$H(z) = \frac{Y_1(z)}{X_1(z)} = t_{11} + \frac{\alpha t_{12} t_{21}}{1 - \alpha t_{22}}.$$

From the above,  $\frac{\partial H(z)}{\partial \alpha} = \frac{t_{12} t_{21} (1 - \alpha t_{22}) + t_{22} \alpha t_{12} t_{21}}{(1 - \alpha t_{22})^2} = \frac{t_{12} t_{21}}{(1 - \alpha t_{22})^2}$ .

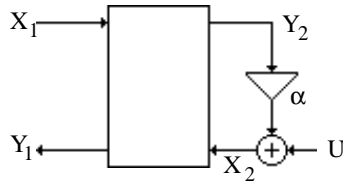
Substituting  $X_2 = \alpha Y_2$  in  $Y_2 = t_{21}X_1 + t_{22}X_2$  we get the expression for the scaling transfer function

$$F_\alpha(z) = \frac{Y_2(z)}{X_1(z)} = \frac{t_{21}}{1 - \alpha t_{22}}. \text{ Now from the structure given below we observe that the noise}$$

transfer function  $G_\alpha(z)$  is given by  $G_\alpha(z) = \left. \frac{Y_1}{U} \right|_{X_1=0}$ . From the structure we also observe that

$X_2 = \alpha Y_2 + U$ . Substituting this in the transfer relations we arrive at with  $X_1 = 0$ ,

$Y_1 = t_{12}X_2 = t_{12}(\alpha Y_2 + U)$ , and  $Y_2 = t_{22}X_2 = t_{22}(\alpha Y_2 + U)$ . From these two equations we obtain



after some algebra  $Y_1 = \frac{\alpha t_{12} t_{22}}{1 - \alpha t_{22}} U + t_{12} U = \frac{t_{12}}{1 - \alpha t_{22}} U$ . Hence the noise transfer function is

given by  $G_\alpha(z) = \left. \frac{Y_1}{U} \right|_{X_1=0} = \frac{t_{12}}{1 - \alpha t_{22}}$ . Therefore,  $\frac{\partial H(z)}{\partial \alpha} = \frac{t_{21}}{1 - \alpha t_{22}} \cdot \frac{t_{12}}{1 - \alpha t_{22}} = F_\alpha(z) G_\alpha(z)$ .

$$9.8 \text{ (a) } W_N(\omega) = \left[ \frac{1}{2N+1} \left( N + \frac{\sin[(N+1)\omega]}{\sin(\omega)} \right) \right]^{1/2} = \left[ \frac{1}{2N+1} \left( 1 + 4 \sum_{n=1}^{N/2} \cos^2(n\omega) \right) \right]^{1/2}.$$

Since the maximum value of  $\cos^2(n\omega) = 1$  and the minimum value  $= 0$ , hence

$$\max\{W_N(\omega)\} = \left[ \frac{1}{2N+1} (1 + 4(N/2)) \right]^{1/2} = 1, \text{ and } \min\{W_N(\omega)\} = \left[ \frac{1}{2N+1} \right]^{1/2} > 0.$$

Hence  $0 < W_N(\omega) \leq 1$ .

$$(b) W_N(0) = \left[ \frac{1}{2N+1} \left( 1 + 4 \sum_{m=1}^{N/2} 1 \right) \right]^{1/2} = 1.$$

$$W_N(\pi) = \left[ \frac{1}{2N+1} \left( 1 + 4 \sum_{m=1}^{N/2} 1 \right) \right]^{1/2} = 1.$$

$$(c) \lim_{N \rightarrow \infty} W_N(\pi) = \left[ \lim_{N \rightarrow \infty} \frac{N}{2N+1} + \lim_{N \rightarrow \infty} \frac{1}{2N+1} \frac{\sin((N+1)\omega)}{\sin(\omega)} \right]^{1/2}.$$

Since  $|\sin((N+1)\omega)| \leq 1$ ,  $\lim_{N \rightarrow \infty} \frac{1}{2N+1} \frac{\sin((N+1)\omega)}{\sin(\omega)} = 0$ , and  $\lim_{N \rightarrow \infty} W_N(\pi) = \frac{1}{\sqrt{2}}$ .

9.9 From Eq. (9.75)  $SNR_{A/D} = 6.0206b + 16.81 - 20 \log_{10}(K)$  dB. For a constant value of  $K$ , an increase in  $b$  by 1 bit increases the  $SNR_{A/D}$  by 6.0206 dB and an increase in  $b$  by 2 bits increases the  $SNR_{A/D}$  by 12.0412 dB.

If  $K = 4$  and  $b = 7$  then  $SNR_{A/D} = 6.0206 \times 7 + 16.81 - 20 \log_{10}(4)$   
 $= 6.0206 \times 7 + 16.81 - 20 \times 0.6021 = 46.9122$  dB. Therefore for  $K = 4$  and  $b = 9$ ,  $SNR_{A/D} = 46.91 + 12.04 = 58.95$  dB; for  $K = 4$  and  $b = 11$ ,  $SNR_{A/D} = 58.95 + 12.04 = 70.99$  dB; for  $K = 4$  and  $b = 13$ ,  $SNR_{A/D} = 70.99 + 12.04 = 83.03$ ; for  $K = 4$  and  $b = 15$ ,  $SNR_{A/D} = 83.03 + 12.04 = 95.08$ .

If  $K = 6$  and  $b = 7$  then  $SNR_{A/D} = 6.0206 \times 7 + 16.81 - 20 \log_{10}(6)$   
 $= 6.0206 \times 7 + 16.81 - 20 \times 0.7782 = 43.3902$  dB. Therefore for  $K = 6$  and  $b = 9$ ,  $SNR_{A/D} = 43.39 + 12.04 = 55.43$  dB; for  $K = 6$  and  $b = 11$ ,  $SNR_{A/D} = 43.39 + 12.04 = 67.47$  dB; for  $K = 6$  and  $b = 13$ ,  $SNR_{A/D} = 67.47 + 12.04 = 79.51$ ; for  $K = 6$  and  $b = 15$ ,  $SNR_{A/D} = 79.51 + 12.04 = 91.56$ .

If  $K = 8$  and  $b = 7$  then  $SNR_{A/D} = 6.0206 \times 7 + 16.81 - 20 \log_{10}(8)$   
 $= 6.0206 \times 7 + 16.81 - 20 \times 0.9031 = 40.8922$  dB. Therefore for  $K = 8$  and  $b = 9$ ,  $SNR_{A/D} = 40.89 + 12.04 = 52.93$  dB; for  $K = 8$  and  $b = 11$ ,  $SNR_{A/D} = 52.93 + 12.04 = 64.97$  dB; for  $K = 8$  and  $b = 13$ ,  $SNR_{A/D} = 64.97 + 12.04 = 77.01$ ; for  $K = 8$  and  $b = 15$ ,  $SNR_{A/D} = 77.01 + 12.04 = 89.05$ .

$$9.10 \text{ (a) } H(z) = \sum_{k=1}^N \frac{A_k}{z + a_k} = \sum_{k=1}^N C_k \frac{a_k z + 1}{z + a_k} - \sum_{k=1}^N C_k a_k = \sum_{k=1}^N C_k F_k(z) + B, \text{ where } C_k = \frac{A_k}{1 - a_k^2},$$

$$F_k(z) = \frac{a_k z + 1}{z + a_k}, \text{ and } B = - \sum_{k=1}^N C_k a_k.$$



$$\begin{aligned}
\text{(b) } \sigma_o^2 &= \frac{1}{2\pi j} \oint_C H(z)H(z^{-1})z^{-1} dz = \frac{1}{2\pi j} \oint_C \left[ \sum_{k=1}^N C_k F_k(z) + B \right] \left[ \sum_{k=1}^N C_k F_k(z^{-1}) + B \right] z^{-1} dz \\
&= \frac{1}{2\pi j} \oint_C B^2 z^{-1} dz + B \sum_{k=1}^N C_k \oint_C F_k(z) z^{-1} dz + B \sum_{l=1}^N C_l \oint_C F_l(z^{-1}) z^{-1} dz \\
&\quad + \sum_{k=1}^N \sum_{l=1}^N C_k C_l \frac{1}{2\pi j} \oint_C F_k(z) F_l(z^{-1}) z^{-1} dz.
\end{aligned}$$

Now,  $\frac{1}{2\pi j} \oint_C B^2 z^{-1} dz = B^2$ ,  $\frac{1}{2\pi j} \oint_C F_k(z) z^{-1} dz = a_k$ ,  $\frac{1}{2\pi j} \oint_C F_l(z^{-1}) z^{-1} dz = a_l$ , and

$$\frac{1}{2\pi j} \oint_C F_k(z) F_l(z^{-1}) z^{-1} dz = \frac{1 - a_k^2 - a_l^2 + a_k a_l}{1 - a_k a_l}. \text{ Therefore,}$$

$$\begin{aligned}
\sigma_o^2 &= B^2 + 2 \sum_{k=1}^N B C_k a_k + \sum_{k=1}^N \sum_{l=1}^N \frac{1 - a_k^2 - a_l^2 + a_k a_l}{1 - a_k a_l}. \text{ Since, } \sum_{k=1}^N C_k a_k = -B, \\
\sigma_o^2 &= -B^2 + \sum_{k=1}^N \sum_{l=1}^N \frac{1 - a_k^2 - a_l^2 + a_k a_l}{1 - a_k a_l} C_k C_l \\
&= -\sum_{k=1}^N \sum_{l=1}^N C_k C_l a_k a_l + \sum_{k=1}^N \sum_{l=1}^N C_k C_l \frac{1 - a_k^2 - a_l^2 + a_k a_l}{1 - a_k a_l} \\
&= \sum_{k=1}^N \sum_{l=1}^N C_k C_l \left[ \frac{1 - a_k^2 - a_l^2 + a_k a_l}{1 - a_k a_l} - a_k a_l \right] = \sum_{k=1}^N \sum_{l=1}^N \frac{A_k}{(1 - a_k^2)} \cdot \frac{A_l}{(1 - a_l^2)} \left[ \frac{1 - a_k^2 - a_l^2 + a_k^2 a_l^2}{1 - a_k a_l} \right] \\
&= \sum_{k=1}^N \sum_{l=1}^N \frac{A_k}{(1 - a_k^2)} \cdot \frac{A_l}{(1 - a_l^2)} \left[ \frac{(1 - a_k^2)(1 - a_l^2)}{1 - a_k a_l} \right] = \sum_{k=1}^N \sum_{l=1}^N \frac{A_k A_l}{1 - a_k a_l}. \tag{A}
\end{aligned}$$

$$\begin{aligned}
\text{Further, } \sigma_o^2 &= \sum_{k=1}^N \frac{A_k^2}{1 - a_k^2} + \sum_{k=1}^{N-1} \sum_{l=k+1}^N \frac{A_k A_l}{1 - a_k a_l} + \sum_{l=1}^{N-1} \sum_{k=l+1}^N \frac{A_k A_l}{1 - a_k a_l} \\
&= \sum_{k=1}^N \frac{A_k^2}{1 - a_k^2} + 2 \sum_{k=1}^{N-1} \sum_{l=k+1}^N \frac{A_k A_l}{1 - a_k a_l}.
\end{aligned}$$

(c) See Eq.(A).

**9.11 (a)**  $H_1(z) = \frac{(z+3)(z-1)}{(z-0.5)(z+0.3)} = 1 + \frac{-2.1875}{z-0.5} + \frac{4.3875}{z+0.3}$ . Making use of Eq. (9.86) and Table

$$\begin{aligned}
9.4, \text{ we get } \sigma_{1,n}^2 &= 1 + \frac{(-2.1875)^2}{1 - (0.5)^2} + \frac{(4.3875)^2}{1 - (0.3)^2} + \frac{2 \times (-2.1875) \times 4.3875}{1 - 0.15} \\
&= 1 + 6.3802 + 21.154 - 16.692 = 11.843.
\end{aligned}$$

Output of Program 9\_4 is 11.843.



The noise transfer function from the noise source  $e_1[n]$  to the filter output is

$$G_1(z) = H(z) = \frac{z^2 + z - 6}{z^2 - 0.1z - 0.12} = 1 + \frac{-7.7714}{z - 0.4} + \frac{8.8714}{z + 0.3}$$

variance at the output is  $\sigma_1^2 = 1 + \frac{(-7.7714)^2}{1 - (0.4)^2} + \frac{(8.8714)^2}{1 - (-0.3)^2} + 2 \left( \frac{-7.7714 \times 8.8714}{1 - (0.4)(-0.3)} \right) = 36.271$ .

The noise transfer function from the noise sources  $e_2[n]$  and  $e_3[n]$  to the filter output is

$$G_2(z) = \frac{z + 3}{z - 0.4} = 1 + \frac{3.4}{z - 0.4}$$

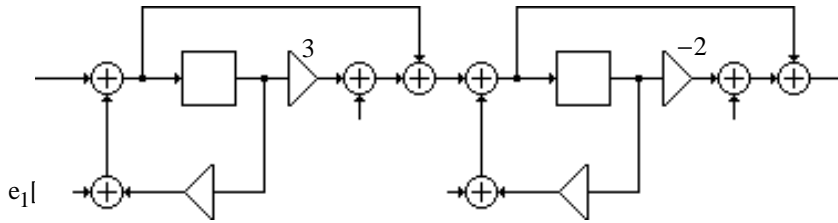
noise sources is  $\sigma_2^2 = 1 + \frac{(3.4)^2}{1 - (0.4)^2} = 14.762$ . The noise transfer function from the noise source

$e_4[n]$  to the filter output is  $G_3(z) = 1$ . The corresponding normalized noise variance at the

output is  $\sigma_3^2 = 1$ . Hence the total noise variance at the output is  $\sigma_o^2 = \sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 66.795$ .

Cascade Structure #2:  $H(z) = \left( \frac{1 + 3z^{-1}}{1 + 0.3z^{-1}} \right) \left( \frac{1 - 2z^{-1}}{1 - 0.4z^{-1}} \right)$ . The noise model of this structure is

shown below:



The noise transfer function from the noise source  $e_1[n]$  to the filter output is

$$G_1(z) = H(z) = \frac{z^2 + z - 6}{z^2 - 0.1z - 0.12} = 1 + \frac{-7.7714}{z - 0.4} + \frac{8.8714}{z + 0.3}$$

variance at the output is  $\sigma_1^2 = 1 + \frac{(-7.7714)^2}{1 - (0.4)^2} + \frac{(8.8714)^2}{1 - (-0.3)^2} + 2 \left( \frac{-7.7714 \times 8.8714}{1 - (0.4)(-0.3)} \right) = 36.271$ .

The noise transfer function from the noise sources  $e_2[n]$  and  $e_3[n]$  to the filter output is

$$G_2(z) = \frac{z - 2}{z - 0.4} = 1 + \frac{-1.6}{z - 0.4}$$

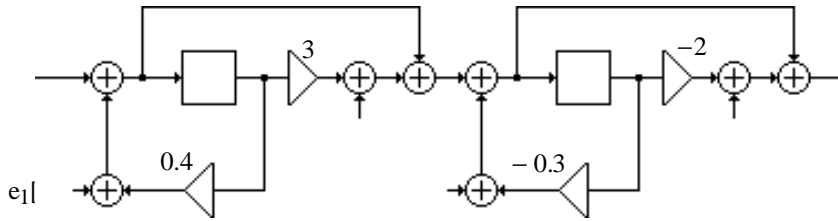
noise sources is  $\sigma_2^2 = 1 + \frac{(-1.6)^2}{1 - (0.4)^2} = 4.0476$ . The noise transfer function from the noise source

$e_4[n]$  to the filter output is  $G_3(z) = 1$ . The corresponding normalized noise variance at the

output is  $\sigma_3^2 = 1$ . Hence the total noise variance at the output is  $\sigma_o^2 = \sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 45.366$ .

Cascade Structure #3:  $H(z) = \left( \frac{1 + 3z^{-1}}{1 - 0.4z^{-1}} \right) \left( \frac{1 - 2z^{-1}}{1 + 0.3z^{-1}} \right)$ . The noise model of this structure is

shown below:



The noise transfer function from the noise source  $e_1[n]$  to the filter output is

$$G_1(z) = H(z) = \frac{z^2 + z - 6}{z^2 - 0.1z - 0.12} = 1 + \frac{-7.7714}{z - 0.4} + \frac{8.8714}{z + 0.3}.$$

The corresponding normalized noise variance at the output is  $\sigma_1^2 = 1 + \frac{(-7.7714)^2}{1 - (0.4)^2} + \frac{(8.8714)^2}{1 - (-0.3)^2} + 2 \left( \frac{-7.7714 \times 8.8714}{1 - (0.4)(-0.3)} \right) = 36.271.$

The noise transfer function from the noise sources  $e_2[n]$  and  $e_3[n]$  to the filter output is

$$G_2(z) = \frac{z - 2}{z + 0.3} = 1 + \frac{-2.3}{z + 0.3}.$$

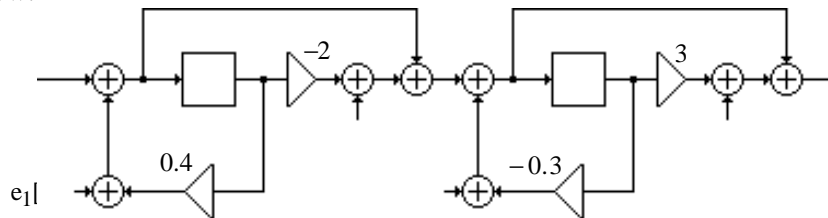
The normalized noise variance at the output due to each of these noise sources is  $\sigma_2^2 = 1 + \frac{(-2.3)^2}{1 - (-0.3)^2} = 6.8132.$  The noise transfer function from the noise source

$e_4[n]$  to the filter output is  $G_3(z) = 1.$  The corresponding normalized noise variance at the

output is  $\sigma_3^2 = 1.$  Hence the total noise variance at the output is  $\sigma_o^2 = \sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 50.897.$

Cascade Structure #4:  $H(z) = \left( \frac{1 - 2z^{-1}}{1 - 0.4z^{-1}} \right) \left( \frac{1 + 3z^{-1}}{1 + 0.3z^{-1}} \right).$  The noise model of this structure is

shown below:



The noise transfer function from the noise source  $e_1[n]$  to the filter output is

$$G_1(z) = H(z) = \frac{z^2 + z - 6}{z^2 - 0.1z - 0.12} = 1 + \frac{-7.7714}{z - 0.4} + \frac{8.8714}{z + 0.3}.$$

The corresponding normalized noise variance at the output is  $\sigma_1^2 = 1 + \frac{(-7.7714)^2}{1 - (0.4)^2} + \frac{(8.8714)^2}{1 - (-0.3)^2} + 2 \left( \frac{-7.7714 \times 8.8714}{1 - (0.4)(-0.3)} \right) = 36.271.$

The noise transfer function from the noise sources  $e_2[n]$  and  $e_3[n]$  to the filter output is

$$G_2(z) = \frac{z + 3}{z + 0.3} = 1 + \frac{2.7}{z + 0.3}.$$

The normalized noise variance at the output due to each of these noise sources is  $\sigma_2^2 = 1 + \frac{(2.7)^2}{1 - (-0.3)^2} = 9.011.$  The noise transfer function from the noise source

$e_4[n]$  to the filter output is  $G_3(z) = 1.$  The corresponding normalized noise variance at the

output is  $\sigma_3^2 = 1.$  Hence the total noise variance at the output is  $\sigma_o^2 = \sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 55.293.$

Hence, the Cascade Structure #2 has the smallest roundoffnoise variance.

**(b) Quantization of products after addition.**

From the results of Part (a) we have here the following roundoff noise variances:

Cascade Structure #1:  $\sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 52.033.$

Cascade Structure #2:  $\sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 41.319.$

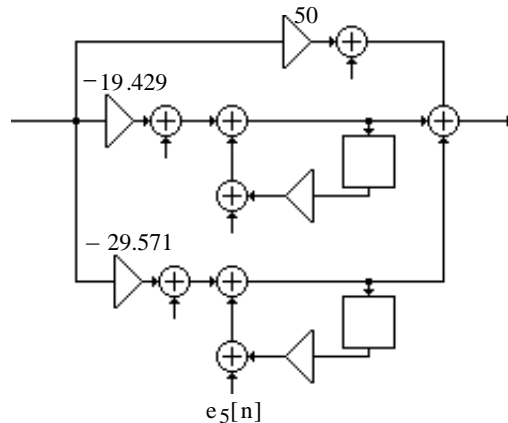
Cascade Structure #3:  $\sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 44.084.$

Cascade Structure #4:  $\sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 46.282.$

In this case also the Cascade Form 2 has the smallest roundoffnoise variance.

**9.14 (a) Quantization of products before addition.**

Parallel Form I:  $H(z) = \frac{(z-2)(z+3)}{(z+0.3)(z-0.2)} = 50 + \frac{-19.429}{1-0.4z^{-1}} + \frac{-29.571}{1+0.3z^{-1}}.$  The noise model is as shown below



The noise transfer function from the noise source  $e_1[n]$  to the filter output is  $G_1(z) = 1$  and its corresponding normalized noise variance at the output is  $\sigma_1^2 = 1$ . The noise transfer function from the noise sources  $e_2[n]$  and  $e_3[n]$  to the filter output is

$G_2(z) = \frac{1}{1-0.4z^{-1}} = \frac{z}{z-0.4} = 1 + \frac{0.4}{z-0.4}.$  The normalized noise variance at the output due to each of these noise sources is  $\sigma_2^2 = 1 + \frac{(0.4)^2}{1-(0.4)^2} = 1.1905.$  The noise transfer function from the

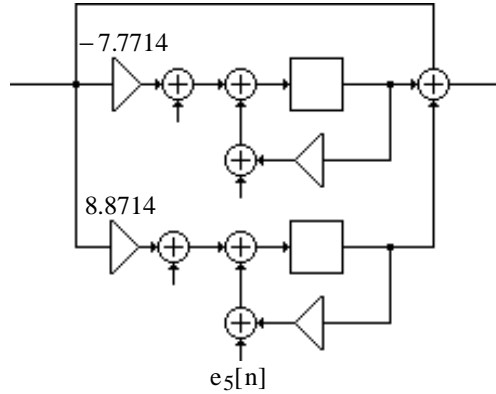
noise sources  $e_4[n]$  and  $e_5[n]$  to the filter output is  $G_3(z) = \frac{1}{1+0.3z^{-1}} = \frac{z}{z+0.3} = 1 + \frac{-0.3}{z+0.3}.$

The normalized noise variance at the output due to each of these noise sources is

$$\sigma_3^2 = 1 + \frac{(-0.3)^2}{1 - (-0.3)^2} = 1.0989. \quad \text{Hence the total noise variance at the output is}$$

$$\sigma_0^2 = \sigma_1^2 + 2\sigma_2^2 + 2\sigma_3^2 = 5.5788.$$

Parallel Form II:  $H(z) = 1 + \frac{-7.7714}{z-0.4} + \frac{8.8714}{z+0.3} = 1 + \frac{-7.7714z^{-1}}{1-0.4z^{-1}} + \frac{8.8714z^{-1}}{1+0.3z^{-1}}$ . The noise model is as shown below



The noise transfer function from the noise sources  $e_2[n]$  and  $e_3[n]$  to the filter output is

$$G_2(z) = \frac{z^{-1}}{1-0.4z^{-1}} = \frac{1}{z-0.4}. \quad \text{The normalized noise variance at the output due to each of these}$$

noise sources is  $\sigma_2^2 = \frac{1}{1-(0.4)^2} = 1.1905$ . The noise transfer function from the noise sources

$$e_4[n] \text{ and } e_5[n] \text{ to the filter output is } G_3(z) = \frac{z^{-1}}{1+0.3z^{-1}} = \frac{1}{z+0.3}. \quad \text{The normalized noise}$$

variance at the output due to each of these noise sources is  $\sigma_3^2 = \frac{1}{1-(-0.3)^2} = 1.0989$ . Hence

the total noise variance at the output is  $\sigma_0^2 = 2\sigma_2^2 + 2\sigma_3^2 = 4.5788$ .

Hence, Parallel Form #2 has the smaller roundoff noise variance.

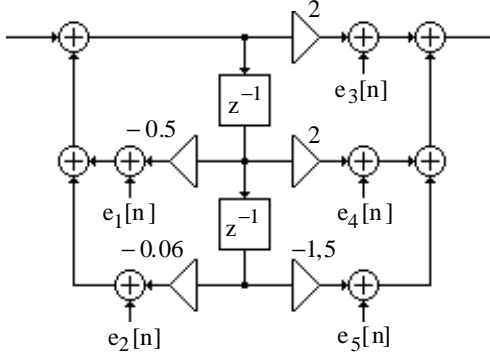
**(b) Quantization of products after addition.**

Parallel Form I: From the results of Part (a) we note that the total noise variance at the output is  $\sigma_0^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 3.2894$ .

Parallel Form II: From the results of Part (a) we note that the total noise variance at the output is  $\sigma_0^2 = \sigma_2^2 + \sigma_3^2 = 2.2894$ .

**9.15**  $H(z) = \frac{2 + 2z^{-1} - 1.5z^{-2}}{1 + 0.5z^{-1} + 0.06z^{-2}}$ .

**(a)** The noise model of the Direct Form II realization of  $H(z)$  is shown below:



The noise transfer function from the noise sources  $e_3[n]$ ,  $e_4[n]$  and  $e_5[n]$  to the filter output is  $G_2(z) = 1$ , and the noise transfer function from the noise sources  $e_1[n]$  and  $e_2[n]$  to the filter

output is  $G_1(z) = H(z) = \frac{2 + 2z^{-1} - 1.5z^{-2}}{1 + 0.5z^{-1} - 0.06z^{-2}} = 2 + \frac{-18.2}{z + 0.2} + \frac{19.2}{z + 0.3}$ . Using Table 9.4, the

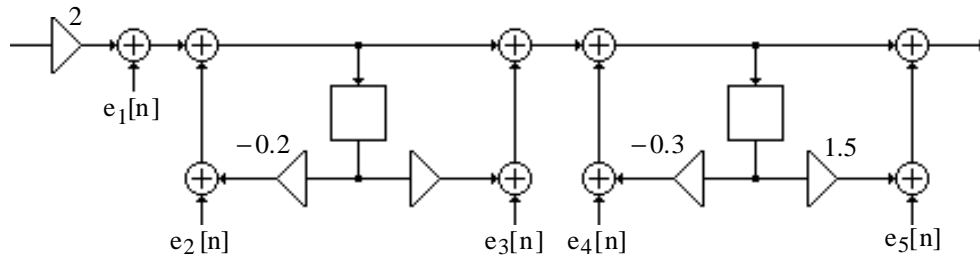
normalized noise variance at the output due to each of the noise sources  $e_1[n]$  and  $e_2[n]$  is then

$$\sigma_1^2 = 2 \left[ 4 + \frac{(-18.2)^2}{1 - (-0.2)^2} + \frac{(19.2)^2}{1 - (-0.3)^2} + 2 \left( \frac{-18.2 \times 19.2}{1 - (-0.2)(-0.3)} \right) \right] = 10.651,$$

and the normalized noise variance at the output due to each of the noise sources  $e_3[n]$ ,  $e_4[n]$  and  $e_5[n]$  is  $\sigma_2^2 = 1$ . Hence the total noise variance at the output is  $\sigma_o^2 = 2\sigma_1^2 + 3\sigma_2^2 = 24.302$ .

**(b) Cascade Form Realization:**  $H(z) = \frac{2(1 - 0.5z^{-1})(1 + 1.5z^{-1})}{(1 + 0.2z^{-1})(1 + 0.3z^{-1})}$ . There are more than 2 possible cascade realizations. We consider here only two such structures.

Cascade Form #1:  $H(z) = 2 \left( \frac{1 - 0.5z^{-1}}{1 + 0.2z^{-1}} \right) \left( \frac{1 + 1.5z^{-1}}{1 + 0.3z^{-1}} \right)$ . The noise model of this realization is shown below:



The noise transfer function from the noise sources  $e_1[n]$  and  $e_2[n]$  to the filter output is

$G_1(z) = \frac{1 + z^{-1} - 0.75z^{-2}}{1 + 0.5z^{-1} - 0.06z^{-2}} = 1 + \frac{-9.1}{z + 0.2} + \frac{9.6}{z + 0.3}$ . Its contribution to the output noise variance

$$\text{is } \sigma_1^2 = 1 + \frac{(-9.1)(-9.1)}{1 - (-0.2)(-0.2)} + \frac{(9.6)(9.6)}{1 - (-0.3)(-0.3)} + 2 \left[ \frac{(-9.1)(9.6)}{1 - (-0.2)(-0.3)} \right] = 2.6628.$$

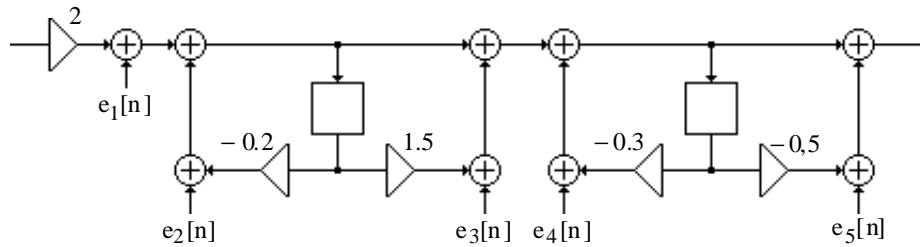
The noise transfer function from the noise sources  $e_3[n]$ , and  $e_4[n]$  to the filter output is

$$G_2(z) = \frac{1+1.5z^{-1}}{1+0.3z^{-1}} = 1 + \frac{1.2}{z+0.3}. \text{ Its contribution to the output noise variance is}$$

$$\sigma_2^2 = 1 + \frac{(1.2)^2}{1-(-0.3)^2} = 2.5824. \text{ Finally, the noise transfer function from the noise source } e_5[n]$$

to the filter output is  $G_3(z) = 1$ . Its contribution to the output noise variance is  $\sigma_3^2 = 1$ . Hence the total noise variance at the output is  $\sigma_o^2 = 2\sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 2(2.6628) + 2(2.5824) + 1 = 11.49$ .

Cascade Form #2:  $H(z) = 2 \left( \frac{1+1.5z^{-1}}{1+0.2z^{-1}} \right) \left( \frac{1-0.5z^{-1}}{1+0.3z^{-1}} \right)$ . The noise model of this realization is shown below:



The noise transfer function from the noise sources  $e_1[n]$  and  $e_2[n]$  to the filter output is

$$G_1(z) = \frac{1+1z^{-1}-0.75z^{-2}}{1+0.5z^{-1}-0.06z^{-2}} = 1 + \frac{-9.1}{z+0.2} + \frac{9.6}{z+0.3}. \text{ Its contribution to the output noise variance is}$$

$$\sigma_1^2 = 1 + \frac{(-9.1)(-9.1)}{1-(-0.2)(-0.2)} + \frac{(9.6)(9.6)}{1-(-0.3)(-0.3)} + 2 \left[ \frac{(-9.1)(9.6)}{1-(-0.2)(-0.3)} \right] = 2.6628.$$

The noise transfer function from the noise sources  $e_3[n]$ , and  $e_4[n]$  to the filter output is

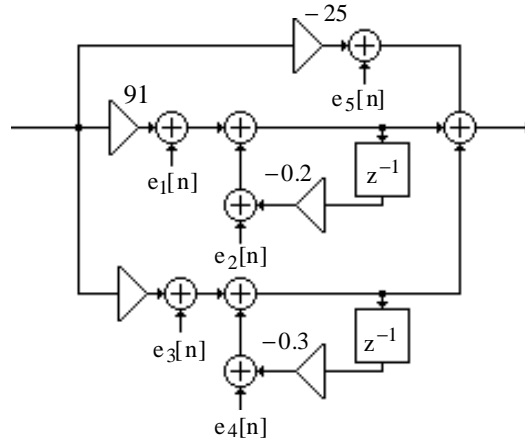
$$G_2(z) = \frac{1-0.5z^{-1}}{1+0.3z^{-1}} = 1 + \frac{-0.8}{z+0.3}. \text{ Its contribution to the output noise variance is}$$

$$\sigma_2^2 = 1 + \frac{(-0.8)^2}{1-(-0.3)^2} = 1.7033, \text{ Finally, the noise transfer function from the noise source } e_5[n]$$

to the filter output is  $G_3(z) = 1$ . Its contribution to the output noise variance is  $\sigma_3^2 = 1$ . Hence the total noise variance at the output is  $\sigma_o^2 = 2\sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 2(2.6628) + 2(1.7033) + 1 = 9.7322$ .

(c) Parallel Form I Realization:  $H(z) = -25 + \frac{91}{1+0.2z^{-1}} + \frac{-64}{1+0.3z^{-1}}$ . The noise model of this realization is shown below:





The noise transfer function from the noise sources  $e_1[n]$  and  $e_2[n]$  to the filter output is

$$G_1(z) = \frac{1}{1 + 0.2z^{-1}} = 1 + \frac{-0.2}{z + 0.2}. \text{ Its contribution to the output noise variance is}$$

$$\sigma_1^2 = 1 + \frac{(-0.2)^2}{1 - (-0.2)^2} = 1.0417.$$

The noise transfer function from the noise sources  $e_3[n]$  and  $e_4[n]$  to the filter output is

$$G_2(z) = \frac{1}{1 + 0.3z^{-1}} = 1 + \frac{-0.3}{z + 0.3}. \text{ Its contribution to the output noise variance is}$$

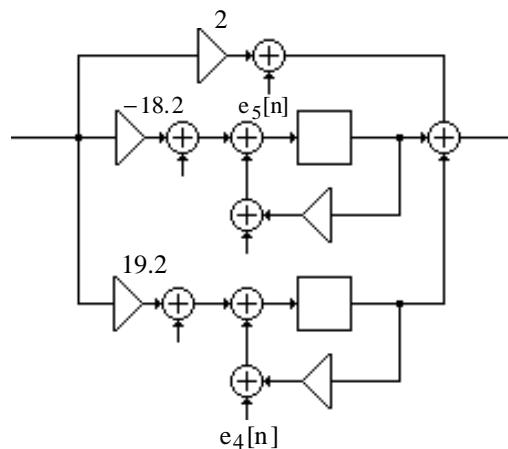
$$\sigma_2^2 = 1 + \frac{(-0.3)^2}{1 - (-0.3)^2} = 1.0989.$$

Finally, the noise transfer function from the noise sources  $e_5[n]$  to the filter output is  $G_3(z) = 1$ .

Its contribution to the output noise variance is  $\sigma_3^2 = 1$ . Hence the total noise variance at the

output is  $\sigma_o^2 = 2\sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 5.2812$ .

Parallel Form II Realization:  $H(z) = 2 + \frac{-18.2}{z + 0.2} + \frac{19.2}{z + 0.3}$ . The noise model of this realization is shown below:



The noise transfer function from the noise sources  $e_1[n]$  and  $e_2[n]$  to the filter output is

$$G_1(z) = \frac{z^{-1}}{1 + 0.2z^{-1}} = \frac{1}{z + 0.2}. \text{ Its contribution to the output noise variance is}$$

$$\sigma_1^2 = \frac{1}{1 - (-0.2)^2} = 1.0417.$$

The noise transfer function from the noise sources  $e_3[n]$  and  $e_4[n]$  to the filter output is

$$G_2(z) = \frac{z^{-1}}{1 + 0.3z^{-1}} = \frac{1}{z + 0.3}. \text{ Its contribution to the output noise variance is}$$

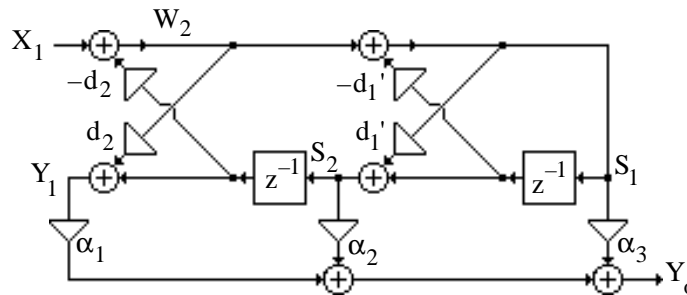
$$\sigma_2^2 = \frac{1}{1 - (-0.3)^2} = 1.0989.$$

Finally, the noise transfer function from the noise sources  $e_5[n]$  to the filter output is  $G_3(z) = 1$ .

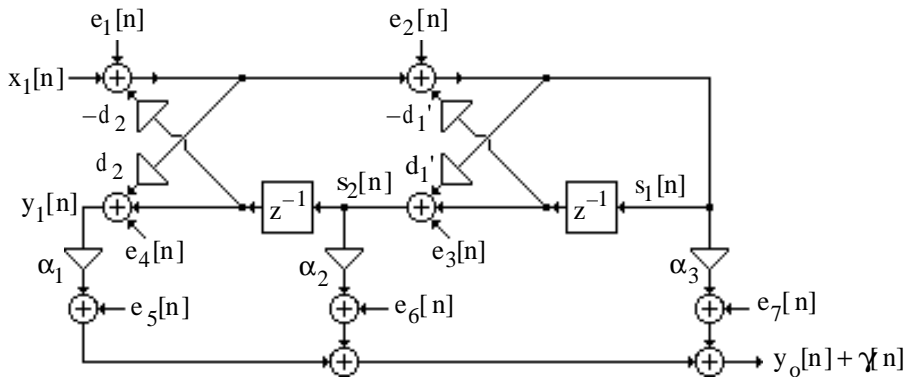
Its contribution to the output noise variance is  $\sigma_3^2 = 1$ . Hence the total noise variance at the output is  $\sigma_o^2 = 2\sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 = 5.2812$ .

As a result, both Parallel Form structures have the smallest product roundoff noise variance.

- 9.16**  $H(z) = \frac{2 + 2z^{-1} - 1.5z^{-2}}{1 + 0.5z^{-1} - 0.06z^{-2}}$ . Using Program 6\_4, the Gray-Markel realization of  $H(z)$  is found to be as shown below where  $d_2 = 0.472$ ,  $d_1' = 0.06$ ,  $\alpha_1 = -1.5$ ,  $\alpha_2 = 2.75$ , and  $\alpha_3 = 0.793$ .



- (a) For quantization of products before addition, the noise model is as shown below:



Analyzing the above structure we obtain the noise transfer functions for each noise source,  $e_i[n]$ ,  $1 \leq i \leq 7$ :

$$G_1(z) = H(z) = \frac{2 + 2z^{-1} - 1.5z^{-2}}{1 + 0.5z^{-1} - 0.06z^{-2}},$$

$$G_2(z) = \frac{(\alpha_2 d_1' + \alpha_3) + [\alpha_1(1 - d_2^2)d_1' + \alpha_2]z^{-1} + \alpha_1(1 - d_2^2)z^{-2}}{1 + 0.5z^{-1} + 0.06z^{-2}} = \frac{2.091 + 2.0445z^{-1} - 1.495z^{-2}}{1 + 0.5z^{-1} + 0.06z^{-2}},$$

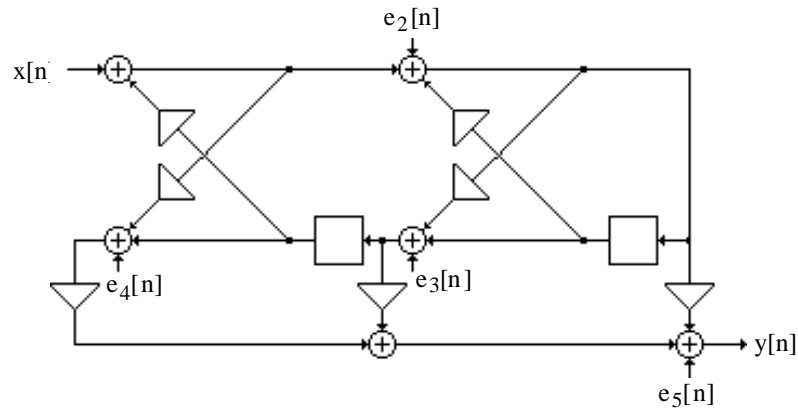
$$G_3(z) = \frac{\alpha_2 + [\alpha_1(1 - d_2^2) - \alpha_2 d_1' d_2 - \alpha_3 d_2]z^{-1} + \alpha_1(1 - d_2^2)d_2 d_1' z^{-2}}{1 + 0.5z^{-1} + 0.06z^{-2}} = \frac{2.75 - 1.6201z^{-1} - 0.7055z^{-2}}{1 + 0.5z^{-1} + 0.06z^{-2}},$$

$$G_4(z) = \alpha_1 = -1.5,$$

$$G_5(z) = G_6(z) = G_7(z) = 1.$$

Using Program 9\_4 we get  $\sigma_{1,n}^2 = 10.651$ ,  $\sigma_{2,n}^2 = 11.021$ ,  $\sigma_{3,n}^2 = 16.945$ ,  $\sigma_{4,n}^2 = 2.25$ . Hence, the total normalized output noise variances =  $10.651 + 11.021 + 16.945 + 2.25 + 3(1) = 43.87$ .

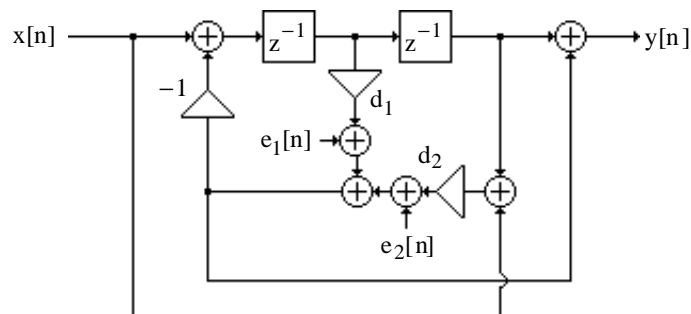
(b) The noise model of the above structure assuming quantization of products after addition is as given below:



The pertinent noise transfer functions here are:  $G_2(z) = \frac{2.091 + 2.0445z^{-1} - 1.495z^{-2}}{1 + 0.5z^{-1} + 0.06z^{-2}}$ ,

$G_3(z) = \frac{2.75 - 1.6201z^{-1} - 0.7055z^{-2}}{1 + 0.5z^{-1} + 0.06z^{-2}}$ ,  $G_4(z) = -1.5$ , and  $G_5(z) = 1$ . Hence, the total normalized output noise variances =  $11.021 + 16.945 + 2.25 + 1 = 31.22$ .

9.17 The noise model of Figure P9.8 is as shown below:



The noise transfer function from the noise sources  $e_1[n]$  and  $e_2[n]$  to the output is given by

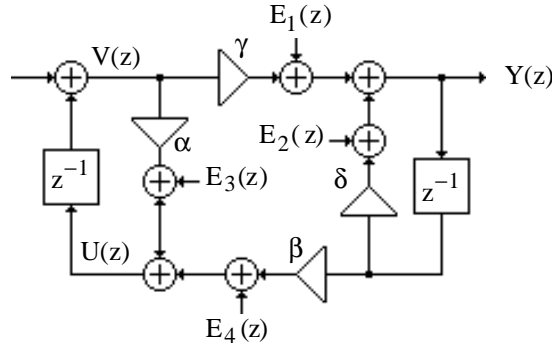
$$G_1(z) = \frac{1-z^{-2}}{1+d_1z^{-1}+d_2z^{-2}} = \frac{z^2-1}{z^2+d_1z+d_2} = 1 - \frac{d_1z+(1+d_2)}{z^2+d_1z+d_2}$$

Using Table 9.4 we arrive at the total normalized output noise power as

$$\begin{aligned} \sigma_n^2 &= 2 \left[ 1 + \frac{(d_1^2 + (1+d_2)^2)(1-d_2^2) - 2(d_1(1+d_2) - d_1(1+d_2)d_2)d_1}{(1-d_2^2)^2 + 2d_2d_1^2 - (1+d_2^2)d_1^2} \right] \\ &= 2 \left[ 1 + \frac{(1-d_2^2)((1+d_2)^2 - d_1^2)}{(1-d_2)^2((1+d_2)^2 - d_1^2)} \right] = \frac{4}{1-d_2}. \end{aligned}$$

### 9.18 (a) Quantization of products before addition

The noise model second-order coupled form structure is shown below.



We observe from Example 9.2 that  $\alpha = \delta = r \cos \theta$ ,  $\beta = -\gamma = r \sin \theta$ ,  $\alpha + \delta = 2r \cos \theta$  and  $\alpha\delta - \beta\gamma = r^2$ .

Analysis yields  $U(z) = \beta z^{-1}Y(z) + \alpha V(z) + E_3(z)$ ,  $V(z) = z^{-1}U(z)$ , and

$Y(z) = \gamma V(z) + \delta z^{-1}Y(z) + E_1(z)$ . Eliminating the variables  $U(z)$  and  $V(z)$  from these equations

we arrive at  $\frac{1 - (\alpha + \delta)z^{-1} + (\alpha\delta - \beta\gamma)z^{-2}}{1 - \alpha z^{-1}} Y(z) = \frac{\gamma z^{-1}}{1 - \alpha z^{-1}} E_3(z) + E_1(z)$ . Hence, the noise

transfer function from the noise sources  $e_1[n]$  and  $e_2[n]$  to the output is given by

$$G_1(z) = \left. \frac{Y(z)}{E_1(z)} \right|_{E_3(z)=E_4(z)=0} = \frac{1 - \alpha z^{-1}}{1 - (\alpha + \delta)z^{-1} + (\alpha\delta - \beta\gamma)z^{-2}} = \frac{z^2 - \alpha z}{z^2 - (\alpha + \delta)z + (\alpha\delta - \beta\gamma)}$$

$= 1 + \frac{\delta z - (\alpha\delta - \beta\gamma)}{z^2 - (\alpha + \delta)z + (\alpha\delta - \beta\gamma)} = 1 + \frac{r \cos \theta z - r^2}{z^2 - 2r \cos \theta z + r^2}$ , and the noise transfer function from the

noise sources  $e_3[n]$  and  $e_4[n]$  to the output is given by

$$\begin{aligned} G_2(z) &= \left. \frac{Y(z)}{E_3(z)} \right|_{E_1(z)=E_2(z)=0} = \frac{\gamma z^{-1}}{1 - (\alpha + \delta)z^{-1} + (\alpha\delta - \beta\gamma)z^{-2}} = \frac{\gamma z}{z^2 - (\alpha + \delta)z + (\alpha\delta - \beta\gamma)} \\ &= \frac{-r \sin \theta z}{z^2 - 2r \cos \theta z + r^2}. \end{aligned}$$

The output roundoff noise variance due to each of the noise sources  $e_1[n]$  and  $e_2[n]$  is

$$\sigma_1^2 = 1 + \frac{(r^2 \cos^2 \theta + r^4)(1 - r^4) - 2(-r^3 \cos \theta + r^3 \cos \theta \cdot r^2)(-2r \cos \theta)}{(1 - r^4)^2 + 2r^2(4r^2 \cos^2 \theta) - (1 + r^4)(4r^2 \cos^2 \theta)}$$

$$= 1 + \frac{r^2(1 + r^2)(r^2 + \cos^2 \theta) - 4r^4 \cos^2 \theta}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]}, \text{ and the output roundoff noise variance due to each of}$$

the noise sources  $e_3[n]$  and  $e_4[n]$  is  $\sigma_2^2 = \frac{r^2(1 + r^2) \sin^2 \theta}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]}$ . Hence, the total roundoff noise variance at the output is

$$\sigma_o^2 = 2(\sigma_1^2 + \sigma_2^2) = 2 \left( 1 + \frac{r^2(1 + r^2)(r^2 + \cos^2 \theta) - 4r^4 \cos^2 \theta}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]} + \frac{r^2(1 + r^2) \sin^2 \theta}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]} \right).$$

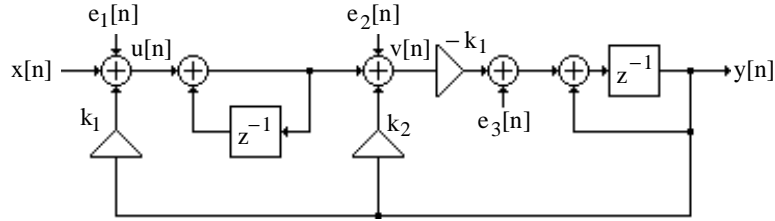
**(b) Quantization of products after addition**

Here, the total roundoff noise variance at the output is

$$\sigma_o^2 = \sigma_1^2 + \sigma_2^2 = 1 + \frac{r^2(1 + r^2)(r^2 + \cos^2 \theta) - 4r^4 \cos^2 \theta}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]} + \frac{r^2(1 + r^2) \sin^2 \theta}{(1 - r^2)[(1 + r^2)^2 - 4r^2 \cos^2 \theta]}.$$

**9.19 (a) Quantization of products before addition**

The noise model of the Kingsbury structure is shown below:



Analysis yields  $U(z) = k_1 Y(z) + E_1(z)$ ,  $V(z) = \frac{1}{1 - z^{-1}} U(z) + k_2 Y(z) + E_2(z)$ , and

$Y(z) = \frac{-k_1 V(z) + E_3(z)}{1 - z^{-1}} = -\frac{k_1 z^{-1}}{1 - z^{-1}} V(z) + \frac{z^{-1}}{1 - z^{-1}} E_3(z)$ . Eliminating  $U(z)$  and  $V(z)$  from these equations we arrive at

$$\frac{1 - [2 - k_1(k_1 + k_2)]z^{-1} + (1 - k_1 k_2)z^{-2}}{(1 - z^{-1})^2} Y(z) = -\frac{k_1 z^{-1}}{(1 - z^{-1})^2} E_1(z) - \frac{k_1 z^{-1}}{1 - z^{-1}} E_2(z) + \frac{z^{-1}}{1 - z^{-1}} E_3(z).$$

Hence, the noise transfer function from the noise source  $e_1[n]$  to the output is given by

$$G_1(z) = \left. \frac{Y(z)}{E_1(z)} \right|_{E_2(z)=E_3(z)=0} = \frac{-k_1 z^{-1}}{1 - [2 - k_1(k_1 + k_2)]z^{-1} + (1 - k_1 k_2)z^{-2}} = \frac{-k_1 z}{z^2 - 2r \cos \theta z + r^2}.$$

Its corresponding output noise variance is thus given by

$$\begin{aligned}\sigma_1^2 &= \frac{k_1^2(1-r^4)}{(1-r^4)^2 + 2r^2(-2r\cos\theta)^2 - (1+r^4)(-2r\cos\theta)^2} = \frac{k_1^2(1-r^2)(1+r^2)}{(1-r^2)^2(1+r^2)^2 - 4r^2\cos^2\theta(1-r^2)^2} \\ &= \frac{k_1^2(1+r^2)}{(1-r^2)[(1+r^2)^2 - 4r^2\cos^2\theta]}.\end{aligned}$$

The noise transfer function from the noise source  $e_2[n]$  to the output is given by

$$G_2(z) = \left. \frac{Y(z)}{E_2(z)} \right|_{E_1(z)=E_3(z)=0} = \frac{-k_1z^{-1}(1-z^{-1})}{1 - [2 - k_1(k_1 + k_2)]z^{-1} + (1 - k_1k_2)z^{-2}} = \frac{-k_1z + k}{z^2 - 2r\cos\theta z + r^2}.$$

Its corresponding output noise variance is thus given by  $\sigma_2^2 = \frac{2k_1^2(1+r^2)}{(1-r^2)[(1+r^2)^2 - 4r^2\cos^2\theta]}$ .

Finally, the noise transfer function from the noise source  $e_3[n]$  to the output is given by

$$G_3(z) = \left. \frac{Y(z)}{E_3(z)} \right|_{E_1(z)=E_2(z)=0} = \frac{z^{-1}(1-z^{-1})}{1 - [2 - k_1(k_1 + k_2)]z^{-1} + (1 - k_1k_2)z^{-2}} = \frac{z-1}{z^2 - 2r\cos\theta z + r^2} \quad \text{Its}$$

corresponding output noise variance is thus given by  $\sigma_3^2 = \frac{2(1+r^2)}{(1-r^2)[(1+r^2)^2 - 4r^2\cos^2\theta]}$ .

Hence, the total normalized output noise variance due to product roundoff before (after)

$$\text{summation is given by } \sigma_0^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \frac{(3k_1^2 + 2)(1+r^2)}{(1-r^2)[(1+r^2)^2 - 4r^2\cos^2\theta]}.$$

**9.20** Let  $e[n]$  be a noise source due to product roundoff inside  $H(z)$  with an associated noise transfer function  $G_e(z)$  to the output of  $H(z)$ . Then the normalized noise power at the output

of the cascade structure due to  $e[n]$  is given by  $\sigma_{e,n}^2 = \frac{1}{2\pi j} \oint_C G_e(z)A_2(z)G_e(z^{-1})A_2(z^{-1})z^{-1}dz$

$$= \frac{1}{2\pi j} \oint_C G_e(z)G_e(z^{-1})z^{-1}dz \quad \text{since } A_2(z)A_2(z^{-1})=1.$$

Assuming  $H(z)$  is realized in a cascade form with the lowest product roundoff noise power  $\sigma_{h,n}^2$ , the normalized noise power at the output of the cascade due to product roundoff in

$H(z)$  still remains  $\sigma_{h,n}^2$ . From the solution of Problem 9.13, we note that the lowest

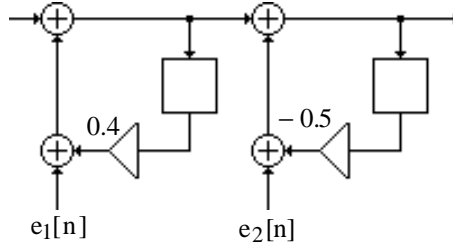
roundoff noise power is  $\sigma_{h,n}^2 = 45.366$  for cascade form #2. Next, from the solution of

Problem 9.17 we note that the normalized output noise power due to noise sources in  $A_2(z)$  is

$$\sigma_{a,n}^2 = \frac{4}{1-d_2}. \quad \text{Substituting } d_2 = 0.8 \text{ we get } \sigma_{a,n}^2 = \frac{4}{1-0.8} = 20. \quad \text{Hence total output noise}$$

variance of the cascade is given by  $\sigma_0^2 = \sigma_{h,n}^2 + \sigma_{a,n}^2 = 45.366 + 20 = 65.366$ .

**9.21** The noise model for the structure is as shown below



The noise transfer function from  $e_1[n]$  to the output is

$$G_1(z) = \frac{1}{(1-0.4z^{-1})(1+0.5z^{-1})} = 1 + \frac{-27.778}{z+0.5} + \frac{17.778}{z-0.4}$$

Thus, the normalized noise power at the

output due to  $e_1[n]$  is  $\sigma_1^2 = 1 + \frac{(-0.27778)^2}{1-(-0.5)^2} + \frac{(0.17778)^2}{1-(0.4)^2} + 2 \times \frac{(-0.27778) \times (0.17778)}{1-(-0.5)(0.4)} = 1.0582$ .

The noise transfer function from  $e_2[n]$  to the output is  $G_2(z) = \frac{1}{1+0.5z^{-1}} = 1 + \frac{-0.5}{z+0.5}$ .

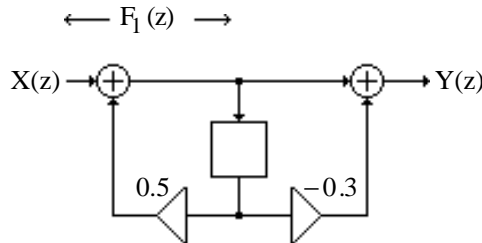
Thus, the normalized noise power at the output due to  $e_2[n]$  is  $\sigma_2^2 = 1 + \frac{(-0.5)^2}{1-(-0.5)^2} = 1.3333$ .

Therefore, the total normalized noise at the output =  $1.0582 + 1.3333 = 2.3915$ .

For a 9-bit signed two's-complement number representation, the quantization level  $\delta = 2^{-8} = 0.0039062$ . Hence, the total output noise power due to product roundoff is

$$\sigma_o^2 = \frac{\delta^2}{12} (2.3915) = 3.0409 \times 10^{-6}$$

**9.22** The unscaled structure is shown below.



Now,  $F_1(z) = \frac{1}{1-0.5z^{-1}} = 1 + \frac{0.5}{z-0.5}$ . Using Table 9.4 we obtain  $\|F_1\|_2^2 = 1 + \frac{(0.5)^2}{1-(0.5)^2} = 1.3333$ .

Next,  $H(z) = \frac{1-0.3z^{-1}}{1-0.5z^{-1}} = \frac{z-0.3}{z-0.5} = 1 + \frac{0.2}{z-0.5}$ . Using Table 9.4 we obtain

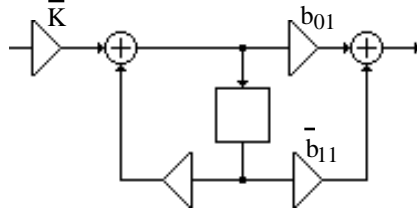
$$\|H\|_2^2 = 1 + \frac{(0.2)^2}{1-(0.5)^2} = 1.0533$$

From Eq. (9.129a),  $\|F_1\|_2 = \alpha_1 = 1.1547$ , and from Eq. (9.129b),

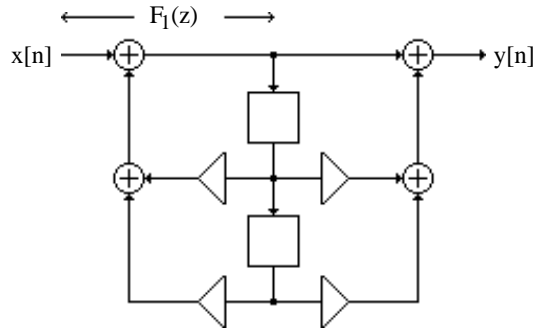
$$\|H\|_2 = \alpha_2 = \sqrt{1.0533} = 1.0263$$

The scaled structure is as shown below, where  $\bar{b}_{01} = \beta_1$  and  $\bar{b}_{11} = -0.3\beta_1$ . From Eqs. (9.130) and (9.133a),  $\bar{K} = \beta_0 K = \beta_0 = \frac{1}{\alpha_1} = 0.86603$ , and from Eq. (9.133b),  $\beta_1 = \frac{\alpha_1}{\alpha_2} = \frac{1.1547}{1.0263} = 1.1251$ .

Therefore,  $\bar{b}_{01} = \beta_1 = 1.1251$ , and  $\bar{b}_{11} = -0.3\beta_1 = -0.3375$ .



**9.23** The unscaled structure is shown below.



Here,  $F_1(z) = \frac{1}{1+0.6z^{-1}+0.4z^{-2}} = \frac{z^2}{z^2+0.6z+0.4} = 1 - \frac{0.6z+0.4}{z^2+0.6z+0.4}$ . Using Table 9.4, we

get  $\|F_1\|_2^2 = 1 + \frac{[0.6^2+0.4^2][1-0.4^2] - 2 \times (1-0.4) \times 0.6 \times 0.4 \times 0.8}{[1-0.4^2]^2 + 2 \times 0.4 \times (0.6)^2 - [1+(0.4^2)] \times (0.6)^2} = 1.4583$ .

Next, we note  $H(z) = \frac{1+0.3z^{-1}+1.5z^{-2}}{1+0.6z^{-1}+0.4z^{-2}} = \frac{z^2+0.3z+1.5}{z^2+0.6z+0.4} = 1 + \frac{-0.3z-1.1}{z^2+0.6z+0.4}$ .

Using Table 9.4 we then get

$\|H\|_2^2 = 1 + \frac{[0.3^2+1.1^2][1-0.4^2] - 2 \times 0.6 \times 0.3 \times (-1.1) \times (1-0.4)}{[1-0.4^2]^2 + 2 \times 0.4 \times (0.6)^2 - [1+0.4^2] \times (0.6)^2} = 3.3083$ .

From Eq. (9.129a),  $\|F_1\|_2 = \alpha_1 = \sqrt{1.4583}$ , and  $\|H\|_2 = \alpha_2 = \sqrt{3.3083}$ . Hence, from

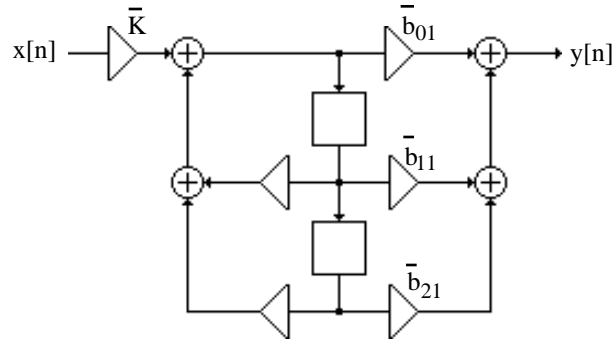
Eq. (9.133a),  $\beta_0 = \frac{1}{\alpha_1} = \frac{1}{\sqrt{1.4583}} = 0.8281$ , and from Eq. (9.133b),

$\beta_1 = \frac{\alpha_1}{\alpha_2} = \frac{\sqrt{1.4583}}{\sqrt{3.3083}} = 0.6639$ .

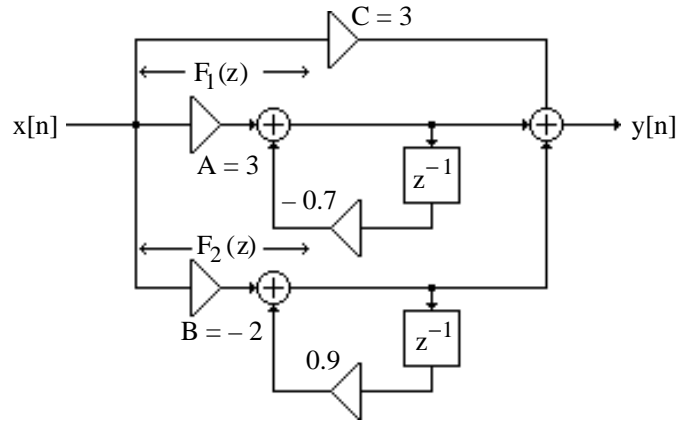
The scaled structure is as shown below, where  $\bar{K} = \beta_0 K = \beta_0 = 0.8281$ ,

$\bar{b}_{01} = (1)(\beta_1) = 0.6639$ ,  $\bar{b}_{11} = (0.3)(\beta_1) = 0.1992$ , and  $\bar{b}_{21} = (1.5)(\beta_1) = 0.9959$ .





9.24  $H(z) = 3 + \frac{3}{1+0.7z^{-1}} + \frac{-2}{1-0.9z^{-1}}$ . The unscaled structure is shown below:



Note that  $F_1(z) = \frac{3}{1+0.7z^{-1}} = 3 + \frac{-2.1}{z+0.7}$ . Hence, using Table 9.4 we get

$\|F_1\|_2^2 = 9 + \frac{(2.1)^2}{1-(0.7)^2} = 17.6471$ . Likewise,  $F_2(z) = \frac{-2}{1-0.9z^{-1}} = -2 + \frac{-1.8}{z-0.9}$ . Hence, using Table

9.4 we get  $\|F_2\|_2^2 = 4 + \frac{(1.8)^2}{1-(0.9)^2} = 21.0526$ . Also,

$H(z) = 3 + \frac{3z}{z+0.7} + \frac{-2z}{z-0.9} = 4 + \frac{-2.1}{z+0.7} + \frac{-1.8}{z-0.9}$ . Hence, using Table 9.4 we get

$\|H\|_2^2 = 4^2 + \frac{(-2.1)^2}{1-(-0.7)^2} + \frac{(-1.8)^2}{1-0.9^2} + 2 \frac{(-2.1)(-1.8)}{1-(-0.7)(0.9)} = 46.338$ . Using Program 9\_4 we get 46.338.

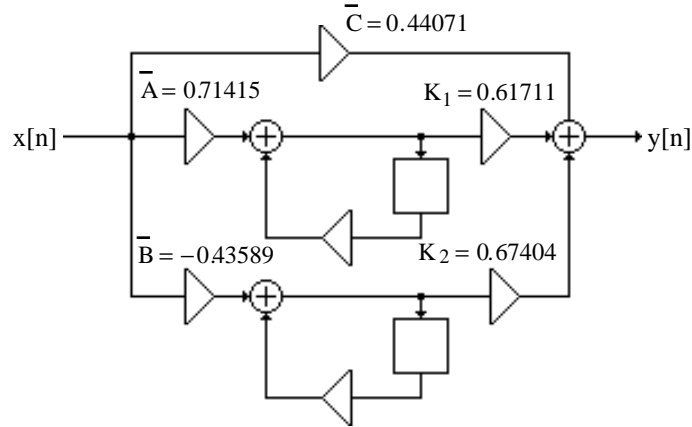
Denote  $\gamma_0 = \|H\|_2 = \sqrt{46.338} = 6.8072$ ,  $\gamma_1 = \|F_1\|_2 = \sqrt{17.6471} = 4.2008$ ,

$\gamma_2 = \|F_2\|_2 = \sqrt{21.0526} = 4.5883$ ,

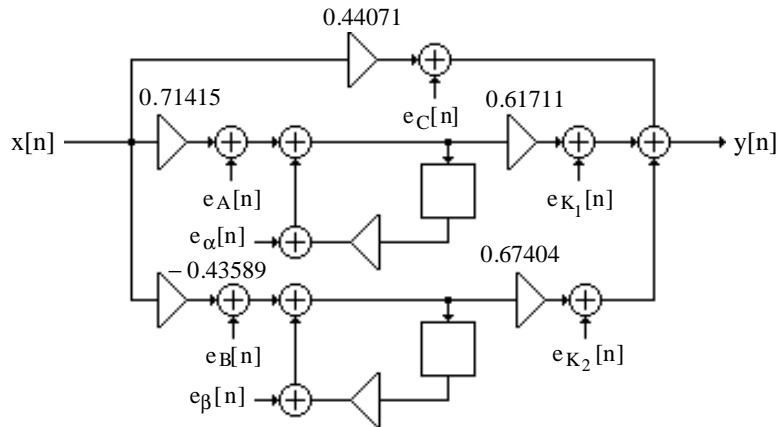
Hence,  $\bar{A} = \frac{A}{\gamma_1} = \frac{3}{4.2008} = 0.71415$ ,  $\bar{B} = \frac{B}{\gamma_1} = \frac{-2}{4.5883} = -0.43589$ ,  $\bar{C} = \frac{C}{\gamma_0} = \frac{3}{6.8072} = 0.44071$ ,

$K_1 = \frac{\gamma_1}{\gamma_0} = \frac{4.2008}{6.8072} = 0.61711$ ,  $K_2 = \frac{\gamma_2}{\gamma_0} = \frac{4.5883}{6.8072} = 0.67404$ .

The scaled structure is shown below:



The noise model for the above structure is shown below:



The noise transfer function from the noise sources  $e_A[n]$  and  $e_\alpha[n]$  is given by:

$$G_1(z) = \frac{0.61171}{1 + 0.7z^{-1}} = 0.61171 + \frac{-0.4282}{1 + 0.7z^{-1}}. \text{ Hence, the normalized output noise variance due to}$$

$$\text{these two noise sources is given by } \sigma_1^2 = 2 \left( (0.61171)^2 + \frac{(-0.4282)^2}{1 - (0.7)^2} \right) = 1.4674.$$

Likewise, the noise transfer function from the noise sources  $e_A[n]$  and  $e_\alpha[n]$  is given by:

$$G_2(z) = \frac{0.60664}{1 - 0.9z^{-1}} = 0.60664 + \frac{0.546}{1 - 0.9z^{-1}}. \text{ Hence, the normalized output noise variance due to}$$

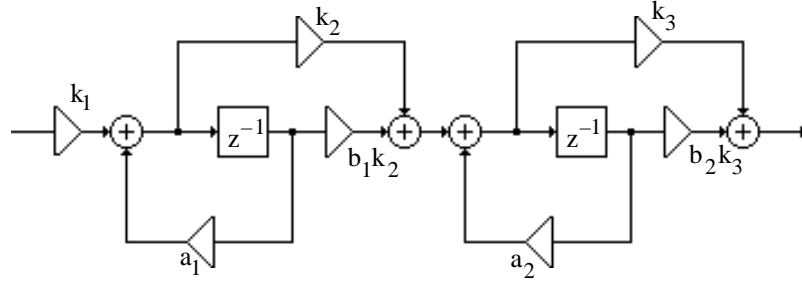
$$\text{these two noise sources is given by } \sigma_2^2 = 2 \left( (0.60664)^2 + \frac{(0.546)^2}{1 - (-0.9)^2} \right) = 3.8738.$$

The noise transfer function from the remaining three noise sources is  $G_3(z) = 1$ . Hence, the normalized output noise variance due to these three noise sources is given by  $\sigma_3^2 = 3$ .

Therefore, the total normalized output noise variance is

$$\sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 1.4674 + 3.8738 + 3 = 8.3412.$$

**9.25 (a)** The scaled structure is shown below. The value of the scaling constants is found below.



Cascade Structure #1:  $H(z) = \left( \frac{1-2z^{-1}}{1+0.3z^{-1}} \right) \left( \frac{1+3z^{-1}}{1-0.4z^{-1}} \right)$ . Here,  $a_1 = -0.3$ ,  $b_1 = -2$ ,  $a_2 = 0.4$ , and  $b_2 = 3$ .

$F_1(z) = \frac{z^{-1}}{1+0.3z^{-1}} = \frac{1}{z+0.3}$ . Thus, using Program 9\_4 we get  $\|F_1\|_2^2 = 1.9089$ . Hence,

$\gamma_1 = \|F_1\|_2 = \sqrt{1.9089} = 1.0483$ .  $F_2(z) = \frac{1-2z^{-1}}{1+0.3z^{-1}} \cdot \frac{z^{-1}}{1-0.4z^{-1}} = \frac{z-2}{z^2-0.1z-0.12}$ . Using Program

9\_4 we get  $\|F_2\|_2^2 = 4.6722$ . Hence,  $\gamma_2 = \|F_2\|_2 = \sqrt{4.6722} = 2.1615$ . Next,

$H(z) = \frac{z^2+z-6}{z^2-0.1z-0.12}$ . Using Program 9\_4 we get  $\|H\|_2^2 = 36.271$ . Hence,

$\gamma_0 = \|H\|_2 = \sqrt{36.271} = 6.0226$ .

The scaling multipliers are therefore given by

$k_1 = \frac{1}{\gamma_1} = 0.95393$ ,  $k_2 = \frac{\gamma_1}{\gamma_2} = 0.48499$ ,  $k_3 = \frac{\gamma_2}{\gamma_0} = 0.3589$ .  $b_1k_2 = -0.96998$ , and  $b_2k_3 = 1.0767$ .

The noise at the output due to the scaling constant  $k_1$  and multiplier  $a_1$  have a variance  $\sigma_1^2 = \gamma_1^2 = 1.9089$ .

Noise at the output due to  $a_2$ ,  $k_2$  and  $b_1k_2$  have variance  $\sigma_2^2$  which is calculated below. The noise transfer function for these noise sources is

$G_2(z) = \frac{0.3589+1.0767z^{-1}}{1-0.4z^{-1}} = \frac{0.3589z+1.0767}{z-0.4}$ . Using Program 9\_4 we get  $\sigma_2^2 = 1.1654$ .

Hence the total noise power (variance) at the output =  $2 \times 1.9089 + 3 \times 1.1654 + 2 = 9.314$ .

In case quantization is carried out after addition, then the total noise power at the output =  $1.9089 + 1.1654 + 1 = 4.0743$ .

Cascade Structure #2:  $H(z) = \left( \frac{1+3z^{-1}}{1+0.3z^{-1}} \right) \left( \frac{1-2z^{-1}}{1-0.4z^{-1}} \right)$ . Here  $a_1 = -0.3$ ,  $b_1 = 3$ ,  $a_2 = 0.4$ , and  $b_2 = -2$ .

$F_1(z) = \frac{z^{-1}}{1+0.3z^{-1}} = \frac{1}{z+0.3}$ . Thus, using Program 9\_4 we get  $\|F_1\|_2^2 = 1.9089$ . Hence,

$$\gamma_1 = \|F_1\|_2 = \sqrt{1.9089} = 1.0483. \quad F_2(z) = \frac{1+3z^{-1}}{1+0.3z^{-1}} \cdot \frac{z^{-1}}{1-0.4z^{-1}} = \frac{z+3}{z^2-0.1z-0.12}. \quad \text{Using}$$

Program 9\_4 we get  $\|F_2\|_2^2 = 10.98$ . Hence,  $\gamma_2 = \|F_2\|_2 = \sqrt{10.98} = 3.3136$ . Next,

$$H(z) = \frac{z^2+z-6}{z^2-0.1z-0.12}. \quad \text{Using Program 9_4 we get } \|H\|_2^2 = 36.271. \quad \text{Hence,}$$

$$\gamma_0 = \|H\|_2 = \sqrt{36.271} = 6.0226.$$

The scaling multipliers are therefore given by

$$k_1 = \frac{1}{\gamma_1} = 0.95393, \quad k_2 = \frac{\gamma_1}{\gamma_2} = 0.31636, \quad k_3 = \frac{\gamma_2}{\gamma_0} = 0.55019. \quad b_1k_2 = 0.94908, \quad \text{and} \\ b_2k_3 = -1.1004.$$

The noise at the output due to the scaling constant  $k_1$  and multiplier  $a_1$  have a variance  $\sigma_1^2 = \gamma_1^2 = 1.9089$ .

Noise at the output due to  $a_2$ ,  $k_2$  and  $b_1k_2$  have variance  $\sigma_2^2$  which is calculated below. The noise transfer function for these noise sources is

$$G_2(z) = \frac{0.55019 - 1.1004z^{-1}}{1 - 0.4z^{-1}} = \frac{0.55019z - 1.1004}{z - 0.4}. \quad \text{Using Program 9_4 we get } \sigma_2^2 = 1.2253.$$

Hence the total noise power (variance) at the output =  $2 \times 1.9089 + 3 \times 1.2253 + 2 = 9.4937$ .

In case quantization is carried out after addition, then the total noise power at the output =  $1.9089 + 1.2253 + 1 = 4.1342$ .

**Cascade Structure #3:**  $H(z) = \left( \frac{1+3z^{-1}}{1-0.4z^{-1}} \right) \left( \frac{1-2z^{-1}}{1+0.3z^{-1}} \right)$ . Here  $a_1 = 0.4$ ,  $b_1 = 3$ ,  $a_2 = -0.3$ , and  $b_2 = -2$ .

$F_1(z) = \frac{z^{-1}}{1-0.4z^{-1}} = \frac{1}{z-0.4}$ . Thus, using Program 9\_4 we get  $\|F_1\|_2^2 = 1.1905$ . Hence,

$$\gamma_1 = \|F_1\|_2 = \sqrt{1.1905} = 1.0911. \quad F_2(z) = \frac{1+3z^{-1}}{1-0.4z^{-1}} \cdot \frac{z^{-1}}{1+0.3z^{-1}} = \frac{z+3}{z^2-0.1z-0.12}. \quad \text{Using}$$

Program 9\_4 we get  $\|F_2\|_2^2 = 10.98$ . Hence,  $\gamma_2 = \|F_2\|_2 = \sqrt{10.98} = 3.3136$ . Next,

$$H(z) = \frac{z^2+z-6}{z^2-0.1z-0.12}. \quad \text{Using Program 9_4 we get } \|H\|_2^2 = 36.271. \quad \text{Hence,}$$

$$\gamma_0 = \|H\|_2 = \sqrt{36.271} = 6.0226.$$

The scaling multipliers are therefore given by

$$k_1 = \frac{1}{\gamma_1} = 0.91652, \quad k_2 = \frac{\gamma_1}{\gamma_2} = 0.32928, \quad k_3 = \frac{\gamma_2}{\gamma_0} = 0.55019. \quad b_1k_2 = 0.98784, \quad \text{and} \\ b_2k_3 = -1.1004.$$

The noise at the output due to the scaling constant  $k_1$  and multiplier  $a_1$  have a variance  $\sigma_1^2 = \gamma_1^2 = 1.1905$ .

Noise at the output due to  $a_2$ ,  $k_2$  and  $b_1k_2$  have variance  $\sigma_2^2$  which is calculated below. The noise transfer function for these noise sources is

$$G_2(z) = \frac{0.55019 - 1.1004z^{-1}}{1 + 0.3z^{-1}} = \frac{0.55019z - 1.1004}{z + 0.3}. \text{ Using Program 9\_4 we get } \sigma_2^2 = 2.0625.$$

Hence the total noise power (variance) at the output =  $2 \times 1.1905 + 3 \times 2.0625 + 2 = 10.568$ .

In case quantization is carried out after addition, then the total noise power at the output =  $1.1905 + 2.0625 + 1 = 4.253$ .

**Cascade Structure #4:**  $H(z) = \left( \frac{1 - 2z^{-1}}{1 - 0.4z^{-1}} \right) \left( \frac{1 + 3z^{-1}}{1 + 0.3z^{-1}} \right)$ . Here  $a_1 = 0.4$ ,  $b_1 = -2$ ,  $a_2 = -0.3$ , and  $b_2 = 3$ .

$$F_1(z) = \frac{z^{-1}}{1 - 0.4z^{-1}} = \frac{1}{z - 0.4}. \text{ Thus, using Program 9\_4 we get } \|F_1\|_2^2 = 1.1905. \text{ Hence,}$$

$$\gamma_1 = \|F_1\|_2 = \sqrt{1.1905} = 1.0911. \quad F_2(z) = \frac{1 - 2z^{-1}}{1 - 0.4z^{-1}} \cdot \frac{z^{-1}}{1 + 0.3z^{-1}} = \frac{z - 2}{z^2 - 0.1z - 0.12}. \text{ Using}$$

Program 9\\_4 we get  $\|F_2\|_2^2 = 4.6722$ . Hence,  $\gamma_2 = \|F_2\|_2 = \sqrt{4.6722} = 2.1615$ . Next,

$$H(z) = \frac{z^2 + z - 6}{z^2 - 0.1z - 0.12}. \text{ Using Program 9\_4 we get } \|H\|_2^2 = 36.271. \text{ Hence,}$$

$$\gamma_0 = \|H\|_2 = \sqrt{36.271} = 6.0226.$$

The scaling multipliers are therefore given by

$$k_1 = \frac{1}{\gamma_1} = 0.91651, \quad k_2 = \frac{\gamma_1}{\gamma_2} = 0.50479, \quad k_3 = \frac{\gamma_2}{\gamma_0} = 0.3589. \quad b_1k_2 = -1.0096, \text{ and } b_2k_3 = 1.0767.$$

The noise at the output due to the scaling constant  $k_1$  and multiplier  $a_1$  have a variance  $\sigma_1^2 = \gamma_1^2 = 1.1905$ .

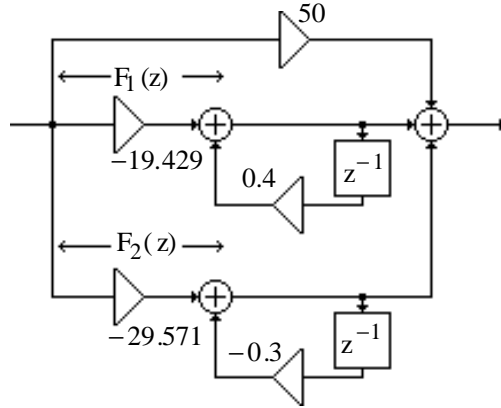
Noise at the output due to  $a_2$ ,  $k_2$  and  $b_1k_2$  have variance  $\sigma_2^2$  which is calculated below. The noise transfer function for these noise sources is

$$G_2(z) = \frac{0.3589 + 1.0767z^{-1}}{1 - 0.4z^{-1}} = \frac{0.3589z + 1.0767}{z - 0.4}. \text{ Using Program 9\_4 we get } \sigma_2^2 = 1.9015.$$

Hence the total noise power (variance) at the output =  $2 \times 1.1905 + 3 \times 1.9015 + 2 = 10.085$ .

In case quantization is carried out after addition, then the total noise power at the output =  $1.1905 + 1.9105 + 1 = 4.092$ .

**9.26 Parallel form I realization:** The corresponding structure is obtained from a partial fraction expansion of  $H(z)$  in  $z^{-1}$  the form:  $H(z) = 50 + \frac{-19.429}{1 - 0.4z^{-1}} + \frac{-29.571}{1 + 0.3z^{-1}}$ , and is shown below:



Here the scaling transfer function  $F_1(z)$  for the middle branch is given by

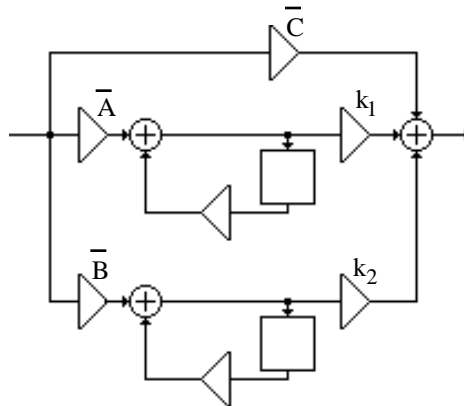
$F_1(z) = \frac{-19.429z}{z-0.4}$ . Hence using Program 9\_4 we get,  $\|F_1\|_2^2 = 449.39$ . This implies

$\gamma_1 = \|F_1\|_2 = \sqrt{449.39} = 21.199$ . Similarly, the scaling transfer function for the bottom branch

is given by  $F_2(z) = \frac{-29.571z}{z+0.3}$ . Hence using Program 9\_4 we get,  $\|F_2\|_2^2 = 960.93$ . This

implies  $\gamma_2 = \|F_2\|_2 = \sqrt{960.93} = 30.999$ . Finally using Program 9\_4 we get,  $\|H\|_2^2 = 36.271$ .

Hence,  $\gamma_0 = \|H\|_2 = \sqrt{36.271} = 6.0226$ . The scaled structure with noise sources is shown below:



where  $\bar{A} = \frac{-19.429}{\gamma_1} = -0.91651$ ,  $\bar{B} = \frac{-29.571}{\gamma_2} = -0.95393$ ,  $\bar{C} = \frac{50}{\gamma_0} = 8.3021$ ,  $k_1 = \frac{\gamma_1}{\gamma_0} = 3.5199$ ,

and  $k_2 = \frac{\gamma_2}{\gamma_0} = 5.1471$ .

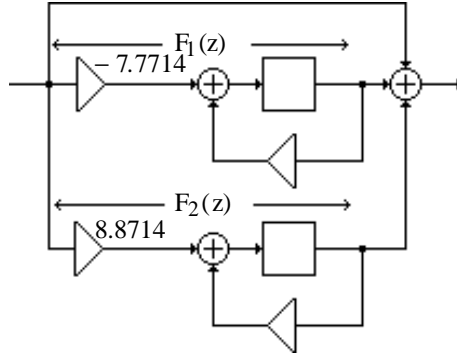
The noise transfer function from each of the noise sources  $e_C[n]$ ,  $e_{k1}[n]$ , and  $e_{k2}[n]$  is  $G_3(z) = 1$ . The corresponding output noise variance is  $\sigma_3^2 = 1$ . The noise transfer function

from each of the noise sources  $e_A[n]$  and  $e_{a1}[n]$  is  $G_1(z) = \frac{k_1}{1-0.4z^{-1}} = \frac{3.5199z}{z-0.4}$ . Using

Program 9\_4 we get the corresponding output noise variance as  $\sigma_1^2 = 14.75$ . Likewise, the noise transfer function from each of the noise sources  $e_B[n]$  and  $e_{a2}[n]$  is

$G_2(z) = \frac{k_2}{1+0.3z^{-1}} = \frac{5.1471z}{z+0.3}$ . Using Program 9\_4 we get the corresponding output noise variance as  $\sigma_2^2 = 29.113$ . Total noise variance at the output is therefore  $\sigma_0^2 = 2\sigma_1^2 + 2\sigma_2^2 + 3\sigma_3^2 = 2(14.75) + 2(29.113) + 3 = 90.726$ .

**Parallel form II realization:** The corresponding unscaled structure is obtained from a partial fraction expansion of  $H(z)$  in  $z$  in the form:  $H(z) = 1 + \frac{-7.7714}{z-0.4} + \frac{8.8714}{z+0.3}$ , and is shown below.



Here the scaling transfer function  $F_1(z)$  for the middle branch is given by  $F_1(z) = -\frac{7.7714}{z-0.4}$ .

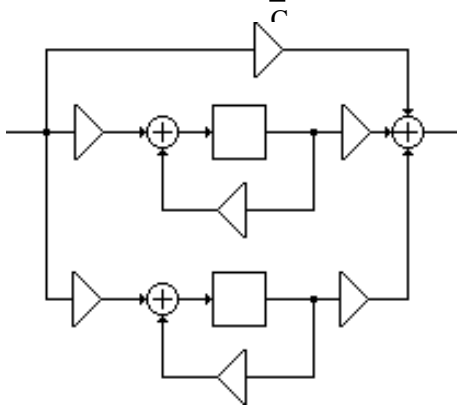
Hence using Program 9\_4 we get,  $\|F_1\|_2^2 = 71.898$ . This implies  $\gamma_1 = \|F_1\|_2 = \sqrt{71.898} = 8.4793$ .

Similarly, the scaling transfer function for the bottom branch is given by  $F_2(z) = \frac{8.8714}{z+0.3}$ .

Hence using Program 9\_4 we get,  $\|F_2\|_2^2 = 86.485$ . Thus,  $\gamma_2 = \|F_2\|_2 = \sqrt{86.485} = 9.2998$ .

Finally using Program 9\_4 we get,  $\|H\|_2^2 = 36.271$ . Hence,  $\gamma_0 = \|H\|_2 = \sqrt{36.271} = 6.0226$ .

The scaled structure with noise sources is shown below:



where  $\bar{A} = \frac{-7.7714}{\gamma_1} = -0.91651$ ,  $\bar{B} = \frac{8.8714}{\gamma_2} = 0.95394$ ,  $\bar{C} = \frac{1}{\gamma_0} = 0.16604$ ,  $k_1 = \frac{\gamma_1}{\gamma_0} = 1.4079$ ,

$k_2 = \frac{\gamma_2}{\gamma_0} = 1.5441$ .

The noise transfer function from each of the noise sources  $e_C[n]$ ,  $e_{k1}[n]$ , and  $e_{k2}[n]$  is  $G_3(z) = 1$ . The corresponding output noise variance is  $\sigma_3^2 = 1$ . The noise transfer function

from each of the noise sources  $e_A[n]$  and  $e_{a1}[n]$  is  $G_1(z) = \frac{k_1 z^{-1}}{1 - 0.4z^{-1}} = \frac{1.4079}{z - 0.4}$ . Using

Program 9\_4 we get the corresponding output noise variance as  $\sigma_1^2 = 2.3597$ . Likewise, the noise transfer function from each of the noise sources  $e_B[n]$  and  $e_{a2}[n]$  is

$G_2(z) = \frac{k_2 z^{-1}}{1 + 0.3z^{-1}} = \frac{1.5441}{z + 0.3}$ . Using Program 9\_4 we get the corresponding output noise

variance as  $\sigma_2^2 = 2.62$ . Total noise variance at the output is therefore

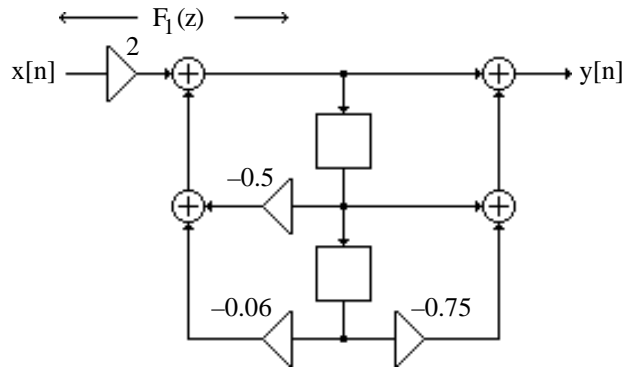
$$\sigma_o^2 = 2\sigma_1^2 + 2\sigma_2^2 + 3\sigma_3^2 = 2(2.3597) + 2(2.62) + 3 = 12.959.$$

For quantization of products after addition, the total output noise variance in the case of Parallel form I structure (after scaling) is  $\sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 14.75 + 29.113 + 1 = 44.863$ . and total output noise variance in the case of Parallel form II structure (after scaling) is

$\sigma_o^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 2.3597 + 2.62 + 1 = 5.9797$ . In either case, Parallel form II structure (after scaling) has the lowest roundoff noise variance.

**9.27** 
$$H(z) = \frac{2 + 2z^{-1} - 1.5z^{-2}}{1 + 0.5z^{-1} + 0.06z^{-2}} = \frac{2z^2 + 2z - 1.5}{z^2 + 0.5 + 0.06} = 2 \left( \frac{z^2 + z - 0.75}{z^2 + 0.5 + 0.06} \right) \dots$$

**(a) Direct Form:** The unscaled structure is shown below:



Scaling:  $F_1(z) = \frac{2}{1 + 0.5z^{-1} + 0.06z^{-2}} = \frac{2z^2}{z^2 + 0.5 + 0.06}$ . Using Program 9\_4 we get,

$\|F_1\|_2^2 = 5.1633$ . This implies,  $\beta_0 = \frac{1}{\|F_1\|_2} = \frac{1}{\sqrt{5.1633}} = 0.44009$ . The other scaling transfer

function is  $H(z) = \frac{2z^2 + 2z - 1.5}{z^2 + 0.5 + 0.06}$ . Using Program 9\_4 we get,  $\|H\|_2^2 = 10.6512$ . This implies

$$\beta_1 = \frac{\|F_1\|_2}{\|H\|_2} = \frac{\sqrt{5.1633}}{\sqrt{10.6512}} = 0.69625.$$





$$F_2(z) = \frac{2(1-0.5z^{-1})}{(1+0.2z^{-1})(1+0.3z^{-1})} = \frac{2z^2 - z}{z^2 + 0.5z + 0.06}. \text{ Using Program 9\_4 we get, } \|F_2\|_2^2 = 8.8896.$$

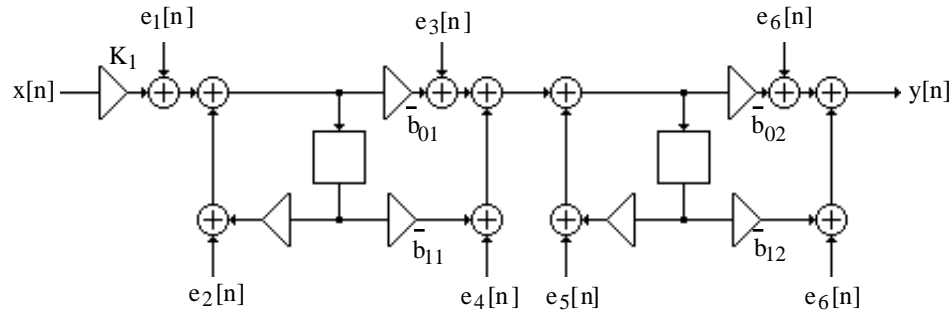
This implies  $\beta_1 = \frac{\|F_1\|_2}{\|F_2\|_2} = \frac{\sqrt{4.1667}}{\sqrt{8.8896}} = 0.68463$ . The third scaling transfer function is

$$H(z) = \frac{2z^2 + 2z - 1.5}{z^2 + 0.5z + 0.06}. \text{ Using Program 9\_4 we get, } \|H\|_2^2 = 10.6512. \text{ This implies}$$

$$\beta_2 = \frac{\|F_2\|_2}{\|H\|_2} = \frac{\sqrt{8.8896}}{\sqrt{10.6512}} = 0.91357.$$

The scaling multipliers are therefore  $K_1 = \beta_0 K = 0.4899 \times 2 = 0.9798$ ,  $\bar{b}_{01} = \beta_1 = 0.68463$ ,  $\bar{b}_{11} = -0.5 \times \beta_1 = -0.34231$ ,  $\bar{b}_{02} = \beta_2 = 0.91357$ , and  $\bar{b}_{12} = 1.5 \times \beta_2 = 1.3704$ .

The scaled structure with the noise sources is shown below:



Product roundoff noise analysis, quantization before addition:

$$G_1(z) = G_2(z) = \frac{(0.68463 - 0.34231z^{-1})(0.91357 + 1.3704z^{-1})}{(1 + 0.2z^{-1})(1 + 0.3z^{-1})} = \frac{0.6255z^2 + 0.6255z - 0.4691}{z^2 + 0.5z + 0.06}.$$

Using Program 9\_4 we get  $\sigma_{1,n}^2 = \sigma_{2,n}^2 = 1.0417$ .

$$G_3(z) = G_4(z) = \frac{0.91357 + 1.3704z^{-1}}{1 + 0.3z^{-1}} = \frac{0.91357z + 1.3704}{z + 0.3}. \text{ Using Program 9\_4 we get}$$

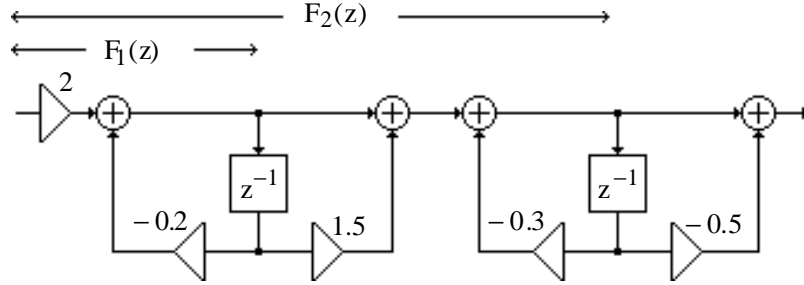
$\sigma_{3,n}^2 = \sigma_{4,n}^2 = 2.1553$ . Finally,  $G_5(z) = G_6(z) = 1$ . This implies,  $\sigma_{5,n}^2 = \sigma_{6,n}^2 = 1$ .

Hence, total normalized output noise power is  $\sigma_n^2 = 2 \times 1.0417 + 2 \times 2.1553 + 2 \times 1 = 8.3940$ .

Product roundoff noise analysis, quantization after addition:

Here, total normalized output noise power is  $\sigma_n^2 = 1.0417 + 2.1553 + 1 = 4.1970$ .

(c) Cascade Structure #2:  $H(z) = \frac{2(1 + 1.5z^{-1})(1 - 0.5z^{-1})}{(1 + 0.2z^{-1})(1 + 0.3z^{-1})}$ . The unscaled structure is shown below:



Scaling:  $F_1(z) = \frac{2}{1+0.2z^{-1}}$ . Using Program 9\_4 we get  $\|F_1\|_2^2 = 4.1667$ . This implies

$\beta_0 = \frac{1}{\|F_1\|_2} = \frac{1}{\sqrt{4.1667}} = 0.4899$ . The second scaling transfer function is

$F_2(z) = \frac{2(1+1.5z^{-1})}{(1+0.2z^{-1})(1+0.3z^{-1})} = \frac{2z^2+3z}{z^2+0.5z+0.06}$ . Using Program 9\_4 we get,  $\|F_2\|_2^2 = 9.4741$ .

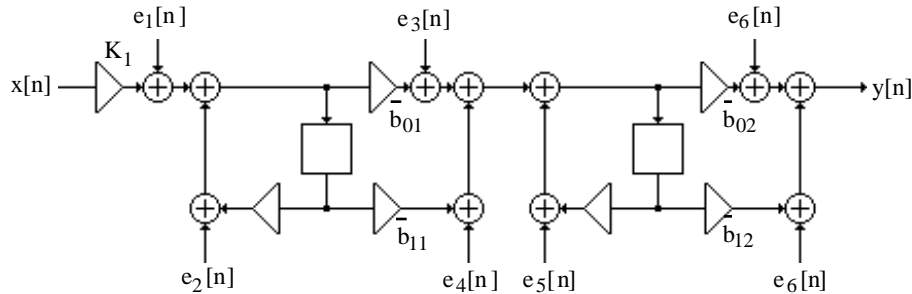
This implies  $\beta_1 = \frac{\|F_1\|_2}{\|F_2\|_2} = \frac{\sqrt{4.1667}}{\sqrt{9.4741}} = 0.6632$ . The third scaling transfer function is

$H(z) = \frac{2z^2+2z-1.5}{z^2+0.5z+0.06}$ . Using Program 9\_4 we get,  $\|H\|_2^2 = 10.6512$ . This implies

$\beta_2 = \frac{\|F_2\|_2}{\|H\|_2} = \frac{\sqrt{9.4741}}{\sqrt{10.6512}} = 0.9431$ .

The scaling multipliers are therefore  $K_1 = \beta_0 K = 0.4899 \times 2 = 0.9798$ ,  $\bar{b}_{01} = \beta_1 = 0.6632$ ,  $\bar{b}_{11} = 1.5 \times \beta_1 = 0.9948$ ,  $\bar{b}_{02} = \beta_2 = 0.9431$ , and  $\bar{b}_{12} = -0.5 \times \beta_2 = -0.4716$ .

The scaled structure with the noise sources is shown below:



Product roundoff noise analysis, quantization before addition:

$G_1(z) = G_2(z) = \frac{(0.6632 + 0.9948z^{-1})(0.9431 - 0.4716z^{-1})}{(1 + 0.2z^{-1})(1 + 0.3z^{-1})} = \frac{0.6255z^2 + 0.6255z - 0.4691}{z^2 + 0.5z + 0.06}$ .

Using Program 9\_4 we get  $\sigma_{1,n}^2 = \sigma_{2,n}^2 = 1.0417$ .

$$G_3(z) = G_4(z) = \frac{0.9431 - 0.4716z^{-1}}{1 + 0.3z^{-1}} = \frac{0.9431z - 0.4716}{z + 0.3}. \text{ Using Program 9\_4 we get}$$

$$\sigma_{3,n}^2 = \sigma_{4,n}^2 = 1.5151. \text{ Finally, } G_5(z) = G_6(z) = 1. \text{ This implies, } \sigma_{5,n}^2 = \sigma_{6,n}^2 = 1.$$

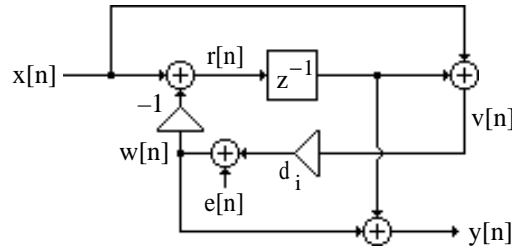
Hence, total normalized output noise power is  $\sigma_n^2 = 2 \times 0.1.0417 + 2 \times 1.5151 + 2 \times 1 = 7.1136$ .

Product roundoff noise analysis, quantization after addition:

Here, total normalized output noise power is  $\sigma_n^2 = 0.1.0417 + 1.5151 + 1 = 3.5568$ .

Hence, the scaled direct form realization has the smallest roundoff noise in both cases.

**9.28** The noise model for the allpass structure is shown below



Analysis yields  $W(z) = E(z) + d_i V(z)$ ,  $R(z) = X(z) - W(z)$ ,  $V(z) = X(z) + z^{-1}R(z)$ , and  $Y(z) = W(z) + z^{-1}R(z)$ .

To determine the noise transfer function we set  $X(z) = 0$  in the above equations. This leads to

$$R(z) = -W(z), \quad V(z) = z^{-1}R(z) = -z^{-1}W(z), \text{ and hence } W(z) = E(z) - d_i z^{-1}W(z) \text{ or}$$

$E(z) = (1 + d_i z^{-1})W(z)$ . As a result,  $Y(z) = W(z) - z^{-1}W(z) = (1 - z^{-1})W(z)$ . Consequently, the noise transfer function is given by

$$G(z) = \frac{Y(z)}{E(z)} = \frac{1 - z^{-1}}{1 + d_i z^{-1}} = \frac{z - 1}{z + d_i} = 1 + \frac{-(1 + d_i)}{z + d_i}.$$

$$\text{Thus } \sigma_e^2 = 1 + \frac{(1 + d_i)^2}{1 - d_i^2} = \frac{2}{1 - d_i}.$$

**(b)** Let  $G_1(z)$  be the noise transfer function for  $d_1$ ,  $G_2(z)$  be the noise transfer function for  $d_2$  and  $G_3(z)$  be the noise transfer function for  $d_3$  then

$$G_1(z) = \frac{z-1}{z+d_1} A_2(z), \quad G_2(z) = \frac{z-1}{z+d_2} \quad \text{and} \quad G_3(z) = \frac{z-1}{z+d_3}. \text{ From the results of part (a) it}$$

follows that the noise variances due to  $d_1$ ,  $d_2$  and  $d_3$  are given by

$$\sigma_k^2 = \frac{2}{1 - d_k} \quad \text{for } k = 1, 2, 3. \text{ Hence the total noise power at the output is given by}$$

$$\sigma^2 = \frac{2}{1 - d_1} + \frac{2}{1 - d_2} + \frac{2}{1 - d_3}.$$

**9.29** Let the total noise power at the output of  $G(z)$  due to product-off be given by  $\sigma_G^2$ . Assuming

a total of  $L$  multipliers in the realization of  $G(z)$  we get  $\sigma_G^2 = \sum_{l=1}^L k_l \left( \frac{1}{2\pi} \int_0^{2\pi} |G_1(e^{j\omega})|^2 d\omega \right)$  where

$G_1(z)$  denotes the noise transfer function due to 1 -th noise source in  $G(z)$ . Now if each delay is replaced by two delays then each of the noise transfer function becomes  $G_1(e^{2j\omega})$ . Thus the total noise power at the output due to noise sources in  $G(z^2)$  is given by  $\hat{\sigma}_G^2 =$

$\sum_{l=1}^L k_l \left( \frac{1}{2\pi} \int_0^{2\pi} |G_1(e^{2j\omega})|^2 d\omega \right)$ . Replacing  $\omega$  by  $\hat{\omega} / 2$  in the integral we get  $\hat{\sigma}_G^2 =$

$\sum_{l=1}^L k_l \left( \frac{1}{2\pi} \int_0^{4\pi} |G_1(e^{j\hat{\omega}})|^2 \left(\frac{1}{2}\right) d\hat{\omega} \right) = \frac{1}{2} \sum_{l=1}^L k_l \left( \frac{1}{2\pi} \int_0^{4\pi} |G_1(e^{j\hat{\omega}})|^2 d\hat{\omega} \right) = \sigma_G^2$ . Since

$\frac{1}{2\pi} \int_0^{2\pi} |G_1(e^{j\omega})|^2 |A(e^{j\omega})|^2 d\omega = \frac{1}{2\pi} \int_0^{2\pi} |G_1(e^{j\omega})|^2 d\omega$ , the total noise power at the output of the

cascade is still equal to  $\sigma_0^2$ .

**9.30** For the first factor in the numerator there are  $R$  possible choices of factors. Once this factor has been chosen, there are  $R - 1$  choices for the next factor and continuing further we get that the total number of possible ways in which the factors in the numerator can be generated equal to  $R(R-1)(R-2)\dots 2 \times 1 = R!$ . Similarly the total number of ways in which the factors in the denominator can be generated =  $R!$ . Since the numerator and denominator are generated independent of each other hence the total number of possible realizations are  $N = (R!)(R!) = (R!)^2$ .

**9.31 (a)** First we pair the poles closest to the unit circle with their nearest zeros resulting in the

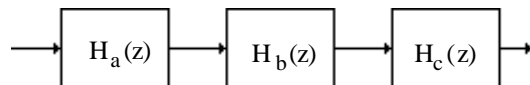
second-order section  $H_a(z) = \frac{z^2 + 0.2z + 0.9}{z^2 + 0.1z + 0.8}$ . Next, the poles that are closest to the poles of

$H_a(z)$  are matched with their nearest zeros resulting in the second-order section  $H_b(z) =$

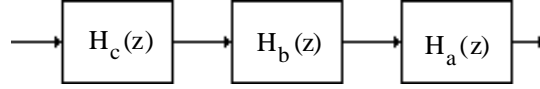
$\frac{z^2 + 0.3z + 0.5}{z^2 + 0.2z + 0.4}$ . Finally, the remaining poles and zeros are matched yielding the second-order

section  $H_c(z) = \frac{z^2 + 0.8z + 0.2}{z^2 + 0.6z + 0.3}$ .

For ordering the sections to yield the smallest peak output noise due to product round-off under an  $L_2$ -scaling rule, the sections should be placed from most peaked to least peaked as shown below.



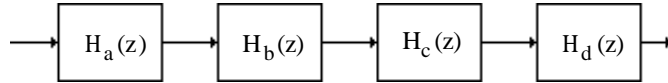
For ordering the sections to yield the smallest peak output noise power due to product round-off under an  $L_\infty$ -scaling rule, the sections should be placed from least peaked to most peaked as shown below.



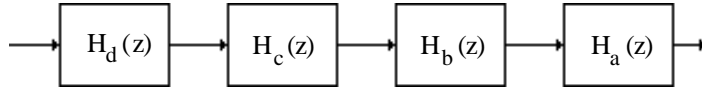
(b) The poles closest to the unit circle is given by the denominator  $(z^2 - 0.2z + 0.9)$ , the next closer pole is given by the factor  $(z + 0.9)$ , followed by the factor  $(z^2 + 0.4z + 0.7)$  and finally the factor  $(z + 0.2)$ . Pairing these poles with their closest zeros we get the following pairings:

$$H_a(z) = \frac{z^2 + 0.1z + 0.7}{z^2 - 0.2z + 0.9}, H_b(z) = \frac{z + 1.1}{z + 0.9}, H_c(z) = \frac{z^2 + 0.5z + 0.6}{z^2 + 0.4z + 0.7} \text{ and } H_d(z) = \frac{z + 0.3}{z + 0.2}.$$

For ordering the sections to yield the smallest peak output noise due to product round-off under an  $L_2$ -scaling rule, the sections should be placed from most peaked to least peaked as shown below.



For ordering the sections to yield the smallest peak output noise power due to product round-off under an  $L_\infty$ -scaling rule, the sections should be placed from least peaked to most peaked as shown below.



**9.32**  $\text{SNR} = \frac{\sigma_x^2}{\sigma_0^2}$ . After scaling  $\sigma_x$  changes to  $K\sigma_x$  where  $K$  is given by Eq. (9.150).

$$\text{Therefore SNR} = \frac{\sigma_x^2(1-|\alpha|)^2}{\sigma_0^2}.$$

(i) For uniform density function  $\sigma_x^2 = \int_{-1}^1 \frac{1}{2} x^2 dx = \frac{1}{3}$ .

Thus  $\text{SNR} = \frac{(1-|\alpha|)^2}{3\sigma_0^2}$ . With  $b = 12$   $\sigma_0^2 = \frac{2^{-24}}{12} = 4.967 \times 10^{-9}$  and  $|\alpha| = 0.95$ ,

$$\text{Hence SNR}_{\text{dB}} = 10 \log_{10} \left( \frac{(1-|\alpha|)^2}{3\sigma_0^2} \right) = 52.24 \text{ dB.}$$

(ii) For Gaussian input with  $\sigma_x^2 = \frac{1}{9}$ . Hence  $\text{SNR} = \frac{(1-|\alpha|)^2}{9\sigma_0^2}$ . Again with  $b = 12$  and

$$|\alpha| = 0.95, \text{SNR}_{\text{dB}} = 10 \log_{10} \left( \frac{(1-|\alpha|)^2}{9\sigma_0^2} \right) = 47.97 \text{ dB.}$$

(iii) For a sinusoidal input of known frequency, i.e.  $x[n] = \sin(\omega_0 n)$ .

In this case average power  $= \sigma_x^2 = \frac{1}{2}$ . Hence  $\text{SNR} = \frac{(1-|\alpha|)^2 \sigma_x^2}{\sigma_0^2} = \frac{(1-|\alpha|)^2}{2\sigma_0^2}$ .

Therefore  $\text{SNR}_{\text{dB}} = 10 \log_{10} \left( \frac{(1-|\alpha|^2)^2}{2\sigma_0^2} \right) = 69.91 \text{ dB}$ .

**9.33 (a)**  $H_{\text{LP}}(z) = \frac{1}{2} \{1 + A_1(z)\}$  where  $A_1(z) = \frac{-\alpha + z^{-1}}{1 - \alpha z^{-1}}$ .

Hence  $|H_{\text{LP}}(e^{j\omega})|^2 = \left( \frac{1-\alpha}{2} \right)^2 \frac{2(1+\cos(\omega))}{1-2\alpha\cos(\omega)+\alpha^2}$ .

Proving  $\left. \frac{\partial |H_{\text{LP}}(e^{j\omega})|^2}{\partial \alpha} \right|_{\omega=0} = 0$  is equivalent to proving  $\left. \frac{\partial |H_{\text{LP}}(e^{j\omega})|^2}{\partial \alpha} \right|_{\omega=0} = 0$ . Now

$$\frac{\partial |H_{\text{LP}}(e^{j\omega})|^2}{\partial \alpha} = -\frac{1-\alpha}{2} \frac{2(1+\cos(\omega))}{1-2\alpha\cos(\omega)+\alpha^2} - \left( \frac{1-\alpha}{2} \right)^2 \frac{2(1+\cos(\omega))(-2\cos(\omega)+\alpha)}{1-2\alpha\cos(\omega)+\alpha^2}$$

$$\text{Thus, } \left. \frac{\partial |H_{\text{LP}}(e^{j\omega})|^2}{\partial \alpha} \right|_{\omega=0} = \frac{-2}{1-\alpha} + \frac{2}{1-\alpha} = 0.$$

**(b)**  $H_{\text{HP}}(z) = \frac{1}{2} \{1 - A_1(z)\}$  where  $A_1(z) = \frac{-\alpha + z^{-1}}{1 - \alpha z^{-1}}$ .

Hence  $|H_{\text{HP}}(e^{j\omega})|^2 = \left( \frac{1+\alpha}{2} \right)^2 \frac{2(1-\cos(\omega))}{1-2\alpha\cos(\omega)+\alpha^2}$ .

$$\text{Now } \frac{\partial |H_{\text{HP}}(e^{j\omega})|^2}{\partial \alpha} = -\frac{1+\alpha}{2} \frac{2(1-\cos(\omega))}{1-2\alpha\cos(\omega)+\alpha^2} - \left( \frac{1+\alpha}{2} \right)^2 \frac{2(1-\cos(\omega))(-2\cos(\omega)+\alpha)}{1-2\alpha\cos(\omega)+\alpha^2}$$

$$\text{Thus, } \left. \frac{\partial |H_{\text{HP}}(e^{j\omega})|^2}{\partial \alpha} \right|_{\omega=\pi} = \frac{-2}{1+\alpha} + \frac{2}{1+\alpha} = 0.$$

**9.34 (a)**  $H_{\text{BP}}(z) = \frac{1}{2} \{1 - A_2(z)\}$  where  $A_2(z) = \frac{\alpha - \beta(1+\alpha)z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$ , with  $\alpha$  and  $\beta$  being real.

Therefore,  $|H_{\text{BP}}(e^{j\omega})|^2 = \left( \frac{1-\alpha}{2} \right)^2 \left( \frac{2(1-\cos(\omega))}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)} \right)$ . Now,

$$\begin{aligned} \frac{\partial |H_{\text{BP}}(e^{j\omega})|^2}{\partial \alpha} &= \left( \frac{-2}{1-\alpha} \right) \left( \frac{1-\alpha}{2} \right)^2 \left( \frac{2(1-\cos(\omega))}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)} \right) \\ &+ \left( \frac{1-\alpha}{2} \right)^2 \left( \frac{2(1-\cos(\omega))}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)} \right) \times \\ &\quad \left( \frac{2\beta^2(1+\alpha)+2\alpha+2\cos(2\omega)-4\beta(1+\alpha)\cos(\omega)}{1+\beta^2(1+\alpha)^2+\alpha^2+2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos(\omega)} \right). \end{aligned}$$

Using the fact  $\beta = \cos(\omega_0)$  we get  $\left. \frac{\partial |H_{BP}(e^{j\omega})|^2}{\partial \alpha} \right|_{\omega=\omega_0} = \frac{-2}{1-\alpha} - \frac{(2\beta^2 + 2\alpha - 2 - 2\alpha\beta^2)}{1 + \alpha^2 - 2\alpha + 2\alpha\beta^2 - \beta^2 - \alpha^2\beta^2}$

$$= \frac{-2}{1-\alpha} + \frac{2}{1-\alpha} = 0.$$

Similarly,  $\frac{\partial |H_{BP}(e^{j\omega})|^2}{\partial \beta} = \left( \frac{1-\alpha}{2} \right)^2 \left( \frac{2(1-\cos(\omega))}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos(\omega)} \right) \times$

$$\left( -\frac{2\beta(1+\alpha)^2 - 2(1+\alpha)^2\cos(\omega)}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos(\omega)} \right).$$

Again using the fact that  $\beta = \cos(\omega_0)$  it can be seen that  $\left. \frac{\partial |H_{BP}(e^{j\omega})|^2}{\partial \beta} \right|_{\omega=\omega_0} = 0.$

**(b)** For bandstop filters  $H_{BS}(z) = \frac{1}{2}\{1 + A_2(z)\}$  where  $A_2(z)$  is as given in (a). Thus

$$|H_{BS}(e^{j\omega})|^2 = \left( \frac{1+\alpha}{2} \right)^2 \left( \frac{4\beta^2 + 2 + 2\cos(2\omega) - 8\beta\cos(\omega)}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos(\omega)} \right). \text{ Thus,}$$

$$\frac{\partial |H_{BS}(e^{j\omega})|^2}{\partial \alpha} = \left( \frac{1+\alpha}{2} \right) \left( \frac{4\beta^2 + 2 + 2\cos(2\omega) - 8\beta\cos(\omega)}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos(\omega)} \right) - \left( \frac{1+\alpha}{2} \right)^2 \left( \frac{4\beta^2 + 2 + 2\cos(2\omega) - 8\beta\cos(\omega)}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos(\omega)} \right) \times \times \left( \frac{2(1+\alpha)\beta^2 + 2\alpha + 2\cos(2\omega) - 4\beta(1+\alpha)\cos(\omega)}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos(\omega)} \right).$$

Substituting  $\beta = \cos(\omega_0)$  it can be seen that

$$\left. \frac{\partial |H_{BS}(e^{j\omega})|^2}{\partial \alpha} \right|_{\omega=0} = \frac{2}{1-\alpha} - \left( \frac{1+\alpha}{2} \right)^2 \left( \frac{2(1+\alpha)(\beta+1)^2}{(1+\alpha)^2(\beta+1)^2} \right) = 0.$$

Similarly,  $\left. \frac{\partial |H_{BS}(e^{j\omega})|^2}{\partial \alpha} \right|_{\omega=\pi} = \left( \frac{2}{1-\alpha} \right) - \left( \frac{1+\alpha}{2} \right)^2 \left( \frac{2(1+\alpha)(\beta+1)^2}{(1+\alpha)^2(\beta+1)^2} \right) = 0.$

Now,  $\frac{\partial |H_{BS}(e^{j\omega})|^2}{\partial \beta} = \left( \frac{1+\alpha}{2} \right)^2 \left( \frac{8\beta - 8\cos(\omega)}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos(\omega)} \right) - \left( \frac{1+\alpha}{2} \right)^2 \left( \frac{4\beta^2 + 2 + 2\cos(2\omega) - 8\beta\cos(\omega)}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos(\omega)} \right) \times \left( \frac{(1+\alpha)^2 2(\beta - \cos(\omega))}{1 + \beta^2(1+\alpha)^2 + \alpha^2 + 2\alpha\cos(2\omega) - 2\beta(1+\alpha)^2\cos(\omega)} \right).$

Again substituting  $\beta = \cos(\omega_0)$  it can be seen that



$$\left. \frac{\partial |H_{BS}(e^{j\omega})|^2}{\partial \beta} \right|_{\omega=0} = 0 \text{ and also } \left. \frac{\partial |H_{BS}(e^{j\omega})|^2}{\partial \beta} \right|_{\omega=\pi} = 0.$$

**9.35** For a BR transfer function  $G(z)$  realized in a parallel allpass form, its power-complementary transfer function  $H(z)$  is also BR satisfying the condition  $|G(e^{j\omega})|^2 = 1 - |H(e^{j\omega})|^2$ . Let  $\omega = \omega_o$  be a frequency where  $|G(e^{j\omega})|$  is a maximum, i.e.  $|G(e^{j\omega_o})| = 1$ . Then, it follows that  $|H(e^{j\omega_o})| = 0$ . From the power-complementary condition it follows that

$$2|G(e^{j\omega})| \left. \frac{\partial |G(e^{j\omega})|}{\partial \omega} \right|_{\omega=\omega_o} = -2|H(e^{j\omega})| \left. \frac{\partial |H(e^{j\omega})|}{\partial \omega} \right|_{\omega=\omega_o}.$$

Therefore at  $\omega = \omega_o$ ,

$$\left. |G(e^{j\omega_o})| \frac{\partial |G(e^{j\omega})|}{\partial \omega} \right|_{\omega=\omega_o} = - \left. |H(e^{j\omega_o})| \frac{\partial |H(e^{j\omega})|}{\partial \omega} \right|_{\omega=\omega_o}, \text{ or } \left. \frac{\partial |G(e^{j\omega})|}{\partial \omega} \right|_{\omega=\omega_o} = 0 \text{ whether or not}$$

$\left. \frac{\partial |H(e^{j\omega})|}{\partial \omega} \right|_{\omega=\omega_o} = 0$ . Hence, lowpassband sensitivity of  $G(z)$  does not necessarily imply low stopband sensitivity of  $H(z)$ .

### 9.36 Without error feedback

The transfer function  $H(z)$  of the structure without feedback is given by

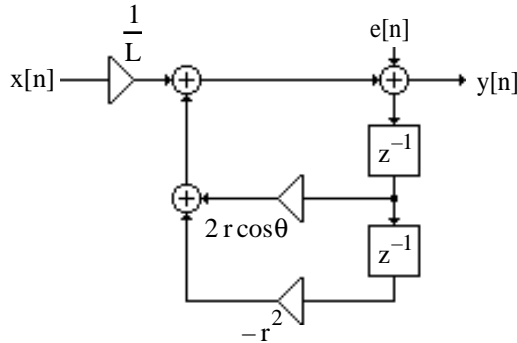
$$H(z) = \frac{1}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}} = \frac{1}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}}, \text{ where } r = 1 - \epsilon.$$

The corresponding impulse response  $h[n]$  is given by  $h[n] = \frac{r^n \sin(n+1)\theta}{\sin \theta} \cdot \mu[n]$ .

To keep  $y[n]$  from overflowing we must insert a multiplier of value  $\frac{1}{L}$  at the input where

$$L = \sum_{n=0}^{\infty} |h[n]|. \text{ From Eq. (9.163) we get } \frac{1}{(1-r)^2(1-2r \cos \theta + r^2)} \leq L^2 \leq \frac{16}{\pi^2(1-r)^2 \sin^2 \theta}. \quad (24)$$

The quantization noise model for  $H(z)$  is as shown below:



The output noise power is given by  $\sigma_n^2 = \sigma_e^2 \sum_{n=0}^{\infty} |h[n]|^2 = \frac{1+r^2}{1-r^2} \cdot \frac{1}{r^4 - 2r^2 \cos \theta + 1} \cdot \sigma_e^2$

The output signal power, assuming an input signal of variance  $\sigma_x^2$  is given by

$\sigma_y^2 = \frac{\sigma_x^2}{L^2} \sum_{n=0}^{\infty} |h[n]|^2$ . Hence, the SNR is given by  $\text{SNR} = \frac{\sigma_y^2}{\sigma_n^2} = \frac{\sigma_x^2}{L^2 \sigma_e^2}$ . For a (b+1)-bit signed representation,  $\sigma_e^2 = \frac{2^{-2b}}{12}$ . Hence,  $\text{SNR} = \frac{S}{N} = \frac{12 \sigma_x^2}{L^2 2^{-2b}}$ . Therefore, from the inequality of Eq.

$$(24) \text{ we get } \frac{2^{-2b}}{12 \sigma_x^2 (1-r)^2 (1-2r \cos \theta + r^2)} \leq \frac{N}{S} \leq \frac{16 \times 2^{-2b}}{12 \sigma_x^2 \pi^2 (1-r)^2 \sin^2 \theta}.$$

(a) For an WSS uniformly distributed input between  $[-1, 1]$ ,  $\sigma_x^2 = \frac{1}{3}$ . Hence,

$$\frac{2^{-2b}}{4(1-r)^2 (1-2r \cos \theta + r^2)} \leq \frac{N}{S} \leq \frac{4}{\pi^2} \frac{2^{-2b}}{(1-r)^2 \sin^2 \theta}.$$

If  $\varepsilon \rightarrow 0$ , and  $\theta \rightarrow 0$ , then  $(1-r)^2 \rightarrow \varepsilon^2$ ,  $\cos 2\theta = 1 - 2 \sin^2 \theta \cong 1 - 2\theta^2$ , and  $\sin^2 \theta \cong \theta^2$ .

In this case we have  $\frac{2^{-2b}}{4\varepsilon^2(\varepsilon^2 + 4\theta^2)} \leq \frac{N}{S} \leq \frac{2^{-2b}}{\pi^2 \varepsilon^2 \theta^2}$ .

(b) For an input with a Gaussian distribution between  $[-1, 1]$ ,  $\sigma_x^2 = \frac{1}{3}$ .

In this case we have  $3 \frac{2^{-2b}}{4\varepsilon^2(\varepsilon^2 + 4\theta^2)} \leq \frac{N}{S} \leq \frac{3}{\pi^2} \frac{2^{-2b}}{\varepsilon^2 \theta^2}$ .

(c) For a sinusoidal input between  $[-1, 1]$  of known frequency  $\omega_o$ ,  $\sigma_x^2 = \frac{1}{2}$ .

The output noise variance here is therefore  $\sigma_n^2 = \sigma_e^2 \sum_{n=0}^{\infty} |h[n]|^2 = \frac{2^{-2b}}{12} \cdot \frac{1+r^2}{1-r^2} \cdot \frac{1}{r^4 - 2r^2 \cos 2\theta + 1}$ .

Thus,  $\frac{N}{S} = \frac{1+r^2}{1-r^2} \cdot \frac{1}{6(r^4 - 2r^2 \cos 2\theta + 1)} = \frac{2^{-2b}}{24\varepsilon(\varepsilon^2 + \theta^2)} \cong \frac{2^{-2b}}{24\varepsilon\theta^2}$ .

With error feedback

$$G(z) = \frac{1 - 2z^{-1} + z^{-1}}{1 - 2r \cos \theta z^{-1} + r^2 z^{-1}} = \frac{z^2 - 2z + 1}{z^2 - 2r \cos \theta z + r^2} = 1 + \frac{2(r \cos \theta - 1)z + 1 - r^2}{z^2 - 2r \cos \theta z + r^2}.$$

$$\text{Thus, } \|G\|^2 = 1 + \frac{(4(1-r \cos \theta)^2 + (1-r^2)^2)(1-r^4) + 8r \cos \theta (1-r^2)^2 (r \cos \theta - 1)}{(1-r^2)^2 + 2r^2(4r^2 \cos^2 \theta) - 4(1+r^4)r^2 \cos^2 \theta}$$

For  $r = 1 - \varepsilon$  with  $\varepsilon \rightarrow 0$ , and  $\theta \rightarrow 0$ , we get after some manipulation  $\|G\|^2 = 1 + \frac{\theta^4}{4\varepsilon(\varepsilon^2 + \theta^2)}$ ,

and  $\sigma_n^2 = \sigma_e^2 \|G\|^2$ .

Now  $L^2$  remains the same as before since it depends only upon the denominator. Also the overall transfer function of the structure remains the same as before. The output noise power

with error feedback is thus  $\hat{N} = \sigma_e^2 \|G\|^2$ , whereas, the output noise power without error feedback is  $N = \sigma_e^2 \|H\|^2$ . Hence,  $\frac{\hat{N}}{N} = \frac{\|G\|^2}{\|H\|^2}$ .

Now,  $\|G\|^2 = 1 + \frac{\theta^4}{4\epsilon(\epsilon^2 + \theta^2)}$ , Since  $\theta \gg \epsilon$ ,  $\|G\|^2 \cong \frac{\theta^4}{4\epsilon\theta^2} = \frac{\theta^2}{4\epsilon}$ .

Also,  $\|H\|^2 = \frac{1+r^2}{1-r^2} \cdot \frac{1}{r^4 - 2r^2 \cos^2 \theta + 1} = \frac{1+(1-\epsilon)^2}{1-(1-\epsilon)^2} \cdot \frac{1}{(1-\epsilon)^4 - 2(1-\epsilon)^2(1-2\theta^2) + 1}$   
 $\approx \frac{1}{4\epsilon(\theta^2 + \epsilon^2)} \approx \frac{1}{4\epsilon\theta^2}$ .

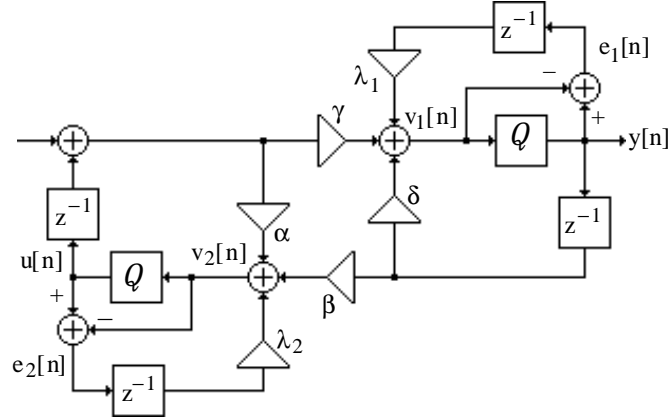
Thus,  $\frac{\hat{N}}{N} \cong \theta^4$ . As a result, with error feedback, the  $\frac{N}{S}$  ratio gets multiplied by  $\theta^4$ .

(a) input with uniform density:  $\frac{2^{-2b}\theta^2}{16\epsilon^2} \leq \frac{N}{S} \leq \frac{\theta^2}{\pi^2} \frac{2^{-2b}}{\epsilon^2}$

(b) Wide-sense stationary, Gaussian density, white:  $\frac{2^{-2b}3\theta^2}{16\epsilon^2} \leq \frac{N}{S} \leq \frac{3\theta^2}{\pi^2} \frac{2^{-2b}}{\epsilon^2}$

(c) sinusoid with known frequency:  $\frac{N}{S} = \frac{2^{-2b}\theta^2}{24\epsilon}$

9.37 The coupled form with error feedback is shown below:



Analysis yields  $V_1(z) = \lambda_1 z^{-1} E_1(z) + \delta z^{-1} Y(z) + \gamma z^{-1} U(z)$ ,  $E_1(z) = Y(z) - V_1(z)$ ,  
 $V_2(z) = \lambda_2 z^{-1} E_2(z) + \beta z^{-1} Y(z) + \alpha z^{-1} U(z)$ , and  $E_2(z) = Y(z) - V_2(z)$ . Eliminating  
 $U_1(z)$ ,  $V_1(z)$ ,  $U_2(z)$ , and  $V_2(z)$  from these equations we arrive at the noise transfer functions

$$G_1(z) = \left. \frac{Y(z)}{E_1(z)} \right|_{E_2(z)=0} = \frac{(1 - \alpha z^{-1})(1 + \lambda_1 z^{-1})}{1 - (\alpha + \delta)z^{-1} + (\alpha\delta - \beta\gamma)z^{-2}}, \text{ and}$$

$$G_2(z) = \left. \frac{Y(z)}{E_2(z)} \right|_{E_1(z)=0} = \frac{\gamma z^{-1}(1 + \lambda_2 z^{-1})}{1 - (\alpha + \delta)z^{-1} + (\alpha\delta - \beta\gamma)z^{-2}}. \text{ The total output noise}$$

power is thus given by  $\sigma_o^2 = \|G_1\|_2^2 \sigma_{e1}^2 + \|G_2\|_2^2 \sigma_{e2}^2$ . Hence, the output noise power, for a lowpass filter design, can be reduced by placing zeros of the noise transfer functions in the passband. For each of the noise transfer functions given above, we can only place a zero at  $z = 1$  by choosing  $\lambda_i = -1$ ,  $i = 1, 2$ ,

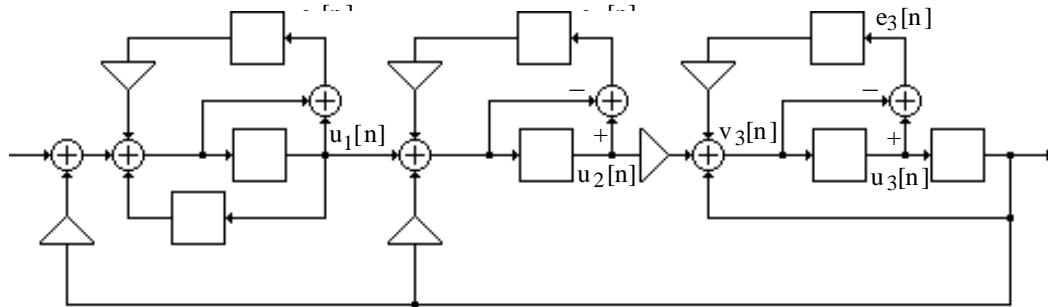
Using the notations  $b = -(\alpha + \delta)$  and  $d = \alpha\delta - \beta\gamma$  we rewrite the noise transfer functions as

$$G_1(z) = \frac{z^2 - (1 + \alpha)z + \alpha}{z^2 + bz + d} = 1 + \frac{-(1 + \alpha + b) + (\alpha - d)}{z^2 + bz + d}, \text{ and } G_2(z) = \frac{\gamma z - \gamma}{z^2 + bz + d}.$$

Using Table 9.4 we obtain  $\|G_1\|_2^2 = 1 + \frac{[(1 + \alpha + b)^2 + (\alpha - d)^2](1 - d^2) + 2(1 + \alpha + b)(\alpha - d)(1 - d)}{(1 - d^2)^2 + 2db^2 - (1 + d^2)b^2}$ , and

$$\|G_2\|_2^2 = \frac{2\gamma^2(1 - d^2)}{(1 - d^2)^2 + 2db^2 - (1 + d^2)b^2}.$$

**9.38** The Kingsbury structure with error feedback is shown below:



Analysis yields:  $V_1(z) = k_1 Y(z) + z^{-1} U_1(z) + \lambda_1 z^{-1} E_1(z)$ ,  $E_1(z) = U_1(z) - V_1(z)$ ,  
 $V_2(z) = k_2 Y(z) + U_1(z) + \lambda_2 z^{-1} E_2(z)$ ,  $E_2(z) = U_2(z) - V_2(z)$ ,  
 $V_3(z) = Y(z) - k_1 U_2(z) + \lambda_3 z^{-1} E_3(z)$ ,  $E_3(z) = U_3(z) - V_3(z)$ , and  $Y(z) = z^{-1} U_1(z)$ . Eliminating  $U_1(z)$ ,  $V_1(z)$ ,  $U_2(z)$ ,  $V_2(z)$ ,  $U_3(z)$  and  $V_3(z)$  from these equations we get

$$G_1(z) = \left. \frac{Y(z)}{E_1(z)} \right|_{E_2(z)=E_3(z)=0} = \frac{-k_1 z^{-1} (1 + \lambda_1 z^{-1})}{1 - [2 - k_1(k_1 + k_2)]z^{-1} + (1 - k_1 k_2)z^{-2}},$$

$$G_2(z) = \left. \frac{Y(z)}{E_2(z)} \right|_{E_1(z)=E_3(z)=0} = \frac{-k_1 z^{-1} (1 - z^{-1})(1 + \lambda_2 z^{-1})}{1 - [2 - k_1(k_1 + k_2)]z^{-1} + (1 - k_1 k_2)z^{-2}}, \text{ and}$$

$$G_3(z) = \left. \frac{Y(z)}{E_3(z)} \right|_{E_1(z)=E_2(z)=0} = \frac{z^{-1} (1 - z^{-1})(1 + \lambda_3 z^{-1})}{1 - [2 - k_1(k_1 + k_2)]z^{-1} + (1 - k_1 k_2)z^{-2}}. \text{ The total output noise}$$

power is thus given by  $\sigma_o^2 = \|G_1\|_2^2 \sigma_{e1}^2 + \|G_2\|_2^2 \sigma_{e2}^2 + \|G_3\|_2^2 \sigma_{e3}^2$ . Hence, the output noise power, for a lowpass filter design, can be reduced by placing zeros of the noise transfer functions in the passband. For each of the noise transfer functions given above, we can only place a zero at  $z = 1$  by choosing  $\lambda_i = -1$ ,  $i = 1, 2, 3$ , in which case, using the notations  $b = -[2 - k_1(k_1 + k_2)]$ ,

and  $d = 1 - k_1 k_2$ . the noise transfer functions reduce to  $G_1(z) = \frac{-k_1 z^{-1} (1 - z^{-1})}{1 + b z^{-1} + d z^{-2}} = \frac{-k_1 z + k_1}{z^2 + b z + d}$ ,

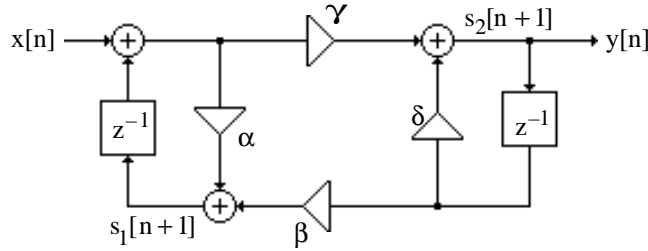
$$G_2(z) = \frac{-k_1 z^{-1}(1-z^{-1})}{1+bz^{-1}+dz^{-2}} = \frac{-k_1(z^2-2z+1)}{z(z^2+bz+d)} = \frac{-k_1 A}{z} + \frac{-k_1(Cz+D)}{z^2+bz+d} \text{ and}$$

$$G_3(z) = \frac{z^{-1}(1-z^{-1})^2}{1+bz^{-1}+dz^{-2}} = \frac{z^2-2z+1}{z(z^2+bz+d)} = \frac{A}{z} + \frac{Cz+D}{z^2+bz+d} \text{ where } A = \frac{1}{d}, C = 1 - \frac{1}{d}, \text{ and}$$

$$D = -\left(2 + \frac{b}{d}\right). \text{ Using Table 9_4 we then obtain } \|G_1\|_2^2 = k_1^2 \frac{2(1-d^2) - 2(1-d)b}{(1-d^2)^2 + 2db^2 - (1+d^2)b^2},$$

$$\|G_3\|_2^2 = A^2 + 2BC + \frac{(C^2 + D^2)(1-d^2) - 2CD(1-d)b}{(1-d^2)^2 + 2db^2 - (1+d^2)b^2}, \text{ and } \|G_2\|_2^2 = k_1^2 \|G_3\|_2^2.$$

**9.39** The transfer function of the coupled-form structure shown below is given by Eq. (9.42) where  $\alpha = \delta = r \cos(\theta)$  and  $\beta = -\gamma = r \sin(\theta)$ . Analysis yields



$s_1[n+1] = \alpha(x[n] + s_1[n]) + \beta s_2[n]$  and  $s_2[n+1] = \gamma(x[n] + s_1[n]) + \delta s_2[n]$ . Rewriting these

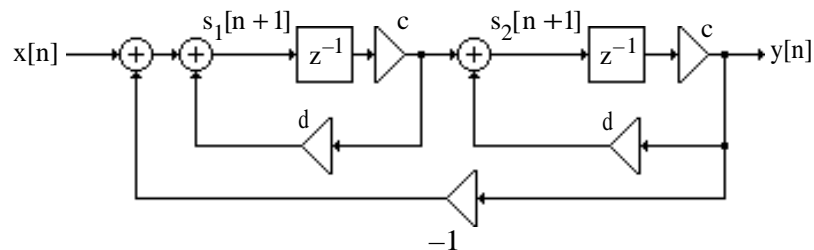
equations in matrix form we get  $\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} x[n]$

Thus  $\mathbf{A} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ . Therefore,  $\mathbf{A}^T \mathbf{A} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha^2 + \gamma^2 & \alpha\beta + \gamma\delta \\ \alpha\beta + \gamma\delta & \beta^2 + \delta^2 \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix}$ .

Likewise,  $\mathbf{A} \mathbf{A}^T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} = \begin{bmatrix} \alpha^2 + \beta^2 & \alpha\gamma + \beta\delta \\ \alpha\gamma + \beta\delta & \gamma^2 + \delta^2 \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \end{bmatrix}$ .

Thus  $\mathbf{A}$  is of normal form and hence, the structure will not support limit cycles.

**9.40** The transfer function of the modified coupled-form structure shown below is given by



$s_1[n+1] = cd s_1[n] - cs_2[n] + x[n]$ ,  $s_2[n+1] = cs_1[n] + cds_2[n]$ , and  $y[n] = cs_2[n]$ .

In matrix form these equations can be written as

$\begin{bmatrix} s_1[n+1] \\ s_2[n+1] \end{bmatrix} = \begin{bmatrix} cd & -c \\ c & cd \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x[n]$ , and  $y[n] = \begin{bmatrix} 0 & c \end{bmatrix} \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} x[n]$ . Thus,



$$\begin{aligned}
&= -\mathbf{s}^T[n] \cdot \mathbf{D} \cdot \mathbf{s}[n] + \mathbf{s}^T[n+1] \cdot \mathbf{A}^T \cdot \mathbf{D} \cdot \mathbf{A} \cdot \mathbf{s}[n+1] - \sum_{k=1}^N \left( v_k^2[n+1] - \mathcal{Q}(v_k[n+1])^2 \right) d_{kk} \\
&= -\mathbf{s}^T[n] (\mathbf{D} - \mathbf{A}^T \cdot \mathbf{D} \cdot \mathbf{A}) \mathbf{s}[n] - \sum_{k=1}^N \left( v_k^2[n+1] - \mathcal{Q}(v_k[n+1])^2 \right) d_{kk}.
\end{aligned}$$

Now if  $\mathbf{s}^T[n] (\mathbf{D} - \mathbf{A}^T \mathbf{D} \mathbf{A}) \mathbf{s}[n] \geq 0$ , and  $v_k[n]$  are quantized such that  $|\mathcal{Q}(v_k[n+1])| \leq |v_k[n+1]|$ , (i.e. using a passive quantizer), then the power stored in the delays will not increase with increasing  $n$ . So under passive quantization, limit cycles will be eliminated if  $\mathbf{s}^T[n] (\mathbf{D} - \mathbf{A}^T \mathbf{D} \mathbf{A}) \mathbf{s}[n] \geq 0$ , or  $\mathbf{D} - \mathbf{A}^T \mathbf{D} \mathbf{A}$  is positive-definite. For second order stable IIR filters, eigen values of  $\mathbf{A} = [a_{ij}]$  are less than 1. This condition is satisfied if

$$a_{12}a_{21} \geq 0,$$

or,  $a_{12}a_{21} < 0$  and  $|a_{11} - a_{22}| + \det(\mathbf{A}) \leq 1$ .

For the given structure,

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & \alpha_1 + 1 \\ \alpha_2 - 1 & \alpha_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \alpha_1 - \beta_1 \\ \alpha_2 - \beta_2 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 1] \quad \text{and} \quad \mathbf{D} = 1.$$

The transfer function of the filter is given by

$$H(z) = \frac{z^2 - (\beta_1 + \beta_2)z + (1 + \beta_1 - \beta_2)}{z^2 - (\alpha_1 + \alpha_2)z + (1 + \alpha_1 - \alpha_2)}.$$

The filter is stable if  $1 + \alpha_1 - \alpha_2 < 1$ , and  $|\alpha_1 + \alpha_2| < 1 + (1 + \alpha_1 - \alpha_2)$ , or equivalently, if  $\alpha_1 - \alpha_2 < 0$ , and  $|\alpha_1 + \alpha_2| < 2 + \alpha_1 - \alpha_2$ .

Now,  $a_1a_2 = (\alpha_1 + 1)(\alpha_2 - 1) = \frac{1}{4}[(\alpha_1 + \alpha_2)^2 - (2 + \alpha_1 - \alpha_2)^2]$ , and since,  $(\alpha_1 + \alpha_2)^2 < (2 + \alpha_1 - \alpha_2)^2$ ,  $a_{12}a_{21} < 0$ .

$$\begin{aligned}
\text{Next, } |a_{11} - a_{22}| + \det(\mathbf{A}) &= |\alpha_1 - \alpha_2| + \alpha_1\alpha_2 - (\alpha_1 + 1)(\alpha_2 - 1) \\
&= -(\alpha_1 - \alpha_2) + \alpha_1\alpha_2 - (\alpha_1\alpha_2 + \alpha_2 - \alpha_1 - 1) = 1.
\end{aligned}$$

Hence, the structure does not support zero-input limit cycles.

**9.42** From Section 8.3.1 we know that each computation of a DFT sample requires  $2N + 4$  real multiplications. Assuming that quantization noise generated from each multiplier is independent

of the noise generated from other multipliers, we get  $\sigma_r^2 = (2N + 4)\sigma_0^2 = \frac{2^{-2b}(N+2)}{6}$ .

**9.43**  $\text{SNR} = \frac{2^{2b}}{N^2}$ . Hence, an SNR of 25 dB implies  $\frac{2^{2b}}{N^2} = 10^{2.5}$  or

$b = \frac{1}{2} \log_2 \left( (10)^{2.5} \times (512)^2 \right) = 13.1524$ . Therefore  $b = 14$  bits should be chosen to get an SNR of 25 dB. Therefore number of bits required for each sample =  $14 + 1 = 15$ .

**9.44** Let  $N = 2^v$ . Consider the  $m^{\text{th}}$  stage. The output sees  $4(2^{v-m})$  noise sources from the  $m^{\text{th}}$  stage.

Each noise source has a variance reduction by a factor of  $\left(\frac{1}{4}\right)^{v-m}$  due to multiplication by  $\frac{1}{2}$  at each stage till the output. Hence the total noise variance at the output due to the noises injected in the  $m^{\text{th}}$  stage is  $4(2^{v-m})(2^{-2(v-m)})\sigma_0^2$ .

Therefore the total noise variance at the output  $= \sigma^2 = \sum_{m=1}^v 4(2^{v-m})(2^{-2(v-m)})\sigma_0^2$

$$4\sigma_0^2 2^{-v} \sum_{m=1}^v 2^m = 4 \frac{2^{-2b}}{12} 2^{-v} \frac{2(2^v - 1)}{2 - 1} = \frac{2}{3} 2^{-2b} (1 - 2^{-v}) \approx \frac{2}{3} 2^{-2b} \text{ for large } N.$$

**9.45**  $\text{SNR} = \frac{2^{2b}}{2N}$ . Hence  $b = \frac{1}{2} \log_2((10)^{2.5} \times 2 \times 512) = 8.652$ .

Hence we choose  $b + 1 = 10$  bits per sample to get an SNR of 25 dB.

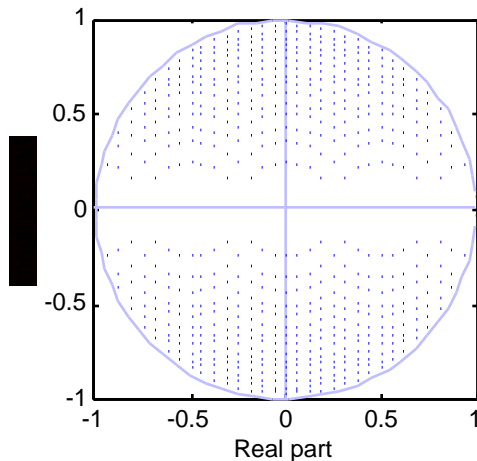
```

M9.1 % This is the function to obtain the pole distribution
% plot of a second order transfer function with a denominator
% of the form Z*Z - KZ + L. For stability 0 < L < 1 and
% abs(K) < 1+L so the range of K is (-2,2).
% for a b-bit wordlength, 1 bit is reserved for the sign
% as the coefficients are sorted in the sign magnitude form.
% Both L and the K are quantized to b-1 bits.
%
function []=pole_plot(bits);
%
% This part prints the unit circle for reference
%
z = [0 1];
p = [0 1];
zplane(z,p);
hold;
axis([-1 1 -1 1]);
%
% One bit is kept for the sign so effectively for the
quantization
% the remaining number of bits is bits-1
%
bits = bits-1;
%
% The quantization step.
%
step = power(2,-bits);
for index_1 = 0:1:(power(2,bits)-1)
L = index_1*step;
for index_2 = -(power(2,bits)-1):1:(power(2,bits)-1)
K = 2*index_2*step;
p1 = (-K+(sqrt(K*K-4*L)))/2;
p2 = (-K-(sqrt(K*K-4*L)))/2;
if abs(p1) < 1
if (imag(p1) ~= 0)
plot(p1, '.');
plot (p2, '.');
end
end
end
end
hold;

```

**M9.2** The pole-distribution plot of the second-order Direct Form structure is obtained by running the MATLAB program given in Exercise M9.1. Running this program using the statement `pole_plot(5)` yields the following plot.





For plotting the pole distribution of the Kingsbury structure, we use the following MATLAB program:

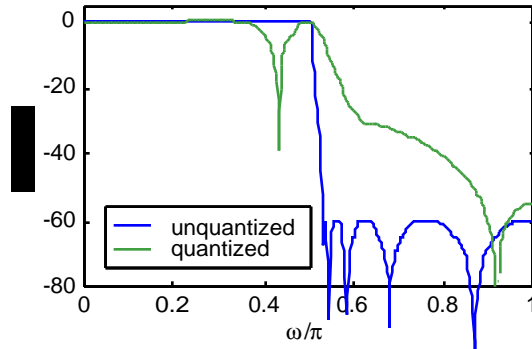
```

% This is the function to obtain the pole distribution plot
% of the Kingsbury
% structure. Here, the transfer function has a denominator
% of the form
%  $Z^2 - (2 - k_1 k_2 - k_1 k_1)Z + (1 - k_1 k_2)$  where the  $k_1$  and
%  $k_2$  are the
% multiplier coefficients of the structure.
% For stability of the structure the range of  $k_1$  is  $(-2,2)$ 
% and the range
% of  $k_2$  is such that  $0 < k_1 k_2 < 2$ ; Hence  $k_2$  can take any
% value.
% In this program the range of  $k_1$  is taken to be  $(-2,2)$  and
% the range of
%  $k_2$  is also  $(-2,2)$ . The wordlength is specified by the
% argument "bits"
% of the function.
%
function[] = pole_plot_kb(bits);
%
% This part prints the unit circle for reference
%
z = [0 1];
p = [0 1];
zplane(z,p);
hold;
axis([-1 1 -1 1]);
%
% The numbers are stored in the sign-magnitude form so one
% bit is
% reserved for the sign. Hence for the quantization, the
% available bits
% = bits - 1
%
bits = bits-1;
step = power(2,-bits);
% This part of the program finds out the various possible
% combinations

```



### M9.3



```

M9.4 N = 5; wn = 0.45; Rp = 0.5; Rs = 45;
[B,A] = ellip(N,Rp,Rs,wn);
zplane(B,A);
z = cplxpair(roots(B)); p = cplxpair(roots(A));
disp('Factors for the numerator');
const = B(1)/A(1);
k = 1;
while k <= length(z),
    if(imag(z(k)) ~=0)
        factor = [1 -2*real(z(k)) abs(z(k))^2]
        k = k+2;
    else
        factor = [1 -z(k)]
        k = k+1;
    end
end
disp('Factors for the denominator');
k = 1;
while k <= length(p),
    if(imag(p(k)) ~=0)
        factor = [1 -2*real(p(k)) abs(p(k))^2]
        k = k+2;
    else
        factor = [1 -p(k)]
        k = k+1;
    end
end
sos = zp2sos(z,p,const)

```

The above program yields

Factors for the numerator

```

factor =
    1.0000e+00    1.0358e+00    1.0000e+00

factor =
    1.0000e+00    3.7748e-01    1.0000e+00

factor =
    1.0000e+00    1.0000e+00

```

Factors for the denominator

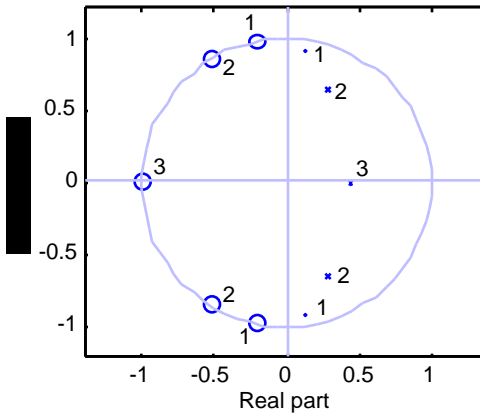
```
factor =
  1.0000e+00  -2.6433e-01  8.6522e-01
```

```
factor =
  1.0000e+00  -5.8090e-01  5.0030e-01
```

```
factor =
  1.0000e+00  -4.4817e-01
```

$$\text{Hence } H(z) = \left( \frac{1 + 1.0358z^{-1} + z^{-2}}{1 - 0.26433z^{-1} + 0.86522z^{-2}} \right) \left( \frac{1 + 0.37748z^{-1} + z^{-2}}{1 - 0.5809z^{-1} + 0.5003z^{-2}} \right) \left( \frac{1 + z^{-1}}{1 - 0.44817z^{-1}} \right).$$

Note that the ordering has no effect if  $L_\infty$ -scaling is used.



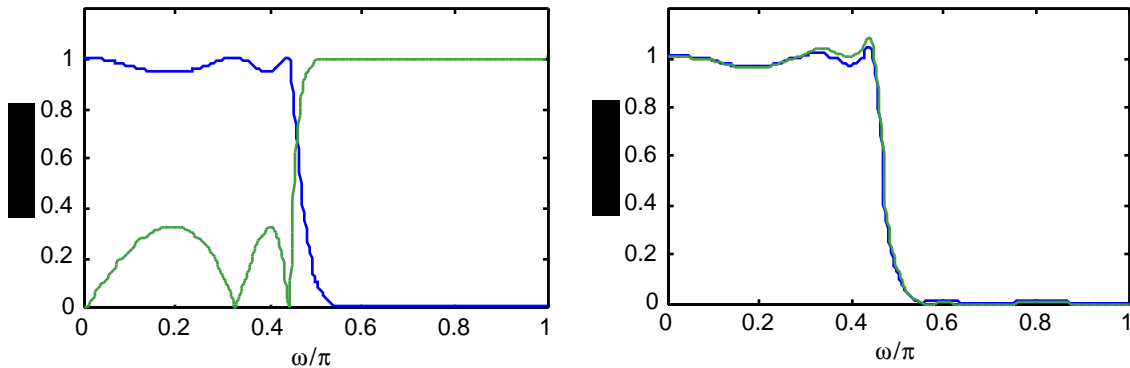
```
sos =
  2.7591e-01  2.7591e-01  0  1.0000e+00  -4.4817e-01  0
  3.0285e-01  3.1370e-01  3.0285e-01  1.0000e+00  -5.8090e-01
  5.0030e-01
  6.7335e-01  2.5418e-01  6.7335e-01  1.0000e+00  -2.6433e-01
  8.6522e-01
```

```
M9.5 [B,A] = ellip(5,0.5,45,0.45);
p = roots(A);
lenp = length(p);
[Y,I] = sort(angle(p));
for k = 1:lenp
    if(rem(k,2)==1)
        p1((k+1)/2) = p(I(k));
    else
        p2(k/2) = p(I(k));
    end
end
b1 = poly(p1); b2 = poly(p2);
a1 = fliplr(b1); a2 = fliplr(b2);
B1 = 0.5*(conv(b2,a1)-conv(b1,a2));
A1 = conv(b1,b2);
[H,W] = freqz(B,A,512); [H1,W] = freqz(B1,A1,512);
plot(W/pi,abs(H),'-',W/pi,abs(H1),'--');axis([0 1 0 1.2]);
xlabel('\omega/\pi');ylabel('Magnitude');
```

```

pause
a1 = 1.01*a1; a2 = 1.01*a2;
b1 = [b1(1) 1.01*b1(2:length(b1))];
b2 = [b2(1) 1.01*b2(2:length(b2))];
A3 = conv(b1,b2);
B3 = 0.5*conv(a1,b2)+0.5*conv(a2,b1);
B4 = 1.01*B;
A4 = [A(1) 1.01*A(2:length(B))];
[H2,W] = freqz(B3,A3,512); [H3,W] = freqz(B4,A4,512);
plot(W/pi,abs(H2),'-',W/pi,abs(H3),'--');axis([0 1 0 1.2]);
xlabel('\omega/\pi');ylabel('Magnitude');

```

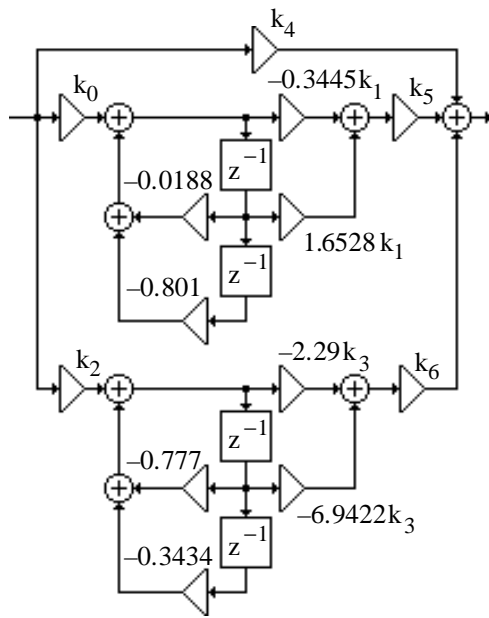


```

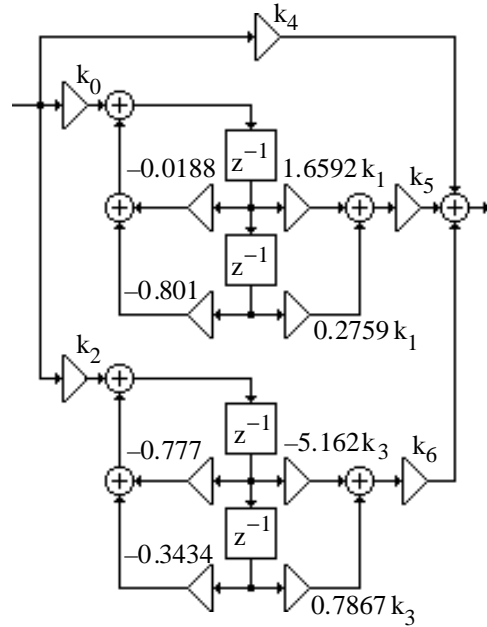
M9.6 num1 = input('The first factor of numerator = ');
num2 = input('The second factor of numerator = ');
den1 = input('The first factor of denominator = ');
den2 = input('The second factor of denominator = ');
% The numerator and denominator of the scaling
% functions f1 and f2 are
f1num = 1; f1den = [den1]; f2num = num1;
f2den = conv(den1,den2); f3num = conv(num1,num2);
f3den = conv(den1,den2);
x = [1 zeros(1,511)];
% Sufficient length for impulse response
% to have decayed to nearly zero
f1 = filter(f1num,f1den,x); f2 = filter(f2num,f2den,x);
f3 = filter(f3num,f3den,x);
k1 = sqrt(sum(f1.*f1)); k2 = sqrt(sum(f2.*f2));
k3 = sqrt(sum(f3.*f3));
disp('The first scaling factor ='); disp(k1);
disp('The second scaling factor ='); disp(k2);
disp('The third scaling factor ='); disp(k3);
% The noise transfer functions
g1num = conv(num1,num2)/(k2*k3);
g1den = conv(den1,den2)/k3;
g2num = num2; g2den = den2;
g1 = filter(g1num,g1den,x); g2 = filter(g2num,g2den,x);
var = sum(f1.*f1)*3+sum(g2.*g2)*5+3;
disp('The normalized noise variance'); disp(var);
% num1 and num2 can be interchanged to come up with the
% second realization

```

**M9.7** The parallel form I structure and the parallel form II structure used for simulation are shown below:



Parallel Form I



Parallel Form II

```

num1 = input('The first factor in the numerator =');
num2 = input('The second factor in the numerator =');
den1 = input('The first factor in the denominator =');
den2 = input('The second factor in the denominator =');
num = conv(num1,num2);
den = conv(den1,den2);
[r1,p1,k11] = residuez(num,den);
[r2,p2,k21] = residue(num,den);
% Simulation of structure for Eq. 9.244
R1 = [r1(1) r1(2)];P1 = [p1(1) p1(2)];
R2 = [r1(3) r1(4)];P2 = [p1(3) p1(4)];
R3 = [r2(1) r2(2)];P3 = [p2(1) p2(2)];
R4 = [r2(3) r2(4)];P4 = [p2(3) p2(4)];
[num11,den11] = residuez(R1,P1,0);
[num12,den12] = residuez(R2,P2,0);
[num21,den21] = residue(R3,P3,0);
[num22,den22] = residue(R4,P4,0);
disp('The numerators for Parallel Form I');
disp(k11); disp(num11); disp(num12);
disp('The denominators for Parallel Form I');
disp(den11); disp(den12);
disp('The numerators for Parallel Form II');
disp(k21); disp(num21); disp(num22);
disp('The denominators for Parallel Form II');
disp(den21); disp(den22);
imp = [1 zeros(1,2000)];
y0 = filter([1 0 0],den11,imp); y1 = filter(num11,den11,imp);
y2 = filter([1 0 0],den12,imp); y3 = filter(num12,den12,imp);
gamma0 = sum(y0.*conj(y0)); gamma1 = sum(y1.*conj(y1));
gamma2 = sum(y2.*conj(y2)); gamma3 = sum(y3.*conj(y3));
k0 = sqrt(1/gamma0); k1 = sqrt(gamma0/gamma1);
k2 = sqrt(1/gamma2); k3 = sqrt(gamma2/gamma3);
y = filter(num,den,imp);
gamma = sum(y.*conj(y));

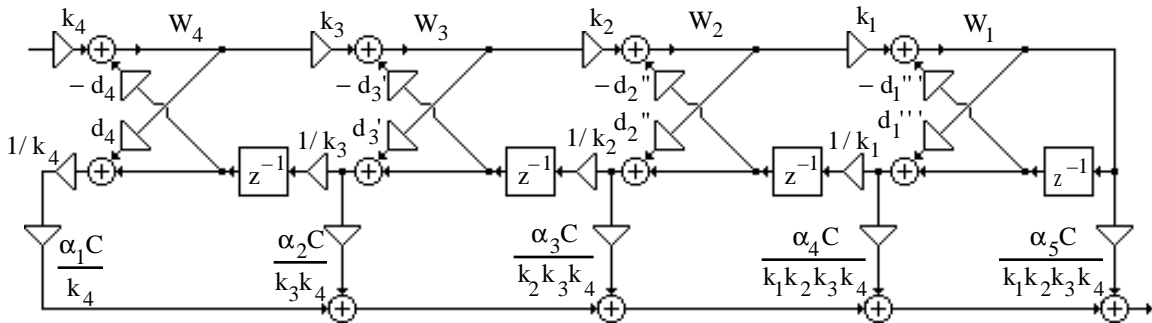
```

```

k4 = sqrt(1/gamma); k5 = k4/(k0*k1); k6 = k4/(k2*k3);
disp('For parallel form I');
disp('The scaling constants are'); disp(k0);disp(k1);
disp(k2);disp(k3);disp(k4);disp(k5);disp(k6);
disp('The product roundoff noise variance');
noise = 3*(k5/k0)^2+3*(k6/k2)^2+2*k5^2+2*k6^2+3;
disp(noise);
%%%%%%%%%%For Parallel Form II %%%%%%%%%%%
y0 = filter([0 0 1],den21,imp);
y1 = filter(fliplr(num21),den21,imp);
y2 = filter([0 0 1],den22,imp);
y3 = filter(fliplr(num22),den22,imp);
gamma0 = sum(y0.*conj(y0)); gamma1 = sum(y1.*conj(y1));
gamma2 = sum(y2.*conj(y2)); gamma3 = sum(y3.*conj(y3));
k0 = sqrt(1/gamma0); k1 = sqrt(gamma0/gamma1);
k2 = sqrt(1/gamma2); k3 = sqrt(gamma2/gamma3);
y = filter(num,den,imp);
gamma = sum(y.*conj(y));
k4 = sqrt(1/gamma); k5 = k4/(k0*k1); k6 = k4/(k2*k3);
disp('For parallel form II');
disp('The scaling constants are');disp(k0);disp(k1);disp(k2);
disp(k3);disp(k4);disp(k5);disp(k6);
disp('The product roundoff noise variance');
noise = 3*(k5/k0)^2+3*(k6/k2)^2+2*k5^2+2*k6^2+3; disp(noise);

```

**M9.8** The scaled Gray-Markel cascaded lattice structure used for simulation is shown below:



```

% Use Program_6_3.m pg. 384 to generate the lattice
% parameters and feedforward multipliers
d = [0.19149189348920 0.75953417365130 0.44348979846264
0.27506340000000];
alpha = [1.00000000000000 -3.50277000000000
4.61511974525466 -1.70693992124303 -0.90009664306164];
imp = [1 zeros([1,499])];
qold1 = 0;
for k = 1:500
    w1 = imp(k)-d(1)*qold1;
    y1(k) = w1;
    qnew1 = w1;
    qold1 = qnew1;
end
k1 = sqrt(1/(sum(y1.*conj(y1))));
imp = [1 zeros([1,499])];
qold1 = 0;qold2 = 0;
for k = 1:500
    w2 = imp(k)-d(2)*qold2*1/k1;

```

```

        w1 = k1*w2-d(1)*qold1;
        y1(k) = w1;y2(k)=w2;
        qnew1 = w1;
        qnew2 = w1*d(1)+qold1;
        qold1 = qnew1;qold2 = qnew2;
    end
    k2 = sqrt(1/(sum(y2.*conj(y2))));
    qold1 = 0;qold2 = 0;qold3 = 0;
    for k = 1:500
        w3 = imp(k)-d(3)*qold3*1/k2;
        w2 = k2*w3-d(2)*qold2*1/k1;
        w1 = k1*w2-d(1)*qold1;
        y3(k) = w3;
        qnew1 = w1;
        qnew2 = w1*d(1)+qold1;
        qnew3 = w2*d(2)+qold2*1/k1;
        qold1 = qnew1;qold2 = qnew2;qold3 = qnew3;
    end
    k3 = sqrt(1/sum(y3.*conj(y3)));
    qold1 = 0;qold2 = 0;qold3 = 0;qold4 = 0;
    for k = 1:500
        w4 = imp(k)-d(4)*qold4/k3;
        w3 = k3*w4-d(3)*qold3/k2;
        w2 = k2*w3-d(2)*qold2/k1;
        w1 = k1*w2-d(1)*qold1;
        y4(k) = w4;
        qnew1 = w1;
        qnew2 = w1*d(1)+qold1;
        qnew3 = w2*d(2)+qold2*1/k1;
        qnew4 = w3*d(3)+qold3*1/k2;
        qold1 = qnew1;qold2 = qnew2;qold3 = qnew3;qold4 = qnew4;
    end
    k4 = sqrt(1/sum(y4.*conj(y4)));
    const = 0.135127668
    % Obtained by scaling the o/p of the actual TF
    disp('The scaling parameters are');
    disp(k1);disp(k2);disp(k3); disp(k4);
    alpha(5) = alpha(5)/(k1*k2*k3*k4);
    alpha(4) = alpha(4)/(k1*k2*k3*k4);
    alpha(3) = alpha(3)/(k2*k3*k4);
    alpha(2) = alpha(2)/(k3*k4);
    alpha(1) = alpha(1)/k4;
    alpha = const*alpha;
    %%% Computation of noise variance %%%
    % Noise variance due to k4 and d4 = 1/k4^2= 1.08185285036642
    % To compute noise variance due to k3,d3' = 1.33858225
    imp = [1 zeros([1,499])];
    for k = 1:500
        w4 = -d(4)*qold4/k3;
        w3 = k3*w4-d(3)*qold3/k2;
        w2 = k2*w3-d(2)*qold2/k1;
        w1 = k1*w2-d(1)*qold1;
        qnew1 = w1;
        qnew2 = w1*d(1)+qold1;
        qnew3 = w2*d(2)+qold2*1/k1;
        qnew4 = w3*d(3)+qold3*1/k2;
        y11 = w4*d(4)+qold4/k3;

```

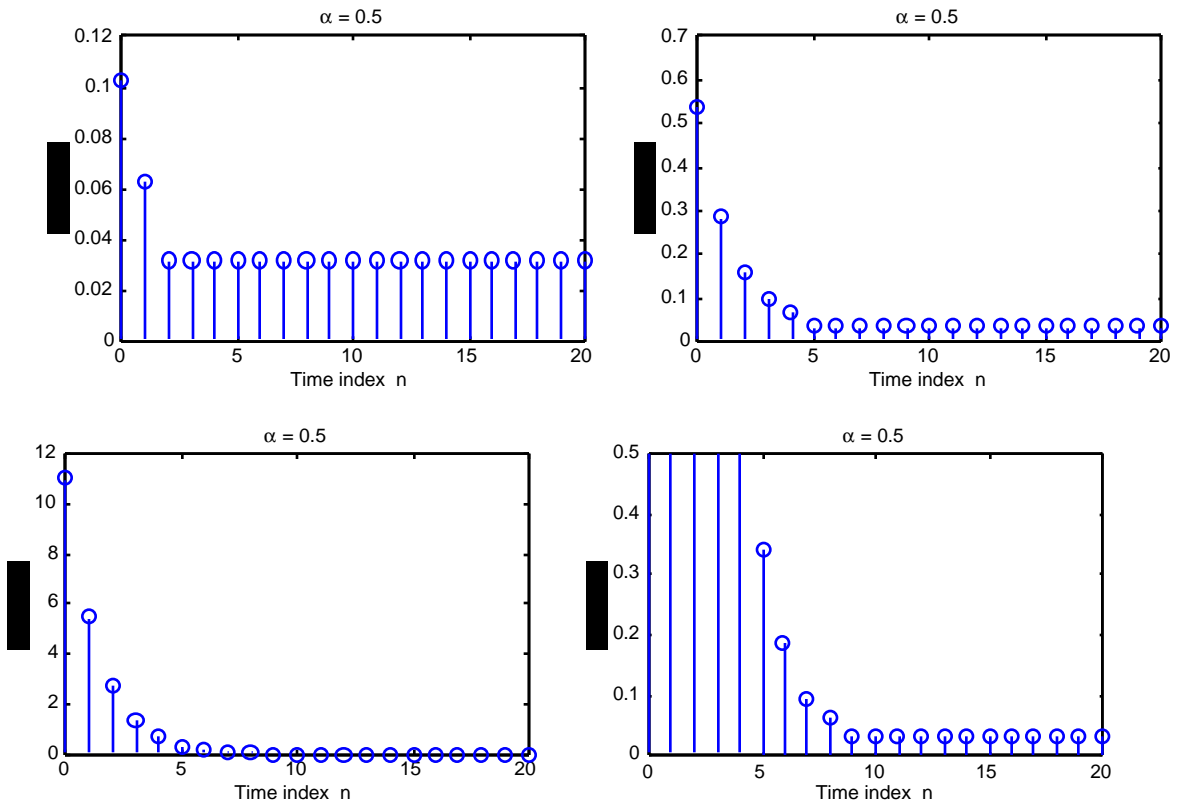


```

y0(k) = alpha(1)*y11+alpha(2)*qnew4+alpha(3)*qnew3 +
alpha(4)*qnew2+alpha(5)*qnew1;
qold1 = qnew1;qold2 = qnew2;qold3 = qnew3;qold4 = qnew4;
end
nv = sum(y0.*conj(y0));
% for k2,-d2'' nv = 3.131899935
% for k1,-d1''' nv = 1.00880596097028
% for d1''' nv = 0.95806646140013
% for d2'' nv = 2.61615077290574
% for d3' nv = 0.41493478856386
% for d4 nv = 0.01975407768839
% for 1/k1 nv = 0.75359663926391
% for 1/k2 nv = 0.58314964498424
% for 1/k3 nv = 0.09095345118133
% Total nv = 23.55888782866100

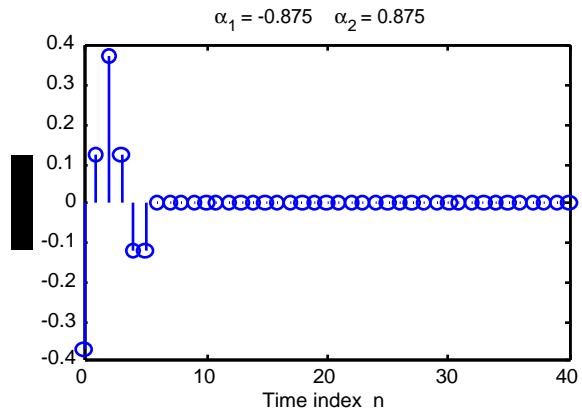
```

### M9.9



In all three cases, the condition of Eq. (9.186) is satisfied and hence, the structure exhibits zero-input granular limit cycles.

### M9.10



## Chapter 10 (2e)

**10.1** For an input  $x_1[n]$  and  $x_2[n]$ , the outputs of the factor-of-L up-sampler are, respectively, given by

$$x_{u1}[n] = \begin{cases} x_1[n/L], & n = 0, \pm L, \pm 2L, \mathbf{K} \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad x_{u2}[n] = \begin{cases} x_2[n/L], & n = 0, \pm L, \pm 2L, \mathbf{K} \\ 0, & \text{otherwise.} \end{cases}$$

Let  $x_2[n] = x_1[n - n_o]$ , where  $n_o$  is an integer. Then  $x_2[n/L] = x_1[(n/L) - n_o]$ . Hence,

$$x_{u2}[n] = \begin{cases} x_1[(n/L) - n_o], & n = 0, \pm L, \pm 2L, \mathbf{K} \\ 0, & \text{otherwise.} \end{cases}$$

But  $x_{u1}[n - n_o] = \begin{cases} x_1[(n - n_o)/L], & n - n_o = 0, \pm L, \pm 2L, \mathbf{K} \\ 0, & \text{otherwise,} \end{cases}$  Since  $x_{u2}[n] \neq x_{u1}[n - n_o]$ , the up-sampler is a time-varying system.

**10.2** Consider first the up-sampler. Let  $x_1[n]$  and  $x_2[n]$  be the inputs with corresponding outputs

$$\text{given by } y_1[n] \text{ and } y_2[n]. \text{ Now, } y_1[n] = \begin{cases} x_1[n/L], & n = 0, \pm L, \pm 2L, \mathbf{K} \\ 0, & \text{otherwise,} \end{cases} \text{ and}$$

$$y_2[n] = \begin{cases} x_2[n/L], & n = 0, \pm L, \pm 2L, \mathbf{K} \\ 0, & \text{otherwise.} \end{cases} \text{ Let us now apply the input } x_3[n] = \alpha x_1[n] + \beta x_2[n],$$

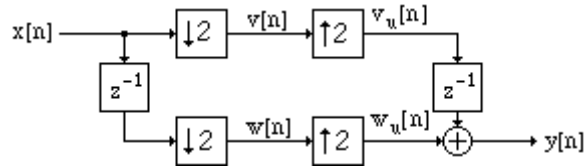
with the corresponding output given by  $y_3[n]$ , where

$$y_3[n] = \begin{cases} \alpha x_1[n/L] + \beta x_2[n/L], & n = 0, \pm L, \pm 2L, \mathbf{K} \\ 0, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \alpha x_1[n/L] \\ 0 \end{cases} + \begin{cases} \beta x_2[n/L] \\ 0 \end{cases} \quad n = 0, \pm L, \pm 2L, \mathbf{K} \quad \text{otherwise,} = \alpha y_1[n] + \beta y_2[n]. \text{ Thus, the up-sampler is a linear system.}$$

Now, consider the down-sampler. Let  $x_1[n]$  and  $x_2[n]$  be the inputs with corresponding outputs given by  $y_1[n]$  and  $y_2[n]$ . Now,  $y_1[n] = x_1[Mn]$ , and  $y_2[n] = x_2[Mn]$ . Let us now apply the input  $x_3[n] = \alpha x_1[n] + \beta x_2[n]$ , with the corresponding output given by  $y_3[n]$ , where  $y_3[n] = x_3[Mn] = \alpha x_1[Mn] + \beta x_2[Mn]$ . Thus, the down-sampler is a linear system.

**10.3**



$$\text{From the figure, } V(z) = \frac{1}{2} X(z^{1/2}) + \frac{1}{2} X(-z^{1/2}), \quad W(z) = \frac{z^{-1/2}}{2} X(z^{1/2}) - \frac{z^{-1/2}}{2} X(-z^{1/2}),$$

$$V_u(z) = \frac{1}{2} X(z) + \frac{1}{2} X(-z), \quad W_u(z) = \frac{z^{-1}}{2} X(z) - \frac{z^{-1}}{2} X(-z). \text{ Hence,}$$

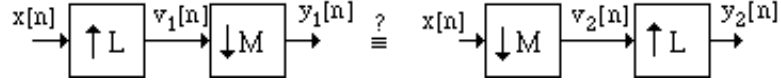
$$Y(z) = z^{-1} V_u(z) + W_u(z) = z^{-1} X(z), \text{ or in other words, } y[n] = x[n-1].$$

**10.4**  $c[n] = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{kn} = \frac{1}{M} \left( \frac{1 - W_M^{nM}}{1 - W_M^n} \right)$ . Hence, if  $n \neq rM$ ,  $c[n] = \frac{1}{M} \left( \frac{1-1}{1-W_M^n} \right) = 0$ . On the other

hand, if  $n = rM$ , then  $c[n] = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{kn} = \frac{1}{M} \sum_{k=0}^{M-1} W_M^{krM} = \frac{1}{M} \sum_{k=0}^{M-1} 1 = 1$ . Thus,

$$c[n] = \begin{cases} 1, & \text{if } n = rM, \\ 0, & \text{otherwise.} \end{cases}$$

**10.5**

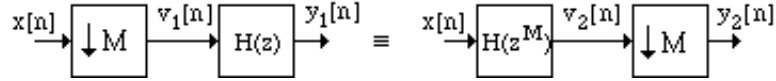


For the left-hand side figure, we have  $V_1(z) = X(z^L)$ ,  $Y_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{L/M} W_M^{kL})$ .

For the right-hand side figure, we have  $V_2(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{1/M} W_M^k)$ ,  $Y_2(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{L/M} W_M^k)$ .

Since  $L$  and  $M$  are relatively prime,  $W_M^k$  and  $W_M^{kL}$  take the same set of values for  $k = 0, 1, \dots, M-1$ . Hence,  $Y_1(z) = Y_2(z)$ .

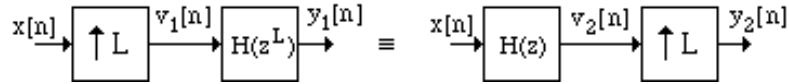
**10.6**



For the left-hand side figure, we have  $V_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{1/M} W_M^k)$ ,  $Y_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} H(z) X(z^{1/M} W_M^k)$ ,

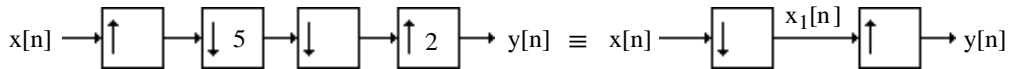
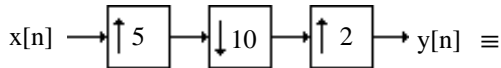
For the right-hand side figure, we have  $V_2(z) = H(z^M) X(z)$ ,  $Y_2(z) = \frac{1}{M} \sum_{k=0}^{M-1} H(z W_M^{kM}) X(z^{1/M} W_M^k)$

$= \frac{1}{M} \sum_{k=0}^{M-1} H(z) X(z^{1/M} W_M^k)$ . Hence,  $Y_1(z) = Y_2(z)$ .



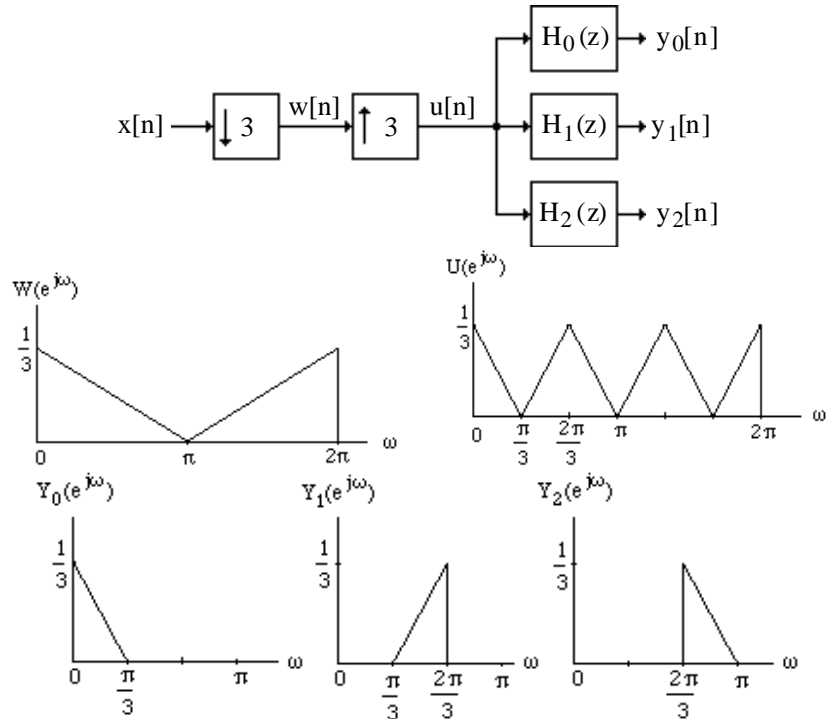
For the left-hand side figure, we have  $V_1(z) = X(z^L)$ ,  $Y_1(z) = H(z^L) X(z^L)$ . For the right-hand side figure, we have  $V_2(z) = H(z) X(z)$ ,  $Y_2(z) = H(z^L) X(z^L)$ . Hence,  $Y_1(z) = Y_2(z)$ .

**10.7**

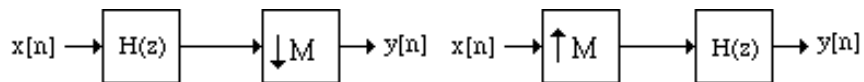


Hence,  $x_1[n] = x[2n]$  and  $y[n] = \begin{cases} x_1[n/2], & \text{for } n = 2r, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} x[n], & \text{for } n = 2r, \\ 0, & \text{otherwise} \end{cases}$  Therefore,  
 $y[n] = \begin{cases} x[n], & \text{for } n = 2r, \\ 0, & \text{otherwise.} \end{cases}$

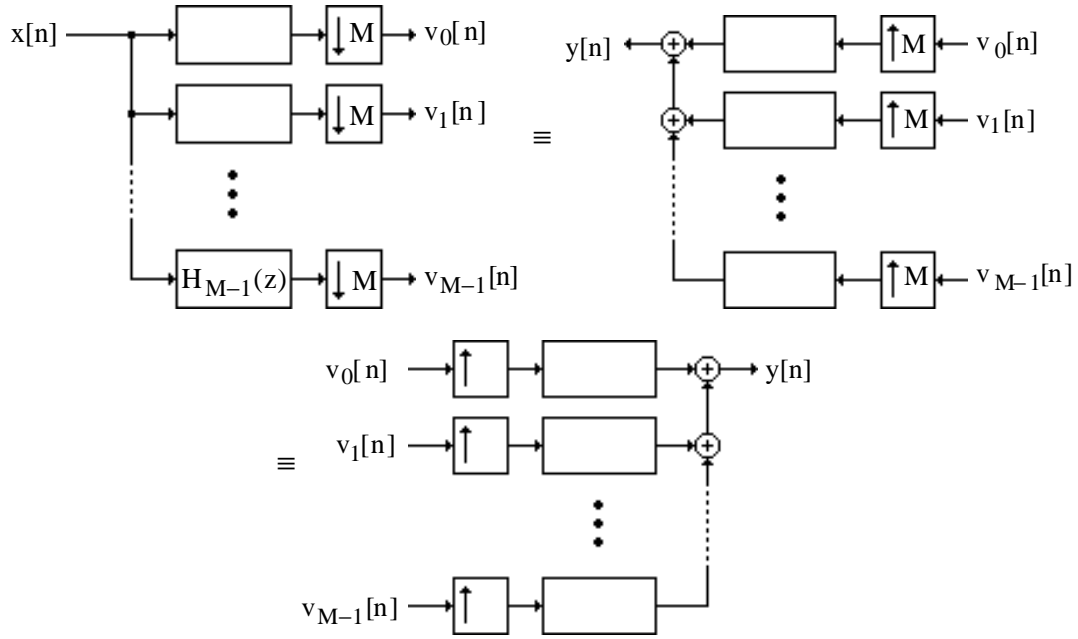
**10.8**



**10.9** As outlined in Section 6.3, the transpose of a digital filter structure is obtained by reversing all paths, replacing the pick-off node with an adder and vice-versa, and interchanging the input and the output nodes. Moreover, in a multirate structure, the transpose of a factor-of-M down-sampler is a factor-of-M up-sampler and vice-versa. Applying these operations to the factor-of-M decimator shown on the left-hand side, we arrive at a factor-of-M interpolator as indicated on the right-hand side in the figure below.

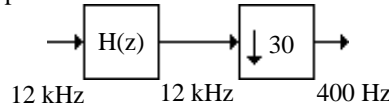


**10.10** Applying the transpose operation to the M-channel analysis filter bank shown below on the left-hand side, we arrive at the M-channel synthesis filter bank shown below on the right-hand side.

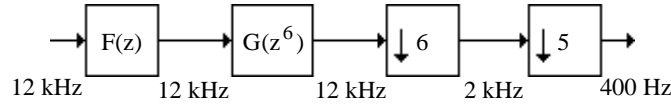


**10.11** Specifications for  $H(z)$  are as follows:

$$F_p = 180 \text{ Hz}, F_s = 200 \text{ Hz}, \delta_p = 0.002, \delta_s = 0.001.$$



We realize  $H(z)$  as  $H(z) = G(z^6)F(z)$ .



Therefore, specifications for  $G(z)$  are as follows:

$$F_p = 1080 \text{ Hz}, F_s = 1200 \text{ Hz}, \delta_p = 0.001, \delta_s = 0.001. \text{ Here, } \Delta f = \frac{120}{12000}.$$

$$\text{Hence, from Eq. (7.15), order of } G(z) \text{ is given by } N_G = \frac{-20 \log_{10} \sqrt{0.001 \times 0.001} - 13}{14.6(120/12000)} = \frac{47 \times 12000}{14.6 \times 120} = 321.92.$$

Likewise, specifications for  $F(z)$  are :  $F_p = 180 \text{ Hz}, F_s = 1800 \text{ Hz}, \delta_p = 0.001, \delta_s = 0.001$ .

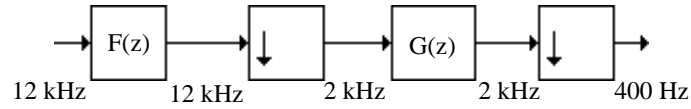
Here,  $\Delta f = \frac{1620}{12000}$ . Hence, order of  $F(z)$  is given by

$$N_F = \frac{-20 \log_{10} \sqrt{10^{-6}} - 13}{14.6(1620/12000)} = \frac{47 \times 12000}{14.6 \times 1620} = 23.846. \text{ Thus, we choose } N_G = 322 \text{ and } N_F = 24.$$

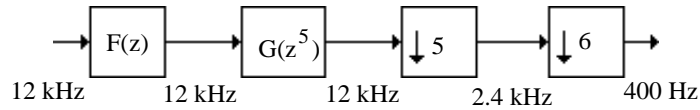
$$R_{M,G} = (322 + 1) \times \frac{2000}{5} = 129,200 \text{ multiplications/second (mps), and}$$

$$R_{M,F} = (24 + 1) \times \frac{12000}{6} = 50,000 \text{ mps}$$

Hence, total no. of mps = 179,200. Hence the computational complexity of this particular IFIR implementation is slightly higher here than that in Example 10.8.



**10.12** We realize  $H(z)$  as  $H(z) = G(z^5)F(z)$ .

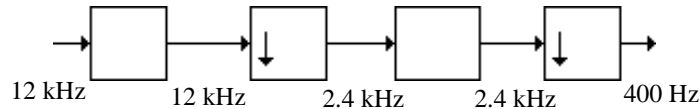


Specifications for  $G(z)$  are:  $F_p = 900$  Hz,  $F_s = 1000$  Hz,  $\delta_p = 0.001$ ,  $\delta_s = 0.001$ . Here,  $\Delta f = \frac{100}{12000}$ . Hence, from Eq. (7.15), order of  $G(z)$  is given by

$$N_G = \frac{-20 \log_{10} \sqrt{0.001 \times 0.001} - 13}{14.6(100/12000)} = \frac{47 \times 12000}{14.6 \times 100} = 386.3.$$

Likewise, specifications for  $F(z)$  are:  $F_p = 180$  Hz,  $F_s = 2200$  Hz,  $\delta_p = 0.001$ ,  $\delta_s = 0.001$ . Here,  $\Delta f = \frac{2020}{12000}$ . Hence, order of

$F(z)$  is given by  $N_F = \frac{-20 \log_{10} \sqrt{10^{-6}} - 13}{14.6(2020/12000)} = \frac{47 \times 12000}{14.6 \times 2020} = 19.124$ . Thus, we choose  $N_G = 387$  and  $N_F = 20$ .

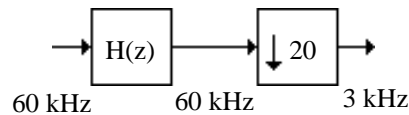


$$R_{M,G} = (387 + 1) \times \frac{2400}{6} = 155,200 \text{ multiplications/second (mps), and}$$

$$R_{M,F} = (20 + 1) \times \frac{12000}{5} = 50,400 \text{ mps}$$

Hence, total no. of mps = 205,600. Hence the computational complexity of this particular IFIR implementation is slightly higher here than that in Example 10.8 and in Problem 10.11.

**10.13**



Specifications for  $H(z)$  are:  $F_p = 1250$  kHz,  $F_s = 1500$  kHz,  $\delta_p = 0.02$  and  $\delta_s = 0.01$ . Hence, from Eq. (7.15), order  $N$  of  $H(z)$  is given by

$$N = \frac{-20 \log_{10} \sqrt{0.02 \times 0.01} - 13}{14.6(250/60000)} = \frac{23.989 \times 60000}{14.6 \times 250} = 394.34.$$

We thus choose  $N = 395$ . Computational complexity is therefore  $= 396 \times \frac{60,000}{20} = 1,188,000$ .

**10.14** Original decimation filter  $H(z)$  specifications:  $F_p = 1250$  Hz,  $F_s = 1500$ ,  $\delta_p = 0.02$ ,  $\delta_s = 0.01$ .

The 4 possible IFIR implementations of  $H(z)$  are as follows: (A)  $H(z) = G(z^5)F(z)$ ,  
 (B)  $H(z) = G(z^4)F(z)$ , (C)  $H(z) = G(z^2)F(z)$ , and (D)  $H(z) = G(z^{10})F(z)$ .

Case A: Specifications for  $G(z)$ :  $F_p = 1250 \times 5 = 6250$  Hz,  $F_s = 1500 \times 5 = 7500$  Hz,

$\delta_p = 0.01$ ,  $\delta_s = 0.01$ . Here  $\Delta f = \frac{1250}{60,000}$ . Hence, using Eq. (10.26), we obtain the order of  $G(z)$

$$\text{as } N_G = \frac{-20 \log_{10} \sqrt{0.01 \times 0.01} - 13}{14.6(1250 / 60000)} = 88.767.$$

Specifications for  $F(z)$ :  $F_p = 1250$  Hz,  $F_s = \frac{60000 - 5 \times 1500}{5} = 10,500$  Hz,

$\delta_p = 0.01$ ,  $\delta_s = 0.01$ . Here  $\Delta f = \frac{9250}{60,000}$ . Hence, using Eq. (10.26), we obtain the order of  $F(z)$

$$\text{as } N_F = \frac{-20 \log_{10} \sqrt{0.01 \times 0.01} - 13}{14.6(9250 / 60000)} = 11.996. \text{ We thus choose } N_G = 89 \text{ and } N_F = 12.$$

Hence,  $R_{M,G} = (89 + 1) \times \frac{12000}{4} = 270,000$  mps and  $R_{M,F} = (12 + 1) \times \frac{60000}{5} = 156,000$  mps.

Hence, total computational complexity = 426,000 mps.

Case B: Specifications for  $G(z)$ :  $F_p = 1250 \times 4 = 5000$  Hz,  $F_s = 1500 \times 4 = 9000$  Hz,

$\delta_p = 0.01$ ,  $\delta_s = 0.01$ . Here  $\Delta f = \frac{1000}{60,000}$ . Hence, using Eq. (10.26), we obtain the order of  $G(z)$

$$N_G = \frac{-20 \log_{10} \sqrt{0.01 \times 0.01} - 13}{14.6(1000 / 60000)} = 110.96$$

Specifications for  $F(z)$ :  $F_p = 1250$  Hz,  $F_s = \frac{60000 - 5 \times 1500}{5} = 10,500$  Hz,

$\delta_p = 0.01$ ,  $\delta_s = 0.01$ . Here  $\Delta f = \frac{12250}{60,000}$ . Hence, using Eq. (10.26), we obtain the order of  $F(z)$

$$N_F = \frac{-20 \log_{10} \sqrt{0.01 \times 0.01} - 13}{14.6(12250 / 60000)} = 0.0579. \text{ We thus choose } N_G = 111 \text{ and } N_F = 10.$$

Hence,  $R_{M,G} = (111 + 1) \times \frac{15000}{5} = 336,000$  mps and  $R_{M,F} = (10 + 1) \times \frac{60000}{4} = 165,000$  mps.

Hence, total computational complexity = 501,000 mps.

Case C: Specifications for  $G(z)$ :  $F_p = 1250 \times 2 = 2500$  Hz,  $F_s = 1500 \times 2 = 3000$  Hz,

$\delta_p = 0.01$ ,  $\delta_s = 0.01$ . Here  $\Delta f = \frac{500}{60,000}$ . Hence, using Eq. (10.26), we obtain the order of  $G(z)$

$$N_G = \frac{-20 \log_{10} \sqrt{0.01 \times 0.01} - 13}{14.6(500 / 60000)} = 221.92.$$



Specifications for  $F(z)$ :  $F_p = 1250$  Hz,  $F_s = \frac{60000 - 2 \times 1500}{2} = 28,500$  Hz,

$\delta_p = 0.01$ ,  $\delta_s = 0.01$ . Here  $\Delta f = \frac{12250}{60,000}$ . Hence, using Eq. (10.26), we obtain the order of  $F(z)$

$$N_F = \frac{-20 \log_{10} \sqrt{0.01 \times 0.01} - 13}{14.6(27250 / 60000)} = 4.0179. \text{ We thus choose } N_G = 222 \text{ and } N_F = 5.$$

Hence,  $R_{M,G} = (222 + 1) \times \frac{30000}{10} = 669,000$  mps and  $R_{M,F} = (5 + 1) \times \frac{60000}{2} = 180,000$  mps.

Hence, total computational complexity = 849,000 mps.

**Case D:** Specifications for  $G(z)$ :  $F_p = 1250 \times 10 = 12500$  Hz,  $F_s = 1500 \times 10 = 15000$  Hz,

$\delta_p = 0.01$ ,  $\delta_s = 0.01$ . Here  $\Delta f = \frac{2500}{60,000}$ . Hence, using Eq. (10.26), we obtain the order of  $G(z)$

$$N_G = \frac{-20 \log_{10} \sqrt{0.01 \times 0.01} - 13}{14.6(2500 / 60000)} = 44.384.$$

Specifications for  $F(z)$ :  $F_p = 1250$  Hz,  $F_s = \frac{60000 - 10 \times 1500}{10} = 4,500$  Hz,

$\delta_p = 0.01$ ,  $\delta_s = 0.01$ . Here  $\Delta f = \frac{3250}{60,000}$ . Hence, using Eq. (10.26), we obtain the order of  $F(z)$

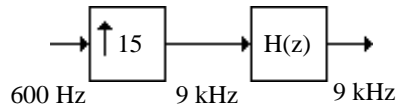
$$N_F = \frac{-20 \log_{10} \sqrt{0.01 \times 0.01} - 13}{14.6(3250 / 60000)} = 34.141. \text{ We thus choose } N_G = 45 \text{ and } N_F = 35.$$

Hence,  $R_{M,G} = (45 + 1) \times \frac{6000}{2} = 138,000$  mps and  $R_{M,F} = (35 + 1) \times \frac{60000}{10} = 216,000$  mps.

Hence, total computational complexity = 354,000 mps.

Thus, Case D has the lowest computational complexity.

**10.15 (a)** Specifications for  $H(z)$  are:  $F_p = 200$  Hz,  $F_s = 300$  Hz,  $\delta_p = 0.002$  and  $\delta_s = 0.004$ .

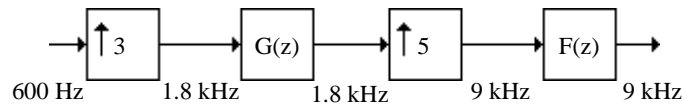


Here,  $\Delta f = \frac{100}{9000}$ . Hence, from Eq. (10.26), order  $N$  of  $H(z)$  is given by

$$N = \frac{-20 \log_{10} \sqrt{0.002 \times 0.004} - 13}{14.6(100 / 9000)} = \frac{37.969 \times 9000}{14.6 \times 100} = 234.06. \text{ We choose } N = 235. \text{ Hence,}$$

computational complexity =  $(235 + 1) \times \frac{9000}{15} = 141,600$  mps.

**(b)** We realize  $H(z) = G(z^5)F(z)$ .



Specifications for  $G(z)$  are:  $F_p = 200$  Hz,  $F_s = 300$  Hz,  $\delta_p = 0.002$  and  $\delta_s = 0.004$ . Here,

$\Delta f = \frac{100}{9000}$ . Hence, from Eq. (7.15), order  $N$  of  $H(z)$  is given by

$$\mathbf{10.16 (a)} \quad H_1(z) = \frac{p_0 + p_1 z^{-1}}{1 + d_1 z^{-1}}. \quad H_1(zW_2^1) = H_1(-z) = \frac{p_0 - p_1 z^{-1}}{1 - d_1 z^{-1}}. \quad \text{Thus,}$$

$$E_0(z^2) = \frac{1}{2} [H_1(z) + H_1(-z)] = \frac{1}{2} \left[ \frac{p_0 + p_1 z^{-1}}{1 + d_1 z^{-1}} + \frac{p_0 - p_1 z^{-1}}{1 - d_1 z^{-1}} \right] = \frac{p_0 - p_1 d_1 z^{-2}}{1 - d_1^2 z^{-2}}, \quad \text{and}$$

$$z^{-1} E_1(z^2) = \frac{1}{2} [H_1(z) - H_1(-z)] = \frac{1}{2} \left[ \frac{p_0 + p_1 z^{-1}}{1 + d_1 z^{-1}} - \frac{p_0 - p_1 z^{-1}}{1 - d_1 z^{-1}} \right] = \frac{(p_1 - p_0 d_1) z^{-1}}{1 - d_1^2 z^{-2}}. \quad \text{Hence, a two-}$$

band polyphase decomposition of  $H_1(z)$  is given by

$$H_1(z) = \left( \frac{p_0 - p_1 d_1 z^{-2}}{1 - d_1^2 z^{-2}} \right) + z^{-1} \left( \frac{p_1 - p_0 d_1}{1 - d_1^2 z^{-2}} \right).$$

$$\mathbf{(b)} \quad H_2(z) = \frac{2 + 3.1z^{-1} + 1.5z^{-2}}{1 + 0.9z^{-1} + 0.8z^{-2}}. \quad H_2(zW_2^1) = H_2(-z) = \frac{2 - 3.1z^{-1} + 1.5z^{-2}}{1 - 0.9z^{-1} + 0.8z^{-2}}. \quad \text{Thus,}$$

$$E_0(z^2) = \frac{1}{2} [H_2(z) + H_2(-z)] = \frac{1}{2} \left[ \frac{2 + 3.1z^{-1} + 1.5z^{-2}}{1 + 0.9z^{-1} + 0.8z^{-2}} + \frac{2 - 3.1z^{-1} + 1.5z^{-2}}{1 - 0.9z^{-1} + 0.8z^{-2}} \right]$$

$$= \frac{4 + 0.62z^{-2} + 2.4z^{-4}}{1 + 0.79z^{-2} + 0.64z^{-4}}.$$

$$z^{-1} E_1(z^2) = \frac{1}{2} [H_2(z) - H_2(-z)] = \frac{1}{2} \left[ \frac{2 + 3.1z^{-1} + 1.5z^{-2}}{1 + 0.9z^{-1} + 0.8z^{-2}} - \frac{2 - 3.1z^{-1} + 1.5z^{-2}}{1 - 0.9z^{-1} + 0.8z^{-2}} \right]$$

$$= \frac{2.6z^{-1} + 2.26z^{-3}}{1 + 0.79z^{-2} + 0.64z^{-4}}. \quad \text{Hence, a two-band polyphase decomposition of } H_2(z) \text{ is given by}$$

$$H_2(z) = \left( \frac{4 + 0.62z^{-2} + 2.4z^{-4}}{1 + 0.79z^{-2} + 0.64z^{-4}} \right) + z^{-1} \left( \frac{2.6 + 2.26z^{-2}}{1 + 0.79z^{-2} + 0.64z^{-4}} \right).$$

$$\mathbf{(c)} \quad H_3(z) = \frac{2 + 3.1z^{-1} + 1.5z^{-2} + 4z^{-3}}{(1 - 0.5z^{-1})(1 + 0.9z^{-1} + 0.8z^{-2})} = \frac{2 + 3.1z^{-1} + 1.5z^{-2} + 4z^{-3}}{1 + 0.4z^{-1} + 0.35z^{-2} - 0.4z^{-3}}.$$

$$H_3(-z) = \frac{2 - 3.1z^{-1} + 1.5z^{-2} - 4z^{-3}}{1 - 0.4z^{-1} + 0.35z^{-2} + 0.4z^{-3}}.$$

$$E_0(z^2) = \frac{1}{2} [H_3(z) + H_3(-z)] = \frac{1}{2} \left[ \frac{2 + 3.1z^{-1} + 1.5z^{-2} + 4z^{-3}}{1 + 0.4z^{-1} + 0.35z^{-2} - 0.4z^{-3}} + \frac{2 - 3.1z^{-1} + 1.5z^{-2} - 4z^{-3}}{1 - 0.4z^{-1} + 0.35z^{-2} + 0.4z^{-3}} \right]$$

$$= \frac{4 + 1.92z^{-2} + 0.33z^{-4} + 3.2z^{-6}}{1 + 0.54z^{-2} + 0.4425z^{-4} - 0.16z^{-6}}.$$

$$z^{-1}E_1(z^2) = \frac{1}{2}[H_3(z) - H_3(-z)] = \frac{1}{2} \left[ \frac{2 + 3.1z^{-1} + 1.5z^{-2} + 4z^{-3}}{1 + 0.4z^{-1} + 0.35z^{-2} - 0.4z^{-3}} - \frac{2 - 3.1z^{-1} + 1.5z^{-2} - 4z^{-3}}{1 - 0.4z^{-1} + 0.35z^{-2} + 0.4z^{-3}} \right]$$

$$= \frac{4.6z^{-1} + 10.57z^{-3} + 4z^{-5}}{1 + 0.54z^{-2} + 0.4425z^{-4} - 0.16z^{-6}}. \text{ Hence, a two-band polyphase decomposition of } H_3(z) \text{ is}$$

$$\text{given by } H_3(z) = \left( \frac{4 + 1.92z^{-2} + 0.33z^{-4} + 3.2z^{-6}}{1 + 0.54z^{-2} + 0.4425z^{-4} - 0.16z^{-6}} \right) + z^{-1} \left( \frac{4.6 + 10.57z^{-2} + 4z^{-4}}{1 + 0.54z^{-2} + 0.4425z^{-4} - 0.16z^{-6}} \right)$$

**10.17 (a)**  $H_1(z) = \frac{p_0 + p_1z^{-1}}{1 + d_1z^{-1}} = \sum_{k=0}^2 z^{-k} E_k(z^3)$ . Thus,

$$\begin{bmatrix} H_1(z) \\ H_1(W_3^1 z) \\ H_1(W_3^2 z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-4} \end{bmatrix} \begin{bmatrix} E_0(z^3) \\ z^{-1}E_1(z^3) \\ z^{-2}E_2(z^3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-1} \end{bmatrix} \begin{bmatrix} E_0(z^3) \\ z^{-1}E_1(z^3) \\ z^{-2}E_2(z^3) \end{bmatrix} \text{ or}$$

$$\begin{bmatrix} E_0(z^3) \\ z^{-1}E_1(z^3) \\ z^{-2}E_2(z^3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-1} \end{bmatrix}^{-1} \begin{bmatrix} H_1(z) \\ H_1(W_3^1 z) \\ H_1(W_3^2 z) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^1 & W_3^2 \\ 1 & W_3^2 & W_3^1 \end{bmatrix} \begin{bmatrix} H_1(z) \\ H_1(W_3^1 z) \\ H_1(W_3^2 z) \end{bmatrix}. \text{ Therefore,}$$

$$E_0(z^3) = \frac{1}{3} [H_1(z) + H_1(zW_3^1) + H_1(zW_3^2)]$$

$$= \frac{1}{3} \left[ \frac{p_0 + p_1z^{-1}}{1 + d_1z^{-1}} + \frac{p_0 + p_1e^{j2\pi/3}z^{-1}}{1 + d_1e^{j2\pi/3}z^{-1}} + \frac{p_0 + p_1e^{j4\pi/3}z^{-1}}{1 + d_1e^{j4\pi/3}z^{-1}} \right] = \frac{p_0 + p_1d_1^2z^{-3}}{1 + d_1^3z^{-3}}.$$

$$z^{-1}E_1(z^3) = \frac{1}{3} [H_1(z) + W_3^1 H_1(zW_3^1) + W_3^2 H_1(zW_3^2)]$$

$$= \frac{1}{3} \left[ \frac{p_0 + p_1z^{-1}}{1 + d_1z^{-1}} + e^{j2\pi/3} \frac{p_0 + p_1e^{j2\pi/3}z^{-1}}{1 + d_1e^{j2\pi/3}z^{-1}} + e^{j4\pi/3} \frac{p_0 + p_1e^{j4\pi/3}z^{-1}}{1 + d_1e^{j4\pi/3}z^{-1}} \right] = z^{-1} \frac{p_1 - p_0d_1}{1 + d_1^3z^{-3}}.$$

$$z^{-2}E_2(z^3) = \frac{1}{3} [H_1(z) + W_3^2 H_1(zW_3^1) + W_3^1 H_1(zW_3^2)]$$

$$= \frac{1}{3} \left[ \frac{p_0 + p_1z^{-1}}{1 + d_1z^{-1}} + e^{j4\pi/3} \frac{p_0 + p_1e^{j2\pi/3}z^{-1}}{1 + d_1e^{j2\pi/3}z^{-1}} + e^{j2\pi/3} \frac{p_0 + p_1e^{j4\pi/3}z^{-1}}{1 + d_1e^{j4\pi/3}z^{-1}} \right] = z^{-2} \frac{-p_1d_1 + p_0d_1^2}{1 + d_1^3z^{-3}}.$$

Hence,  $E_0(z) = \frac{p_0 + p_1d_1^2z^{-1}}{1 + d_1^3z^{-1}}$ ,  $E_1(z) = \frac{p_1 - p_0d_1}{1 + d_1^3z^{-1}}$ , and  $E_2(z) = \frac{-p_1d_1 + p_0d_1^2}{1 + d_1^3z^{-1}}$ .

**(b)**  $H_2(z) = \frac{2 + 3.1z^{-1} + 1.5z^{-2}}{1 + 0.9z^{-1} + 0.8z^{-2}}$ .

$$\begin{bmatrix} H_2(z) \\ H_2(W_3^1 z) \\ H_2(W_3^2 z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-1} \end{bmatrix} \begin{bmatrix} E_0(z^3) \\ z^{-1}E_1(z^3) \\ z^{-2}E_2(z^3) \end{bmatrix} \text{ or } \begin{bmatrix} E_0(z^3) \\ z^{-1}E_1(z^3) \\ z^{-2}E_2(z^3) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^1 & W_3^2 \\ 1 & W_3^2 & W_3^1 \end{bmatrix} \begin{bmatrix} H_2(z) \\ H_2(W_3^1 z) \\ H_2(W_3^2 z) \end{bmatrix}.$$

$$\begin{aligned} \text{Thus, } E_0(z^3) &= \frac{1}{3} \left[ H_2(z) + H_2(z W_3^1) + H_2(z W_3^2) \right] \\ &= \frac{1}{3} \left[ \frac{2+3.1z^{-1}+1.5z^{-2}}{1+0.9z^{-1}+0.8z^{-2}} + \frac{2+3.1e^{j2\pi/3}z^{-1}+1.5e^{j4\pi/3}z^{-2}}{1+0.9e^{j2\pi/3}z^{-1}+0.8e^{j4\pi/3}z^{-2}} + \frac{2+3.1e^{j4\pi/3}z^{-1}+1.5e^{j2\pi/3}z^{-2}}{1+0.9e^{j4\pi/3}z^{-1}+0.8e^{j2\pi/3}z^{-2}} \right] \\ &= \frac{6-8.277z^{-3}+2.88z^{-6}}{1-1.431z^{-3}+0.512z^{-6}}. \end{aligned}$$

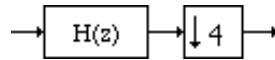
$$\begin{aligned} z^{-1}E_1(z^3) &= \frac{1}{3} \left[ H_2(z) + W_3^1 H_2(z W_3^1) + W_3^2 H_2(z W_3^2) \right] \\ &= \frac{1}{3} \left[ \frac{2+3.1z^{-1}+1.5z^{-2}}{1+0.9z^{-1}+0.8z^{-2}} + e^{j2\pi/3} \frac{2+3.1e^{j2\pi/3}z^{-1}+1.5e^{j4\pi/3}z^{-2}}{1+0.9e^{j2\pi/3}z^{-1}+0.8e^{j4\pi/3}z^{-2}} \right. \\ &\quad \left. + e^{j4\pi/3} \frac{2+3.1e^{j4\pi/3}z^{-1}+1.5e^{j2\pi/3}z^{-2}}{1+0.9e^{j4\pi/3}z^{-1}+0.8e^{j2\pi/3}z^{-2}} \right] = z^{-1} \frac{3.9-2.811z^{-3}}{1-1.431z^{-3}+0.512z^{-6}}. \end{aligned}$$

$$\begin{aligned} z^{-2}E_2(z^3) &= \frac{1}{3} \left[ H_3(z) + W_3^2 H_3(z W_3^1) + W_3^1 H_3(z W_3^2) \right] \\ &= \frac{1}{3} \left[ \frac{2+3.1z^{-1}+1.5z^{-2}}{1+0.9z^{-1}+0.8z^{-2}} + e^{j4\pi/3} \frac{2+3.1e^{j2\pi/3}z^{-1}+1.5e^{j4\pi/3}z^{-2}}{1+0.9e^{j2\pi/3}z^{-1}+0.8e^{j4\pi/3}z^{-2}} \right. \\ &\quad \left. + e^{j2\pi/3} \frac{2+3.1e^{j4\pi/3}z^{-1}+1.5e^{j2\pi/3}z^{-2}}{1+0.9e^{j4\pi/3}z^{-1}+0.8e^{j2\pi/3}z^{-2}} \right] = z^{-2} \frac{-3.81+2.712z^{-3}}{1-1.431z^{-3}+0.512z^{-6}}. \text{ Hence,} \end{aligned}$$

$$E_0(z) = \frac{6-8.277z^{-1}+2.88z^{-2}}{1-1.431z^{-1}+0.512z^{-2}}, \quad E_1(z) = \frac{3.9-2.811z^{-1}}{1-1.431z^{-1}+0.512z^{-2}}, \text{ and}$$

$$E_2(z) = \frac{-3.81+2.712z^{-1}}{1-1.431z^{-1}+0.512z^{-2}}.$$

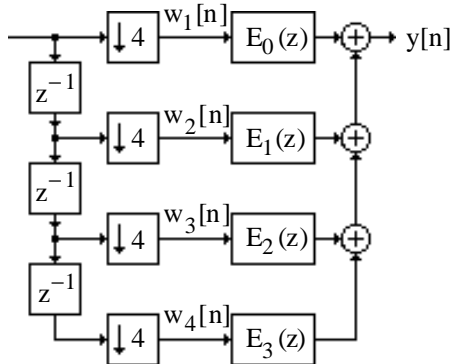
**10.18** A computationally efficient realization of the factor-of-4 decimator



is obtained by applying a 4-branch polyphase decomposition to  $H(z)$ :

$$H(z) = E_0(z^4) + z^{-1}E_1(z^4) + z^{-2}E_2(z^4) + z^{-3}E_3(z^4).$$

and then moving the down-sampler through the polyphase filters resulting in



Further reduction in computational complexity is achieved by sharing common multipliers if  $H(z)$  is a linear-phase FIR filter. For example, for a length-16 Type I FIR transfer function

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7} + h[7]z^{-8} \\ + h[6]z^{-9} + h[5]z^{-10} + h[4]z^{-11} + h[3]z^{-12} + h[2]z^{-13} + h[1]z^{-14} + h[0]z^{-15},$$

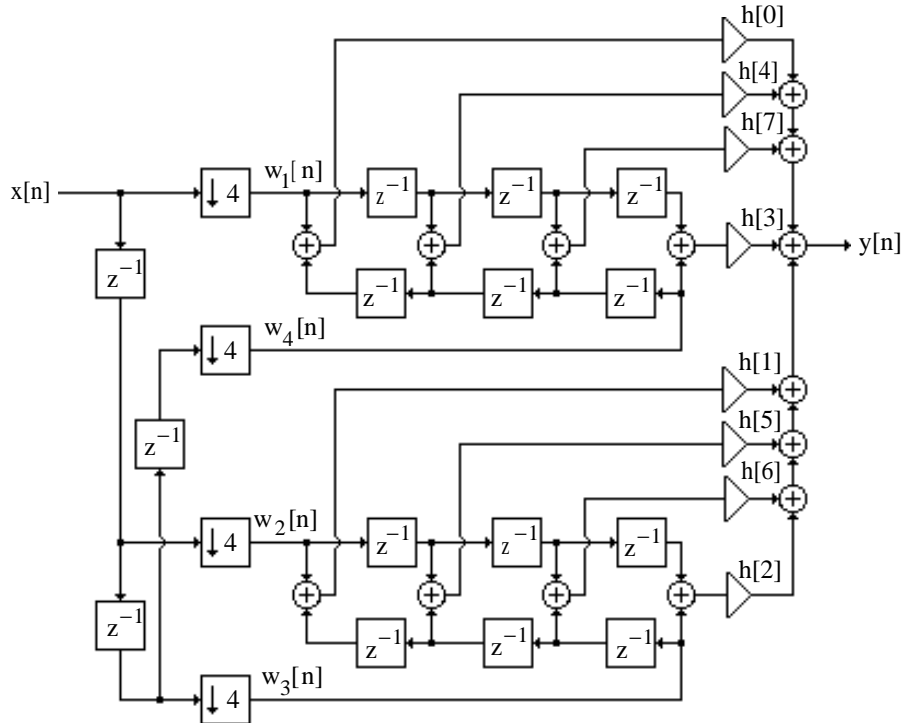
for which  $E_0(z) = h[0] + h[4]z^{-1} + h[7]z^{-2} + h[3]z^{-3}$ ,  $E_1(z) = h[1] + h[5]z^{-1} + h[6]z^{-2} + h[2]z^{-3}$ ,

$E_2(z) = h[2] + h[6]z^{-1} + h[5]z^{-2} + h[1]z^{-3}$ , and  $E_3(z) = h[3] + h[7]z^{-1} + h[4]z^{-2} + h[0]z^{-3}$ .

From the above figure it follows that  $Y(z) = E_0(z)W_1(z) + E_1(z)W_2(z) + E_2(z)W_3(z) + E_3(z)W_4(z)$

$$= h[0](W_1(z) + z^{-3}W_4(z)) + h[4](z^{-1}W_1(z) + z^{-2}W_4(z)) \\ + h[7](z^{-2}W_1(z) + z^{-1}W_4(z)) + h[3](z^{-3}W_1(z) + W_4(z)) \\ + h[1](W_2(z) + z^{-3}W_3(z)) + h[5](z^{-1}W_2(z) + z^{-2}W_3(z)) \\ + h[6](z^{-2}W_2(z) + z^{-1}W_3(z)) + h[2](z^{-3}W_2(z) + W_3(z)).$$

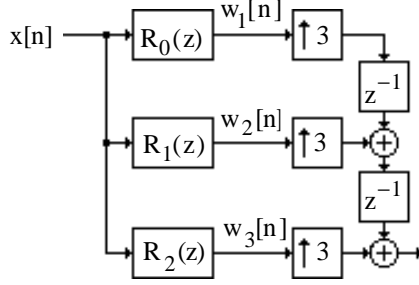
A computationally efficient factor-of-4 decimator structure based on the above equation is then as shown below:



**10.19** A computationally efficient realization of the factor-of-3 interpolator is obtained by applying a 3-branch Type II polyphase decomposition to the interpolation filter  $H(z)$ :

$$H(z) = R_2(z^3) + z^{-1}R_1(z^3) + z^{-2}R_0(z^3),$$

and then moving the up-sampler through the polyphase filters resulting in



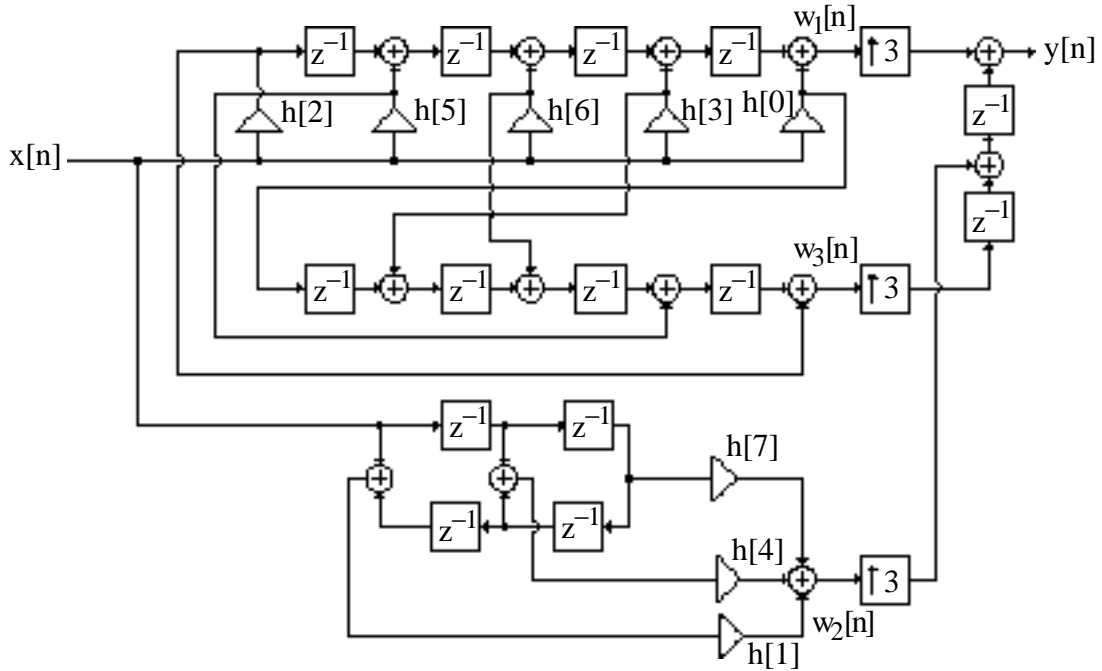
From the above figure it follows that

$$W_3(z) = h[0]X(z) + h[3]z^{-1}X(z) + h[6]z^{-2}X(z) + h[5]z^{-3}X(z) + h[2]z^{-4}X(z),$$

$$W_1(z) = h[2]X(z) + h[5]z^{-1}X(z) + h[6]z^{-2}X(z) + h[3]z^{-3}X(z) + h[0]z^{-4}X(z), \text{ and}$$

$$W_2(z) = h[1](X(z) + z^{-4}X(z)) + h[4](z^{-1}X(z) + z^{-3}X(z)) + h[7]z^{-2}X(z).$$

A computationally efficient factor-of-3 interpolator structure based on the above equations is then as shown below:



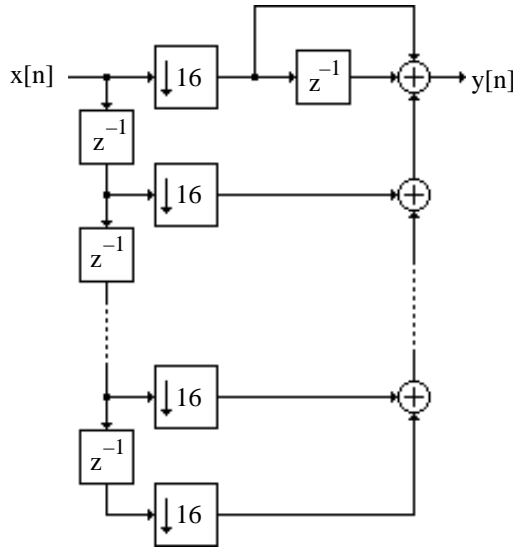
$$10.20 \quad H(z) = \sum_{i=0}^{N-1} z^{-i} = (1 + z^{-1}) + (z^{-2} + z^{-3}) + \mathbf{L} + (z^{-(N-2)} + z^{-(N-1)})$$

$$= (1 + z^{-1})(1 + z^{-2} + z^{-4} + \mathbf{L} + z^{-(N-2)}) = (1 + z^{-1})G(z^2) \text{ where } G(z) = \sum_{i=0}^{(N/2)-1} z^{-2i}.$$

Using a similar technique we can show that  $G(z) = (1 + z^{-1}) \left( \sum_{i=0}^{(N/4)-1} z^{-2i} \right)$ . Therefore we can write

$$H(z) = (1 + z^{-1})(1 + z^{-2}) \left( \sum_{i=0}^{(N/4)-1} z^{-2i} \right) = (1 + z^{-1})(1 + z^{-2})F(z^4) \text{ where } F(z) = \sum_{i=0}^{(N/4)-1} z^{-i}.$$

this decomposition process further we arrive at  $H(z) = (1+z^{-1})(1+z^{-2})\mathbf{L}(1+z^{-2^{K-1}})$ , where  $N = 2^K$ . Realization of a factor-of-16 decimator using a box-car decimation filter is as indicated below:



**10.21** Let  $u[n]$  denote the output of the factor-of- $L$  interpolator. Then,

$$E = \frac{\sum_{n=-\infty}^{\infty} (u[n] - u[n-1])^2}{\sum_{n=-\infty}^{\infty} u^2[n]} \quad (1)$$

and

$$C = \frac{\sum_{n=-\infty}^{\infty} u[n]u[n-1]}{\sum_{n=-\infty}^{\infty} u^2[n]} \quad (2)$$

Substituting Eq. (2) in Eq. (1) we get  $E = 2(1 - C)$ . Hence, as  $C \rightarrow 1$ , i.e., as the signal  $u[n]$  becomes highly correlated,  $E \rightarrow 0$ .

Now, by Parseval's relation,  $\sum_{n=-\infty}^{\infty} u[n]v[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} U(e^{j\omega})V^*(e^{j\omega})d\omega$ , where  $U(e^{j\omega})$  and

$V(e^{j\omega})$  are the DTFTs of  $u[n]$  and  $v[n]$ , respectively. If we let  $v[n] = u[n-1]$  in the numerator of Eq. (1) and  $v[n] = u[n]$  in the denominator of Eq. (1), then we can write

$$E = \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} |U(e^{j\omega})|^2 e^{j\omega} d\omega}{\frac{1}{2\pi} \int_{-\pi}^{\pi} |U(e^{j\omega})|^2 d\omega} = \frac{\int_{-\pi}^{\pi} |U(e^{j\omega})|^2 \cos(\omega) d\omega}{\int_{-\pi}^{\pi} |U(e^{j\omega})|^2 d\omega},$$

assuming  $u[n]$  to be a real sequence. If  $x[n]$  is assumed to be a broadband signal with a flat magnitude spectrum, i.e.,  $|X(e^{j\omega})| = 1$  for  $0 \leq |\omega| \leq \pi$ , then the magnitude spectrum of  $u[n]$  is

bandlimited to the range  $0 \leq |\omega| \leq \pi/L$ , i.e.,  $|U(e^{j\omega})| = \begin{cases} 1, & \text{for } 0 \leq |\omega| < \pi/L, \\ 0, & \text{otherwise.} \end{cases}$  Therefore,

$$C = \frac{\int_0^{\pi/L} \cos(\omega) d\omega}{\int_0^{\pi/L} d\omega} = \frac{\sin(\pi/L)}{(\pi/L)}. \text{ Hence, as } L \rightarrow \infty, C \rightarrow 1.$$

**10.22** Analysis of the structure of Figure P10.4 yields

$$W(e^{j\omega}) = X(e^{j\omega/M}), \quad R(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega/M}),$$

$$S(e^{j\omega}) = e^{-j\omega L}R(e^{j\omega}) = e^{-j\omega L}H(e^{j\omega})X(e^{j\omega/M}), \quad Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} S(e^{-j\omega k/M} e^{j\omega/M}).$$

If the lowpass filter is assumed to be close to an ideal lowpass filter with a cutoff at  $\omega/M$ , we can assume that all images of  $X(e^{j\omega})$  are suppressed leaving only the  $k=0$  term in the expression for  $Y(e^{j\omega})$ . Hence, we can write

$$Y(e^{j\omega}) \cong \frac{1}{M} S(e^{j\omega/M}) \cong \frac{1}{M} H(e^{j\omega/M}) e^{-j\omega L/M} X(e^{j\omega}).$$

Since  $H(z)$  is a Type 1 FIR filter with exact linear phase and a delay of  $(N-1)/2 = KM$  samples and a magnitude response equal to  $M$  in the passband, we have

$$Y(e^{j\omega}) = e^{-j\omega K} e^{-j\omega L/M} X(e^{j\omega}).$$

Thus, the structure of Figure P10.4 is approximately an allpass filter with a fixed delay of  $K$  samples and a variable noninteger delay of  $L/M$  samples.

**10.23** An ideal  $M$ -th band lowpass filter  $H(z)$  is characterized by a frequency response

$$H(e^{j\omega}) = \begin{cases} 1, & -\frac{\pi}{M} \leq \omega \leq \frac{\pi}{M}, \\ 0, & \text{otherwise.} \end{cases}$$

$H(z)$  can be expressed in an  $M$ -branch polyphase form as:  $H(z) = \sum_{k=0}^{M-1} z^{-k} H_k(z^M)$ .

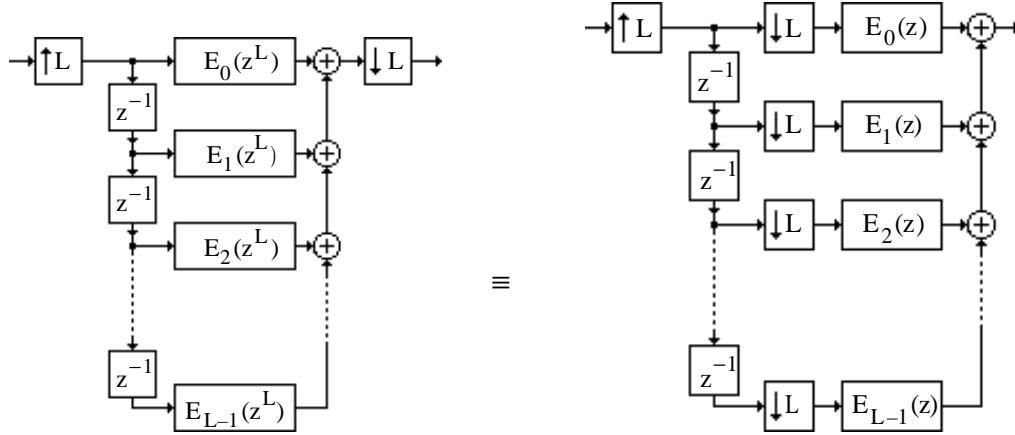
From above  $\sum_{r=0}^{M-1} H(zW_M^r) = MH_0(z^M)$ . Therefore,  $H_0(e^{j\omega M}) = \frac{1}{M} \sum_{r=0}^{M-1} H(e^{j(\omega-2\pi r/M)}) = \frac{1}{M}$ . Or

in other words,  $H_0(z^M)$  is an allpass function.

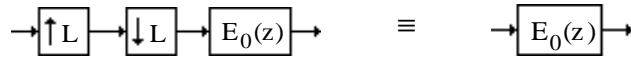
Similarly for the remaining subfilters.

**10.24** An equivalent realization of the structure of Figure 10.34 obtained by realizing the filter  $H(z)$  in a Type I polyphase form is as shown below on the left. By moving the down-sampler through the system and invoking the cascade equivalence #1 of Figure 10.14 we arrive at the structure shown below on the right.





which reduces to the structure shown below on the left from which we arrive at the simplified equivalent structure shown below on the right.



**10.25 (a)**  $H(z) = \sum_{n=0}^{N-1} h[n]z^{-n}$ .

Let  $H_0(z^2) = \frac{1}{2} \sum_{i=0}^{\frac{N}{2}-1} (h[2i] + h[2i+1])z^{-2i}$ , and  $H_1(z^2) = \frac{1}{2} \sum_{i=0}^{\frac{N}{2}-1} (h[2i] - h[2i+1])z^{-2i}$ . Then

$$(1+z^{-1})H_0(z^2) + (1-z^{-1})H_1(z^2) = \sum_{i=0}^{\frac{N}{2}-1} h[2i]z^{-2i} + \sum_{i=0}^{\frac{N}{2}-1} h[2i+1]z^{-2i+1} = H(z).$$

**(b)**  $H(z) = (1+z^{-1})H_0(z^2) + (1-z^{-1})H_1(z^2)$   
 $= (H_0(z^2) + H_1(z^2)) + z^{-1}(H_0(z^2) - H_1(z^2)) = E_0(z^2) + z^{-1}E_1(z^2)$ . Therefore,  
 $E_0(z) = H_0(z) + H_1(z)$ , and  $E_1(z) = H_0(z) - H_1(z)$ .

**(c)** Now  $H(z) = \begin{bmatrix} 1 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} H_0(z^2) \\ H_1(z^2) \end{bmatrix} = \begin{bmatrix} 1 & z^{-1} \end{bmatrix} \begin{bmatrix} H_0(z^2) + H_1(z^2) \\ H_0(z^2) - H_1(z^2) \end{bmatrix}$   
 $= (H_0(z^2) + H_1(z^2)) + z^{-1}(H_0(z^2) - H_1(z^2)) = (1+z^{-1})H_0(z^2) + (1-z^{-1})H_1(z^2)$ .

**(d)** If  $L = 2$ , i.e.  $N = 2^2$ , then we can express  $H_0(z^2) = (1+z^{-2})H_{00}(z^4) + (1-z^{-2})H_{01}(z^4)$ , and  $H_1(z^2) = (1+z^{-2})H_{10}(z^4) + (1-z^{-2})H_{11}(z^4)$ . Substituting these expressions in  $H(z) = (1+z^{-1})H_0(z^2) + (1-z^{-1})H_1(z^2)$  we get  
 $H(z) = (1+z^{-1})[(1+z^{-2})H_{00}(z^4) + (1-z^{-2})H_{01}(z^4)] + (1-z^{-1})[(1+z^{-2})H_{10}(z^4) + (1-z^{-2})H_{11}(z^4)]$   
 $= (1+z^{-1})(1+z^{-2})H_{00}(z^4) + (1+z^{-1})(1-z^{-2})H_{01}(z^4)$

$$\begin{aligned}
& +(1-z^{-1})(1+z^{-2})H_{10}(z^4) + (1-z^{-1})(1-z^{-2})H_{11}(z^4) \\
= & \begin{bmatrix} 1 & z^{-1} & z^{-2} & z^{-3} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} H_{00}(z^4) \\ H_{01}(z^4) \\ H_{10}(z^4) \\ H_{11}(z^4) \end{bmatrix} = \begin{bmatrix} 1 & z^{-1} & z^{-2} & z^{-3} \end{bmatrix} \mathbf{R}_4 \begin{bmatrix} \hat{H}_0(z^4) \\ \hat{H}_1(z^4) \\ \hat{H}_2(z^4) \\ \hat{H}_3(z^4) \end{bmatrix}.
\end{aligned}$$

Continuing this process it is easy to establish that for  $N = 2^L$ , we have

$$\mathbf{H}(z) = \begin{bmatrix} 1 & z^{-1} & \mathbf{L} & z^{-(L-1)} \end{bmatrix} \mathbf{R}_L \begin{bmatrix} \hat{H}_0(z^L) \\ \hat{H}_1(z^L) \\ \vdots \\ \hat{H}_{L-1}(z^L) \end{bmatrix}.$$

**10.26** Now  $\mathbf{H}(z) = \begin{bmatrix} 1 & z^{-1} & z^{-2} & z^{-3} \end{bmatrix} \mathbf{R}_4 \begin{bmatrix} \hat{H}_0(z^4) \\ \hat{H}_1(z^4) \\ \hat{H}_2(z^4) \\ \hat{H}_3(z^4) \end{bmatrix} = \begin{bmatrix} 1 & z^{-1} & z^{-2} & z^{-3} \end{bmatrix} \begin{bmatrix} E_0(z^4) \\ E_1(z^4) \\ E_2(z^4) \\ E_3(z^4) \end{bmatrix}.$

Therefore, 
$$\begin{bmatrix} \hat{H}_0(z) \\ \hat{H}_1(z) \\ \hat{H}_2(z) \\ \hat{H}_3(z) \end{bmatrix} = \mathbf{R}_4^{-1} \begin{bmatrix} E_0(z) \\ E_1(z) \\ E_2(z) \\ E_3(z) \end{bmatrix} = \frac{1}{4} \mathbf{R}_4 \begin{bmatrix} E_0(z) \\ E_1(z) \\ E_2(z) \\ E_3(z) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} E_0(z) \\ E_1(z) \\ E_2(z) \\ E_3(z) \end{bmatrix}.$$

A length-16 Type 1 linear-phase FIR transfer function is of the form

$$\begin{aligned}
\mathbf{H}(z) = & h_0 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + h_4 z^{-4} + h_5 z^{-5} + h_6 z^{-6} + h_7 z^{-7} \\
& + h_7 z^{-8} + h_6 z^{-9} + h_5 z^{-10} + h_4 z^{-11} + h_3 z^{-12} + h_2 z^{-13} + h_1 z^{-14} + h_0 z^{-15}.
\end{aligned}$$

Hence,  $E_0(z) = h_0 + h_4 z^{-1} + h_7 z^{-2} + h_3 z^{-3}$ ,  $E_1(z) = h_1 + h_5 z^{-1} + h_6 z^{-2} + h_2 z^{-3}$ ,

$$E_2(z) = h_2 + h_6 z^{-1} + h_5 z^{-2} + h_1 z^{-3}, \quad E_3(z) = h_3 + h_7 z^{-1} + h_4 z^{-2} + h_0 z^{-3}.$$

Thus,  $\hat{H}_0(z) = g_0 + g_1 z^{-1} + g_1 z^{-2} + g_0 z^{-3}$ ,  $\hat{H}_1(z) = g_2 + g_3 z^{-1} - g_3 z^{-2} - g_2 z^{-3}$ ,

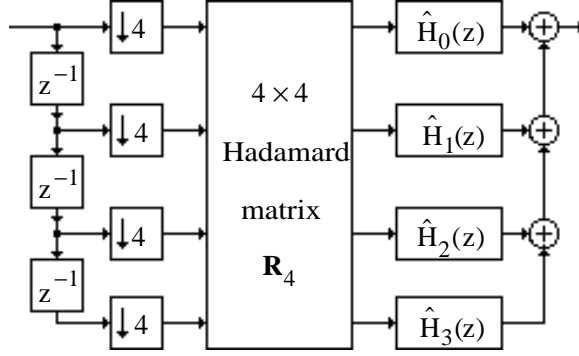
$$\hat{H}_2(z) = g_4 + g_5 z^{-1} - g_4 z^{-2} - g_5 z^{-3}, \quad \hat{H}_3(z) = g_6 + g_7 z^{-1} + g_7 z^{-2} + g_6 z^{-3},$$

where  $g_0 = \frac{1}{4}(h_0 + h_1 + h_2 + h_3)$ ,  $g_1 = \frac{1}{4}(h_4 + h_5 + h_6 + h_7)$ ,  $g_2 = \frac{1}{4}(h_0 - h_1 + h_2 - h_3)$ ,

$$g_3 = \frac{1}{4}(h_4 - h_5 + h_6 - h_7), \quad g_4 = \frac{1}{4}(h_0 + h_1 - h_2 - h_3), \quad g_5 = \frac{1}{4}(h_4 + h_5 - h_6 - h_7),$$

$$g_6 = \frac{1}{4}(h_0 - h_1 - h_2 + h_3), \quad \text{and} \quad g_7 = \frac{1}{4}(h_4 - h_5 - h_6 + h_7). \quad \text{Observe that } \hat{H}_0(z) \text{ and}$$

$\hat{H}_3(z)$  are Type 1 linear-phase FIR transfer functions, where as,  $\hat{H}_1(z)$  and  $\hat{H}_2(z)$  are Type 2 linear-phase FIR transfer functions. A computationally efficient realization of a factor-of-4 decimator using a four-band structural subband decomposition of the decimation filter  $\mathbf{H}(z)$  is developed below.



Because of the symmetric or antisymmetric impulse response, each subband filter  $\hat{H}_i(z)$  can be realized using only 2 multipliers. Hence, the final realization uses only 8 multipliers. Note also that by delay-sharing, the total number of delays in implementing the four subband filters can be reduced to 3.

**10.27**  $y[n] = P_{-2}(\alpha)x[n-2] + P_{-1}(\alpha)x[n-1] + P_0(\alpha)x[n] + P_1(\alpha)x[n+1] + P_2(\alpha)x[n+2]$  where

$$P_{-2}(\alpha) = \frac{(\alpha+1)\alpha(\alpha-1)(\alpha-2)}{(-2+1)(-2)(-2-1)(-2-2)} = \frac{1}{24}(\alpha^4 - 2\alpha^3 - \alpha^2 + 2\alpha),$$

$$P_{-1}(\alpha) = \frac{(\alpha+2)\alpha(\alpha-1)(\alpha-2)}{(-1+2)(-1)(-1-1)(-1-2)} = -\frac{1}{6}(\alpha^4 - \alpha^3 - 4\alpha^2 + 4\alpha),$$

$$P_0(\alpha) = \frac{(\alpha+2)(\alpha+1)(\alpha-1)(\alpha-2)}{(0+2)(0+1)(0-1)(0-2)} = \frac{1}{4}(\alpha^4 - 5\alpha^2 + 4),$$

$$P_1(\alpha) = \frac{(\alpha+2)(\alpha+1)\alpha(\alpha-2)}{(1+2)(1+1)(1-0)(1-2)} = -\frac{1}{6}(\alpha^4 + \alpha^3 - 4\alpha^2 - 4\alpha),$$

$$P_2(\alpha) = \frac{(\alpha+2)(\alpha+1)\alpha(\alpha-1)}{(2+2)(2+1)(2-0)(2-1)} = \frac{1}{24}(\alpha^4 + 2\alpha^3 - \alpha^2 - 2\alpha).$$

We consider the computation of  $y[n]$ ,  $y[n+1]$ ,  $y[n+2]$ ,  $y[n+3]$  using 5 input samples:  $x[n-2]$  through  $x[n+2]$ .

For  $\alpha_0 = 0$ ,  $P_{-2}(\alpha_0) = 0$ ,  $P_{-1}(\alpha_0) = 0$ ,  $P_0(\alpha_0) = 1$ ,  $P_1(\alpha_0) = 0$ , and  $P_2(\alpha_0) = 0$ .

For  $\alpha_1 = 5/4$ ,  $P_{-2}(\alpha_1) = -0.022$ ,  $P_{-1}(\alpha_1) = 0.127$ ,  $P_0(\alpha_1) = -0.3428$ ,  $P_1(\alpha_1) = 1.1426$ , and  $P_2(\alpha_1) = 0.0952$ .

For  $\alpha_2 = 10/4$ ,  $P_{-2}(\alpha_2) = 0.2734$ ,  $P_{-1}(\alpha_2) = -1.4062$ ,  $P_0(\alpha_2) = 2.9531$ ,  $P_1(\alpha_2) = -3.2812$ , and  $P_2(\alpha_2) = 2.4609$ .

For  $\alpha_3 = 15/4$ ,  $P_{-2}(\alpha_3) = 3.5718$ ,  $P_{-1}(\alpha_3) = -17.2949$ ,  $P_0(\alpha_3) = 32.86$ ,  $P_1(\alpha_3) = -29.873$ , and  $P_2(\alpha_3) = 11.7358$ .

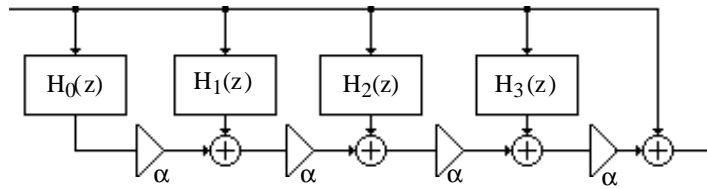
The block filter implementation is thus given by

$$\begin{bmatrix} y[n] \\ y[n+1] \\ y[n+2] \\ y[n+3] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -0.022 & 0.127 & -0.3428 & 1.1426 & 0.0952 \\ 0.2734 & -1.4062 & 2.9531 & -3.2812 & 2.4609 \\ 3.5718 & -17.2949 & 32.86 & -29.873 & 11.7358 \end{bmatrix} \begin{bmatrix} x[n-2] \\ x[n-1] \\ x[n] \\ x[n+1] \\ x[n+2] \end{bmatrix}.$$

Another implementation is given by

$$\begin{aligned} y[n] = & \alpha^4 \left( \frac{1}{24} x[n-2] - \frac{1}{6} x[n-1] + \frac{1}{4} x[n] - \frac{1}{6} x[n+1] + \frac{1}{24} x[n+2] \right) \\ & + \alpha^3 \left( -\frac{1}{12} x[n-2] + \frac{1}{6} x[n-1] - \frac{1}{6} x[n+1] + \frac{1}{12} x[n+2] \right) \\ & + \alpha^2 \left( -\frac{1}{24} x[n-2] + \frac{4}{6} x[n-1] - \frac{5}{4} x[n] + \frac{4}{6} x[n+1] - \frac{1}{24} x[n+2] \right) \\ & + \alpha \left( \frac{1}{12} x[n-2] - \frac{4}{6} x[n-1] + \frac{4}{6} x[n+1] - \frac{1}{12} x[n+2] \right) + x[n]. \end{aligned}$$

The Farrow structure implementation of the interpolator is shown below:



where  $H_0(z) = \frac{1}{24} z^{-2} - \frac{1}{6} z^{-1} + \frac{1}{4} - \frac{1}{6} z + \frac{1}{24} z^2$ ,  $H_1(z) = -\frac{1}{12} z^{-2} + \frac{1}{6} z^{-1} + \frac{1}{6} z - \frac{1}{24} z^2$ ,

$H_2(z) = -\frac{1}{24} z^{-2} + \frac{4}{6} z^{-1} - \frac{5}{4} + \frac{4}{6} z - \frac{1}{24} z^2$ , and  $H_3(z) = \frac{1}{12} z^{-2} - \frac{4}{6} z^{-1} + \frac{4}{6} z - \frac{1}{12} z^2$ .

**10.28** For a half-band zero-phase lowpass filter, the transfer function is of the form

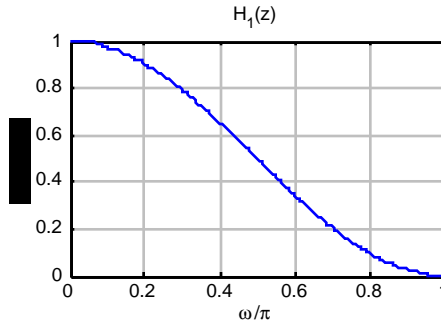
$$H(z) = h[0] + z^{-1} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} h[2n] z^{-2n}, \text{ where } h[0] = \frac{1}{2}. \text{ If the half-band filter has a zero at}$$

$$z = -1, \text{ then } H(-1) = h[0] - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} h[2n] = 0 \text{ or } h[0] = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} h[2n].$$

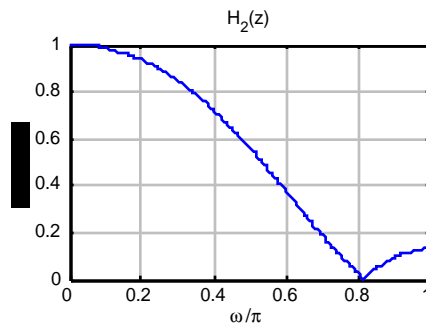
**10.29** From Eq. (10.81), a zero-phase half-band filter  $H(z)$  satisfies the condition  $H(z) + H(-z) = \text{a constant}$ .

(a) The zero-phase equivalent here is given by  $H_1(z) = z + 2 + z^{-1}$ . Hence,

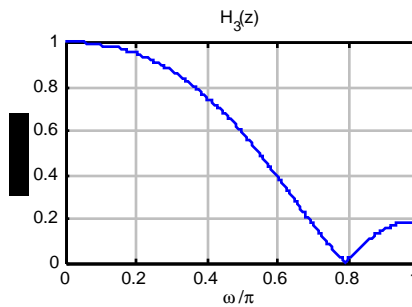
$H_1(z) + H_1(-z) = z + 2 + z^{-1} - z + 2 - z^{-1} = 4$ . A plot of the magnitude response of  $H_1(z)$  is given below:



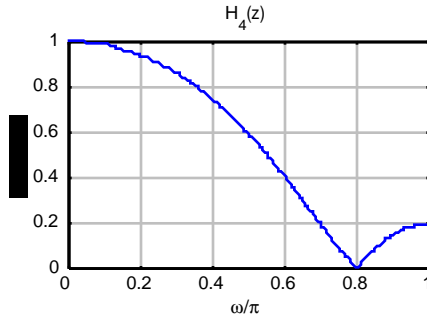
(b) The zero-phase equivalent here is given by  $H_2(z) = -z^3 + 9z + 16 + 9z^{-1} - z^{-3}$ . Hence,  $H_2(z) + H_2(-z) = -z^3 + 9z + 16 + 9z^{-1} - z^{-3} + z^3 - 9z + 16 - 9z^{-1} + z^{-3} = 32$ . A plot of the magnitude response of  $H_2(z)$  is given below:



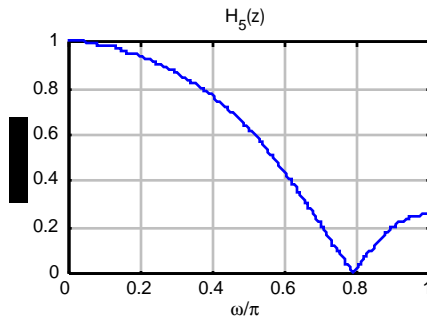
(c) The zero-phase equivalent here is given by  $H_3(z) = -3z^3 + 19z + 32 + 19z^{-1} - 3z^{-3}$ . Hence  $H_3(z) + H_3(-z) = -3z^3 + 19z + 32 + 19z^{-1} - 3z^{-3} + 3z^3 - 19z + 32 - 19z^{-1} + 3z^{-3} = 64$ . A plot of the magnitude response of  $H_3(z)$  is given below:



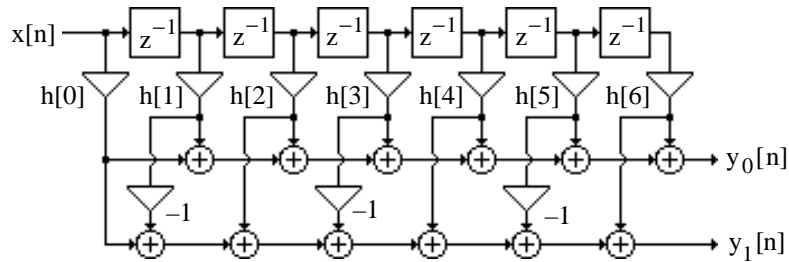
(d) The zero-phase equivalent here is given by  $H_4(z) = 3z^5 - 25z^3 + 150z + 256 + 150z^{-1} - 25z^{-3} + 3z^{-5}$ . Hence,  $H_4(z) + H_4(-z) = 512$ . A plot of the magnitude response of  $H_4(z)$  is given below:



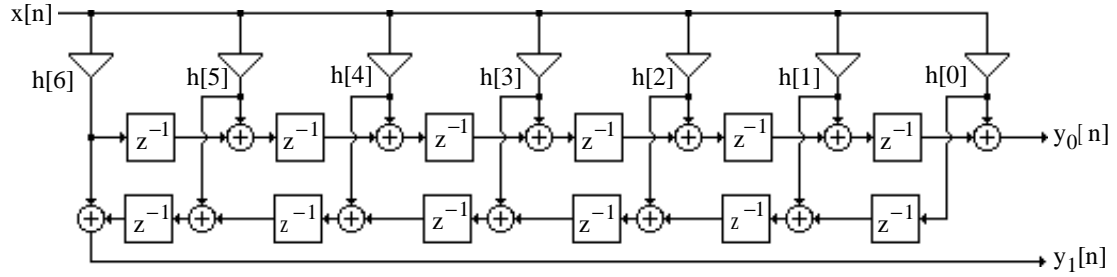
(e) The zero-phase equivalent here is given by  $H_5(z) = 9z^5 - 44z^3 + 208z + 346 + 208z^{-1} - 44z^{-3} + 9z^{-5}$ . Hence  $H_5(z) + H_5(-z) = 692$ . A plot of the magnitude response of  $H_5(z)$  is given below:



**10.30**  $H_0(z) = \frac{Y_0(z)}{X(z)} = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6}$ ,  
 $H_1(z) = \frac{Y_1(z)}{X(z)} = h[0] - h[1]z^{-1} + h[2]z^{-2} - h[3]z^{-3} + h[4]z^{-4} - h[5]z^{-5} + h[6]z^{-6}$ .



**10.31**  $H_0(z) = \frac{Y_0(z)}{X(z)} = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6}$ ,  
 $H_1(z) = \frac{Y_1(z)}{X(z)} = h[6] + h[5]z^{-1} + h[4]z^{-2} + h[3]z^{-3} + h[2]z^{-4} + h[1]z^{-5} + h[0]z^{-6}$ .



**10.31 (a)** 
$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \\ H_3(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} E_0(z^4) \\ z^{-1}E_1(z^4) \\ z^{-2}E_2(z^4) \\ z^{-3}E_3(z^4) \end{bmatrix}. \text{ Hence,}$$

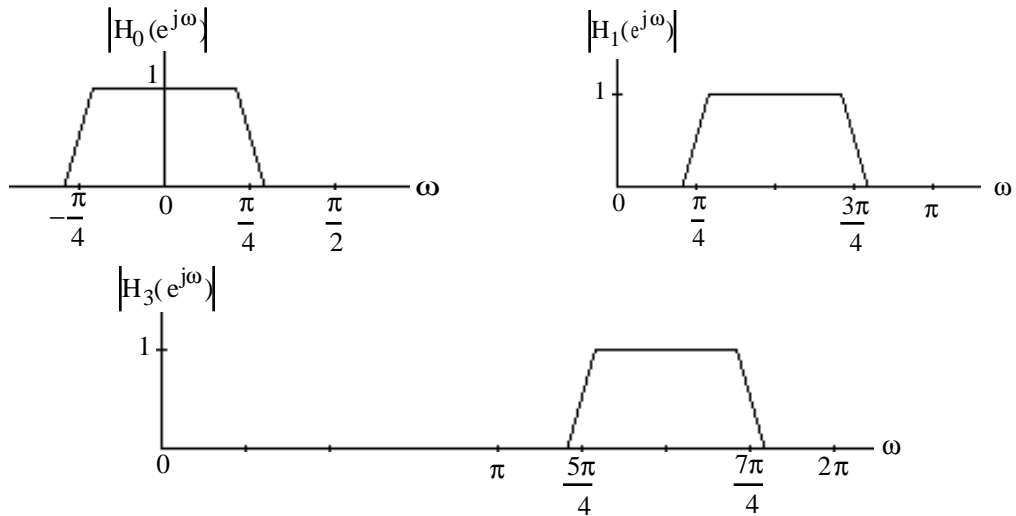
$$H_0(z) = E_0(z^4) + z^{-1}E_1(z^4) + z^{-2}E_2(z^4) + z^{-3}E_3(z^4) \\ = 1 + 2z^{-1} + 4z^{-2} + z^{-3} + 0.3z^{-4} - 1.5z^{-5} - 0.9z^{-6} + 3.7z^{-7} - 0.8z^{-8} + 3.1z^{-9} + 2.3z^{-10} + 1.7z^{-11},$$

$$H_1(z) = E_0(z^4) - jz^{-1}E_1(z^4) - z^{-2}E_2(z^4) + jz^{-3}E_3(z^4) = 1 - j2z^{-1} - 4z^{-2} \\ + jz^{-3} + 0.3z^{-4} + j1.5z^{-5} - 0.9z^{-6} + j3.7z^{-7} - 0.8z^{-8} - j3.1z^{-9} + 2.3z^{-10} + j1.7z^{-11},$$

$$H_2(z) = E_0(z^4) - z^{-1}E_1(z^4) + z^{-2}E_2(z^4) - z^{-3}E_3(z^4) = 1 - 2z^{-1} + 4z^{-2} \\ - z^{-3} + 0.3z^{-4} + 1.5z^{-5} - 0.9z^{-6} - 3.7z^{-7} - 0.8z^{-8} - 3.1z^{-9} + 2.3z^{-10} - 1.7z^{-11},$$

$$H_3(z) = E_0(z^4) + jz^{-1}E_1(z^4) - z^{-2}E_2(z^4) - jz^{-3}E_3(z^4) = 1 + j2z^{-1} - 4z^{-2} \\ - jz^{-3} + 0.3z^{-4} - j1.5z^{-5} + 0.9z^{-6} - j3.7z^{-7} + 0.8z^{-8} + j3.1z^{-9} - 2.3z^{-10} - j1.7z^{-11}$$

**(b)**



**10.33**  $Y(z) = (H_0(z)G_0(z) + H_1(z)G_1(z))X(z)$ . Now,  $H_0(z) = \frac{1+z^{-1}}{2}$ , and  $H_1(z) = \frac{1-z^{-1}}{2}$ . Choose

$$G_0(z) = \frac{1+z^{-1}}{2}, \text{ and } G_1(z) = -\frac{1-z^{-1}}{2}. \text{ Then,}$$

$$Y(z) = \left( \frac{1}{4}(1+z^{-1})^2 - \frac{1}{4}(1-z^{-1})^2 \right) X(z) = \frac{1}{4}(1+2z^{-1}+z^{-2} - 1+2z^{-1}-z^{-2})X(z) = z^{-1}X(z).$$

Or in other words,  $y[n] = x[n-1]$  indicating that the structure of Figure P10.7 is a perfect reconstruction filter bank.

**10.34 (a)** Since  $H_0(z)$  and  $H_1(z)$  are power-complementary,  $H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = 1$ .

$$\text{Now } Y(z) = (H_0(z)G_0(z) + H_1(z)G_1(z))X(z) = z^{-N}(H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}))X(z)$$

$= z^{-N}X(z)$ . Or in other words,  $y[n] = x[n-N]$  indicating that the structure of Figure P10.7 is a perfect reconstruction filter bank.

**(b)** If  $H_0(z)$  and  $H_1(z)$  are causal FIR transfer functions of order  $N$  each,  $H_0(z)$  and  $H_1(z)$  are polynomials in  $z^{-1}$ . As a result,  $H_0(z^{-1})$  and  $H_1(z^{-1})$  are polynomials in  $z$  with the highest power being  $z^N$ . Hence,  $z^{-N}H_0(z^{-1})$  and  $z^{-N}H_1(z^{-1})$  are polynomials in  $z^{-1}$ , making the synthesis filters  $G_0(z)$  and  $G_1(z)$  causal FIR transfer functions of order  $N$  each.

**(c)** From Figure P10.7, for perfect reconstruction we require  $H_0(z)G_0(z) + H_1(z)G_1(z) = z^{-N}$ .

From part (a) we note that the PR condition is satisfied with  $G_0(z) = z^{-N}H_0(z^{-1})$  and

$G_1(z) = z^{-N}H_1(z^{-1})$ , if  $H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = 1$ , i.e. if  $H_0(z)$  and  $H_1(z)$  are power-

complementary. The last condition is satisfied if and only if  $H_0(z) = \frac{z^{-N/2}}{2}(z^{-n_0} + z^{n_0})$ ,

and  $H_1(z) = \frac{z^{-N/2}}{2}(z^{-n_0} - z^{n_0})$ . As a result,  $G_0(z)$  and  $G_1(z)$  are also of the form

$$G_0(z) = \frac{z^{-N/2}}{2}(z^{-n_0} + z^{n_0}), \text{ and } G_1(z) = -\frac{z^{-N/2}}{2}(z^{-n_0} - z^{n_0}).$$

**10.35** From Eq. (10.97),  $Y(z) = T(z)X(z) + A(z)X(-z)$ . Let  $Z^{-1}\{T(z)\} = t[n]$ , and  $Z^{-1}\{A(z)\} = a[n]$ . Then an inverse z-transform of Eq. (10.97) then yields

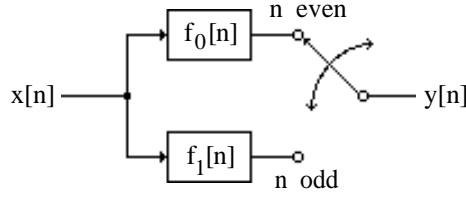
$$y[n] = \sum_{l=-\infty}^{\infty} t[l]x[n-1] + \sum_{l=-\infty}^{\infty} a[l](-1)^{n-1}x[n-1] = \sum_{l=-\infty}^{\infty} (t[l] + (-1)^{n-1}a[l])x[n-1].$$

Define  $f_0[n] = t[n] + (-1)^{-n}a[n]$ , and  $f_1[n] = t[n] - (-1)^{-n}a[n]$ . Then we can write

$$y[n] = \begin{cases} \sum_{l=-\infty}^{\infty} f_0[l]x[n-1], & \text{for } n \text{ even,} \\ \sum_{l=-\infty}^{\infty} f_1[l]x[n-1], & \text{for } n \text{ odd.} \end{cases}$$

The corresponding equivalent realization of the 2-channel QMF bank is therefore as indicated below:





As can be seen from the above representation, the QMF bank is, in general, a linear, time-varying system with a period 2. Note that if  $A(z) = 0$ , then it becomes a linear, time-invariant system.

**10.36 (a)**  $H_0(z) = \frac{1}{4} \{A_0(z^2) + z^{-1}A_1(z^2)\}$ , and  $H_1(z) = \frac{1}{4} \{A_0(z^2) - z^{-1}A_1(z^2)\}$ . Hence,

$$\mathbf{E}(z) = \begin{bmatrix} A_0(z) & A_1(z) \\ A_0(z) & -A_1(z) \end{bmatrix}, \text{ and } \mathbf{R}(z) = \begin{bmatrix} A_1(z) & A_1(z) \\ A_0(z) & -A_0(z) \end{bmatrix}.$$

**(b)** To prove  $\mathbf{E}(z)$  is lossless, we need to show  $\mathbf{E}^\dagger(e^{j\omega})\mathbf{E}(e^{j\omega}) = c^2 \mathbf{I}$  for some constant  $c$ :

$$\begin{aligned} \mathbf{E}^\dagger(e^{j\omega})\mathbf{E}(e^{j\omega}) &= \begin{bmatrix} A_0^*(e^{j\omega}) & A_0^*(e^{j\omega}) \\ A_1^*(e^{j\omega}) & -A_1^*(e^{j\omega}) \end{bmatrix} \begin{bmatrix} A_0(e^{j\omega}) & A_1(e^{j\omega}) \\ A_0(e^{j\omega}) & -A_1(e^{j\omega}) \end{bmatrix} \\ &= \begin{bmatrix} |A_0(e^{j\omega})|^2 + |A_0(e^{j\omega})|^2 & 0 \\ 0 & |A_1(e^{j\omega})|^2 + |A_1(e^{j\omega})|^2 \end{bmatrix} = 2\mathbf{I}. \text{ Hence } \mathbf{E}(z) \text{ is lossless.} \end{aligned}$$

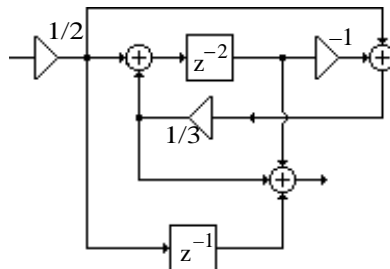
**(c)**  $\mathbf{R}(z)\mathbf{E}(z) = \frac{1}{4} \begin{bmatrix} 2A_1(z)A_0(z) & 0 \\ 0 & 2A_1(z)A_0(z) \end{bmatrix} = \frac{A_1(z)A_0(z)}{2} \mathbf{I}$ . Therefore,

$$\mathbf{R}(z) = \frac{A_1(z)A_0(z)}{2} \mathbf{E}^{-1}(z).$$

**(d)** As in (c),  $\mathbf{R}(z)\mathbf{E}(z) = \frac{A_1(z)A_0(z)}{2}$ .

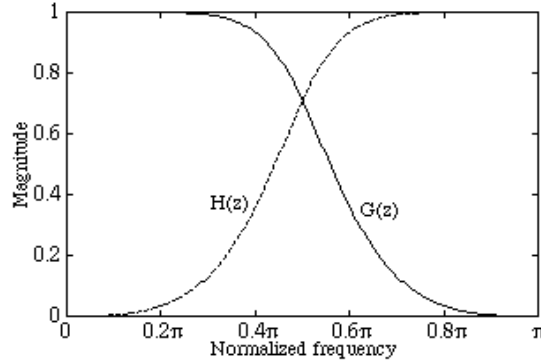
**10.37 (a)**  $G(z) = \frac{1+3z^{-1}+3z^{-2}+z^{-3}}{6+2z^{-2}} = \frac{1}{2} \left( \frac{1+3z^{-2}}{3+z^{-2}} + \frac{3z^{-1}+z^{-3}}{3+z^{-2}} \right) = \frac{1}{2} \left( \frac{1+3z^{-2}}{3+z^{-2}} + z^{-1} \right)$   
 $= \frac{1}{2} (A_0(z^2) + z^{-1}A_1(z^2))$ , where  $A_0(z) = \frac{1+3z^{-1}}{3+z^{-1}}$ , and  $A_1(z) = z^{-1}$ .

**(b)**



$$(c) H(z) = \frac{1}{2}(A_0(z^2) - z^{-1}A_1(z^2)) = \frac{1}{2}\left(\frac{1+3z^{-2}}{3+z^{-2}} - z^{-1}\right) = \frac{1}{2}\left(\frac{1-3z^{-1}+3z^{-2}-z^{-3}}{3+z^{-2}}\right) = \frac{(1-z^{-1})^3}{6+2z^{-2}}.$$

(d)



**10.38** Now the magnitude square function of an N-th order analog lowpass Butterworth transfer function  $H_a(s)$  is given by  $|H_a(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$ , where  $\Omega_c$  is the 3-dB cutoff frequency.

For  $\Omega_c = 1$ , then  $|H_a(j\Omega)|^2 = \frac{1}{1+\Omega^{2N}}$ . The corresponding transfer function of an N-th order

analog highpass transfer function is simply  $H_a\left(\frac{1}{s}\right)$ , whose magnitude square function is given

by  $\left|H_a\left(\frac{1}{j\Omega}\right)\right|^2 = \frac{\Omega^{2N}}{1+\Omega^{2N}}$ . As a result,  $|H_a(j\Omega)|^2 + \left|H_a\left(\frac{1}{j\Omega}\right)\right|^2 = \frac{1}{1+\Omega^{2N}} + \frac{\Omega^{2N}}{1+\Omega^{2N}} = 1$ .

Now the bilinear transformation maps the analog frequency  $\Omega$  to the digital frequency  $\omega$

through the relation  $e^{j\omega} = \frac{1-j\Omega}{1+j\Omega}$ . As,  $-e^{j\omega} = \frac{1-\frac{1}{j\Omega}}{1+\frac{1}{j\Omega}}$ , the analog frequency  $1/\Omega$  is mapped to

the digital frequency  $\pi + \omega$ . Hence the relation  $|H_a(j\Omega)|^2 + \left|H_a\left(\frac{1}{j\Omega}\right)\right|^2 = 1$ , becomes

$$\left|H_0(e^{j\omega})\right|^2 + \left|H_0(e^{j(\pi+\omega)})\right|^2 = \left|H_0(e^{j\omega})\right|^2 + \left|H_0(-e^{j\omega})\right|^2 = \left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 1, \text{ where}$$

$H_0(e^{j\omega})$  is the frequency response of the digital lowpass filter  $H_0(z)$  obtained by applying a bilinear transformation to  $H_a(s)$  and  $H_1(e^{j\omega})$  is the frequency response of the digital highpass filter  $H_1(z)$  obtained by applying a bilinear transformation to  $H_a(1/s)$ . Note that a transfer function  $H_0(z)$  satisfying the condition  $\left|H_0(e^{j\omega})\right|^2 + \left|H_0(-e^{j\omega})\right|^2 = 1$ , is called power-symmetric.

Moreover, from the relation  $\Omega = \tan(\omega/2)$  it follows that the analog 3-dB cutoff frequency  $\Omega_c = 1$  is mapped into the digital 3-dB cutoff frequency  $\omega_c = \pi/2$ . Hence,  $H(z)$  is a digital half-band filter.

Let  $H_0(z) = \frac{P_0(z)}{D_0(z)}$ , where  $P_0(z)$  and  $D_0(z)$  are polynomials in  $z^{-1}$ . Hence,

$$H_0(e^{j\omega}) = \frac{P_0(e^{j\omega})}{D_0(e^{j\omega})}. \text{ Now, } \frac{P_0(-e^{j\omega})P_0^*(-e^{j\omega})}{D_0(-e^{j\omega})D_0^*(-e^{j\omega})} = \frac{D_0(e^{j\omega})D_0^*(e^{j\omega}) - P_0(e^{j\omega})P_0^*(e^{j\omega})}{D_0(e^{j\omega})D_0^*(e^{j\omega})}.$$

Note that there are no common factors between  $P_0(e^{j\omega})$  and  $D_0(e^{j\omega})$ , and between  $P_0^*(e^{j\omega})$  and  $D_0^*(e^{j\omega})$ . As a result there are no common factors between  $P_0(-e^{j\omega})P_0^*(-e^{j\omega})$  and  $D_0(-e^{j\omega})D_0^*(-e^{j\omega})$ . This implies then  $D_0(e^{j\omega})D_0^*(e^{j\omega}) = D_0(-e^{j\omega})D_0^*(-e^{j\omega})$ . As a result,  $D_0(e^{j\omega}) = D_0(-e^{j\omega})$ , or  $D_0(e^{j\omega}) = d_0(e^{j2\omega})$ . Hence,  $D_0(z) = d_0(z^2)$ .

Since  $H_0(z) = \frac{P_0(z)}{D_0(z)} = \frac{P_0(z)}{d_0(z^2)}$ , it follows then  $H_1(z) = \frac{P_0(-z)}{d_0(z^2)}$ . We have shown earlier that

$H_0(z)$  and  $H_1(z)$  are power-complementary. Also  $P_0(z)$  is a symmetric polynomial of odd order and  $P_0(-z)$  is an anti-symmetric polynomial of odd order. As a result we can express

$$H_0(z) = \frac{1}{2}(A_0(z) + A_1(z)), \text{ and } H_1(z) = \frac{1}{2}(A_0(z) - A_1(z)). \text{ But } H_1(z) = H_0(-z). \text{ Hence}$$

$$H_0(z) = \frac{1}{2}(A_0(-z) - A_1(-z)). \text{ It therefore follows that } A_0(z) = A_0(-z) = A_0(z^2), \text{ and}$$

$$A_1(z) = -A_1(-z) = z^{-1}A_1(z^2). \text{ Thus, } H_0(z) = \frac{1}{2}(A_0(z^2) + z^{-1}A_1(z^2)).$$

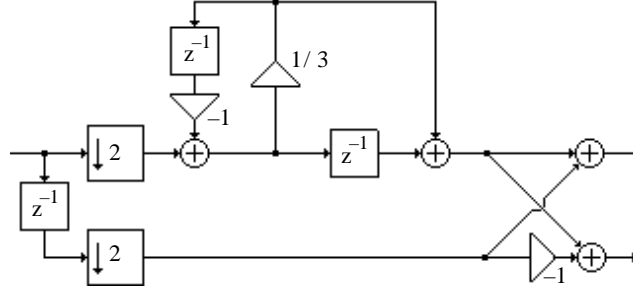
**10.39**  $G_a(s) = \frac{1}{(s+1)(s^2+s+1)}$ . The corresponding digital transfer function  $H_0(z)$  obtained by a bilinear transformation is given by

$$\begin{aligned} H_0(z) &= G_a(s) \Big|_{s=\frac{z-1}{z+1}} = \frac{(z+1)^3}{(z-1+z+1)((z-1)^2+(z^2-1)+(z+1)^2)} = \frac{(z+1)^3}{2z(3z^2+1)} = \frac{(1+z^{-1})^3}{6+2z^{-2}} \\ &= \frac{1+3z^{-1}+3z^{-2}+z^{-3}}{6+2z^{-2}} = \frac{1}{2} \left[ \frac{1+3z^{-2}}{3+z^{-2}} + z^{-1} \right] = \frac{1}{2} [A_0(z^2) + z^{-1}A_1(z^2)] \text{ where } A_0(z) = \frac{1+3z^{-1}}{3+z^{-1}} \\ &\text{and } A_1(z) = 1. \end{aligned}$$

The corresponding power-complementary transfer function is given by

$$H_1(z) = \frac{1}{2} \left[ \frac{1+3z^{-2}}{3+z^{-2}} - z^{-1} \right] = \frac{(1-z^{-1})^3}{6+2z^{-2}}.$$

A realization of the QMF bank is as shown below:



**10.40**  $G_a(s) = \frac{1}{s^5 + 3.2361s^4 + 5.2361s^3 + 5.2361s^2 + 3.2361s + 1}$ . The corresponding digital transfer

function obtained by a bilinear transformation is given by

$$\begin{aligned}
 H_0(z) &= G_a(s) \Big|_{s=\frac{z-1}{z+1}} = \frac{(1+z^{-1})^5}{18.9443 + 12z^{-2} + 1.0557z^{-4}} = \frac{0.0528(1+z^{-1})^5}{1 + 0.6334z^{-2} + 0.0557z^{-4}} \\
 &= \frac{0.0528 + 0.2639z^{-1} + 0.5279z^{-2} + 0.5279z^{-3} + 0.2639z^{-4} + 0.0528z^{-5}}{1 + 0.6334z^{-2} + 0.0557z^{-4}} \\
 &= \frac{0.0528 + 0.5279z^{-2} + 0.2639z^{-4}}{1 + 0.6334z^{-2} + 0.0557z^{-4}} + \frac{0.2639z^{-1} + 0.5279z^{-3} + 0.0528z^{-5}}{1 + 0.6334z^{-2} + 0.0557z^{-4}} \\
 &= \frac{0.0528(1 + 9.4704z^{-2})}{1 + 0.1056z^{-2}} + z^{-1} \frac{0.2639(1 + 1.8948z^{-2})}{1 + 0.5278z^{-2}} \\
 &= \frac{1}{2} \left\{ \frac{0.1056 + z^{-2}}{1 + 0.1056z^{-2}} + z^{-1} \frac{0.5278 + z^{-2}}{1 + 0.5278z^{-2}} \right\} = \frac{1}{2} \{ A_0(z^2) + z^{-1} A_1(z^2) \}, \text{ where} \\
 A_0(z) &= \frac{0.1056 + z^{-1}}{1 + 0.1056z^{-1}}, \text{ and } A_1(z) = \frac{0.5278 + z^{-1}}{1 + 0.5278z^{-1}}.
 \end{aligned}$$

The corresponding power-complementary transfer function is given by

$$H_1(z) = \frac{1}{2} \{ A_0(z^2) - z^{-1} A_1(z^2) \} = \frac{1}{2} \left\{ \frac{0.1056 + z^{-2}}{1 + 0.1056z^{-2}} - z^{-1} \frac{0.5278 + z^{-2}}{1 + 0.5278z^{-2}} \right\}.$$

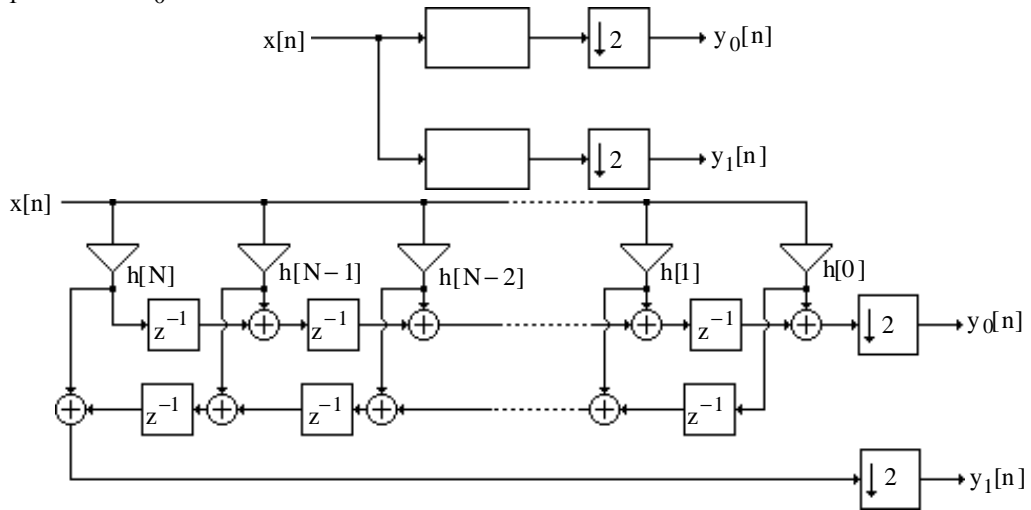
In the realization of a magnitude-preserving QMF bank as shown in Figure 10.47, the realization of the allpass filters  $A_0(z)$  and  $A_1(z)$  require 1 multiplier each, and hence the realization of the analysis (and the synthesis) filter bank requires a total of 2 multipliers.

**10.41 (a)** Total number of multipliers required is  $4(2N-1)$ . Hence, total number of multiplications per second is equal to  $4(2N-1)F_T = 4(2N-1)/T$  where  $F_T = 1/T$  is the sampling frequency in Hz.

**(b)** In Figure 10.47,  $H_0(z) = \frac{1}{2} \{ A_0(z^2) + z^{-1} A_1(z^2) \}$ , and  $H_1(z) = \frac{1}{2} \{ A_0(z^2) - z^{-1} A_1(z^2) \}$ . If order of  $A_0(z)$  is  $K$  and the order of  $A_1(z)$  is  $L$ , then the order of  $H_0(z)$  is  $2K + 2L + 1 = N$ . Hence  $K+L = (N-1)/2$ . The total number of multipliers needed to implement  $A_0(z)$  is  $K$  while the total number of multipliers needed to implement  $A_1(z)$  is  $L$ . Hence the total number of multipliers required to implement the structure of Figure 10.47 is  $2(K+L) = N-1$ . However, the

multipliers here are operating at half of the sampling rate of the input  $x[n]$ . As a result, the total number of multiplications per second in this case  $(N-1)F_T/2 = (N-1)/2T$ .

**10.42**  $H_1(z) = z^{-N}H_0(z^{-1})$ .



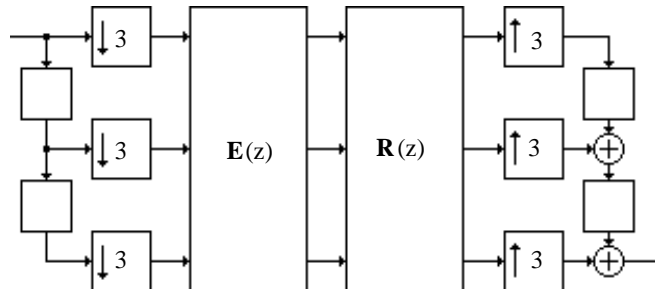
**10.43**  $\mathbf{E}(z) = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 13 & 9 & 7 \\ 3 & 9 & 11 & 10 \\ 2 & 7 & 10 & 15 \end{bmatrix}$ . For  $y[n] = 3x[n-3]$ , we require  $\mathbf{R}(z)\mathbf{E}(z) = 3\mathbf{I}$  or

$$\mathbf{R}(z) = 3\mathbf{E}^{-1}(z) = 3 \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 13 & 9 & 7 \\ 3 & 9 & 11 & 10 \\ 2 & 7 & 10 & 15 \end{bmatrix}^{-1} = \begin{bmatrix} -3.8333 & -1.5 & 4.8333 & -2.333 \\ -1.5833 & 0.25 & 0.5833 & -0.333 \\ 4.5 & 0.5 & -2.5 & 1.0 \\ -1.75 & -0.25 & 0.75 & 0 \end{bmatrix}$$

**10.44**  $\mathbf{E}(z) = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ . For perfect reconstruction,

$$\mathbf{R}(z) = \mathbf{E}^{-1}(z) = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & 1.5 & -2.5 \\ -0.5 & -0.5 & 1.5 \\ 0.5 & -0.5 & 0.5 \end{bmatrix}$$

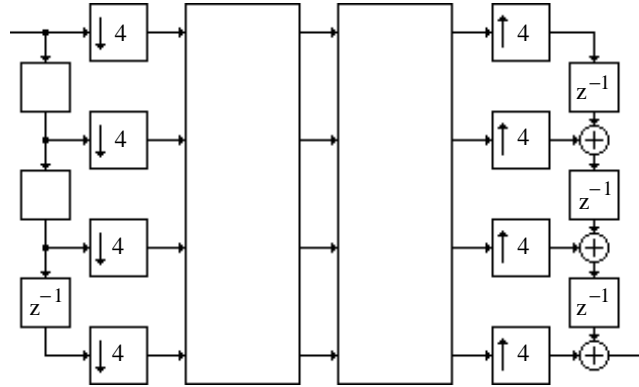
A computationally efficient realization of the filter bank is shown below:



**10.45** For perfect reconstruction we require

$$\mathbf{R}(z) = \mathbf{E}^{-1}(z) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 5 \\ 2 & 1 & 4 & 3 \\ 4 & 2 & 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -0.4 & 0.2 & 0.16 & 0.12 \\ 0.7333 & 0.2 & -0.76 & 0.3467 \\ 0.0667 & 0.2 & 0.24 & 0.0133 \\ 0.0667 & 0.2 & 0.16 & -0.2133 \end{bmatrix}. \text{ A computationally}$$

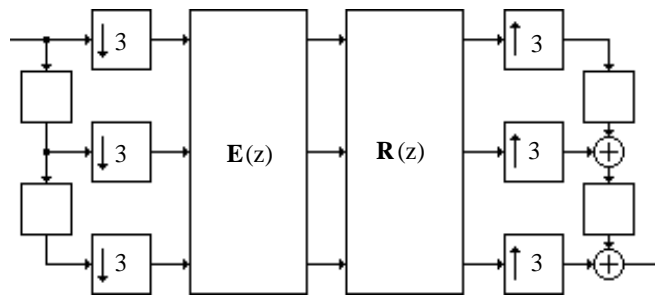
efficient realization of the filter bank is shown below:



**10.46** For perfect reconstruction we require

$$\mathbf{R}(z) = \mathbf{E}^{-1}(z) = \begin{bmatrix} 4 & 2 & 3 \\ 5 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1.2222 & -0.3333 & -1.1111 \\ -1.4444 & 0.6667 & 1.2222 \\ -0.3333 & 0 & 0.6667 \end{bmatrix}. \text{ A computationally}$$

efficient realization of the filter bank is shown below:

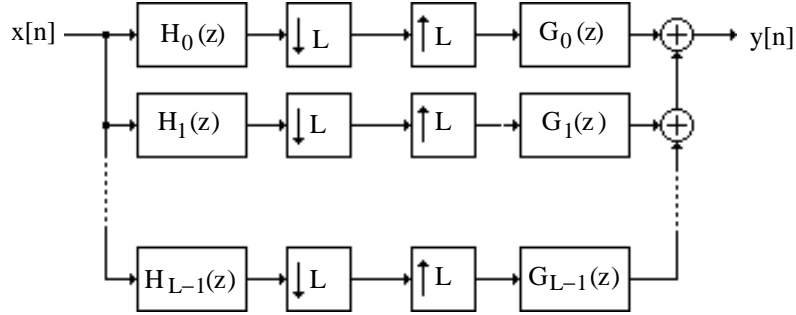


$$\mathbf{10.47} \quad \hat{X}(z) = \begin{bmatrix} G_0(z) & G_1(z) & \mathbf{L} & G_{L-1}(z) \end{bmatrix} \begin{bmatrix} H_0(z) & H_0(zW) & \mathbf{L} & H_0(zW^{L-1}) \\ H_1(z) & H_1(zW) & \mathbf{L} & H_1(zW^{L-1}) \\ \mathbb{M} & \mathbb{M} & \mathbf{O} & \mathbb{M} \\ H_{L-1}(z) & H_{L-1}(zW) & \mathbf{L} & H_{L-1}(zW^{L-1}) \end{bmatrix} \begin{bmatrix} X(z) \\ X(zW) \\ \mathbb{M} \\ X(zW^{L-1}) \end{bmatrix}.$$

To show the system of Figure 10.63 is, in general, periodic with a period  $L$ , we need to show that if  $\hat{X}_1(z)$  is the output for an input  $X_1(z)$ , and  $\hat{X}_2(z)$  is the output for an input  $X_2(z)$ , then if  $X_2(z) = z^{-L}X_1(z)$ , i.e.  $x_2[n] = x_1[n-L]$ , then the corresponding output satisfies  $\hat{X}_2(z) = z^{-L}\hat{X}_1(z)$ , i.e.  $\hat{x}_2[n] = \hat{x}_1[n-L]$ . Now,

$$\begin{aligned}
\hat{X}_2(z) &= [G_0(z) \ G_1(z) \ \mathbf{L} \ G_{L-1}(z)] \begin{bmatrix} H_0(z) & H_0(zW) & \mathbf{L} & H_0(zW^{L-1}) \\ H_1(z) & H_1(zW) & \mathbf{L} & H_1(zW^{L-1}) \\ \parallel & \parallel & \mathbf{O} & \parallel \\ H_{L-1}(z) & H_{L-1}(zW) & \mathbf{L} & H_{L-1}(zW^{L-1}) \end{bmatrix} \begin{bmatrix} X_2(z) \\ X_2(zW) \\ \parallel \\ X_2(zW^{L-1}) \end{bmatrix} \\
&= [G_0(z) \ G_1(z) \ \mathbf{L} \ G_{L-1}(z)] \begin{bmatrix} H_0(z) & H_0(zW) & \mathbf{L} & H_0(zW^{L-1}) \\ H_1(z) & H_1(zW) & \mathbf{L} & H_1(zW^{L-1}) \\ \parallel & \parallel & \mathbf{O} & \parallel \\ H_{L-1}(z) & H_{L-1}(zW) & \mathbf{L} & H_{L-1}(zW^{L-1}) \end{bmatrix} \begin{bmatrix} z^{-L} X_1(z) \\ z^{-L} W^{-L} X_1(zW) \\ \parallel \\ z^{-L} W^{-L(L-1)} X_1(zW^{L-1}) \end{bmatrix} \\
&= z^{-L} [G_0(z) \ G_1(z) \ \mathbf{L} \ G_{L-1}(z)] \begin{bmatrix} H_0(z) & H_0(zW) & \mathbf{L} & H_0(zW^{L-1}) \\ H_1(z) & H_1(zW) & \mathbf{L} & H_1(zW^{L-1}) \\ \parallel & \parallel & \mathbf{O} & \parallel \\ H_{L-1}(z) & H_{L-1}(zW) & \mathbf{L} & H_{L-1}(zW^{L-1}) \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_1(zW) \\ \parallel \\ X_1(zW^{L-1}) \end{bmatrix} \\
&= z^{-L} \hat{X}_1(z). \text{ As a result, the structure of Figure 10.50 is a time-varying system with a period } L.
\end{aligned}$$

### 10.48



If the filter bank is alias-free, then

$$\begin{bmatrix} H_0(z) & H_1(z) & \mathbf{L} & H_{L-1}(z) \\ H_0(zW) & H_1(zW) & \mathbf{L} & H_{L-1}(zW) \\ \parallel & \parallel & \mathbf{O} & \parallel \\ H_0(zW^{L-1}) & H_1(zW^{L-1}) & \mathbf{L} & H_{L-1}(zW^{L-1}) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \\ \parallel \\ G_{L-1}(z) \end{bmatrix} = \begin{bmatrix} T(z) \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The above L equations are

$$(1): H_0(z)G_0(z) + H_1(z)G_1(z) + \mathbf{K} + H_{L-1}(z)G_{L-1}(z) = T(z),$$

$$(2): H_0(zW^r)G_0(z) + H_1(zW^r)G_1(z) + \mathbf{K} + H_{L-1}(zW^r)G_{L-1}(z) = 0, \quad 1 \leq r \leq L-1.$$

Replacing  $z$  by  $zW^{L-r}$  in the above equation we get

$$H_0(zW^L)G_0(zW^{L-r}) + H_1(zW^L)G_1(zW^{L-r}) + \mathbf{K} + H_{L-1}(zW^L)G_{L-1}(zW^{L-r}) = 0,$$

or equivalently.

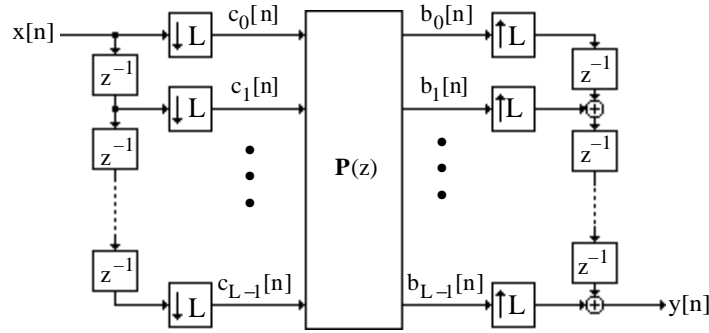
$$(3): H_0(z)G_0(zW^{L-r}) + H_1(z)G_1(zW^{L-r}) + \mathbf{K} + H_{L-1}(z)G_{L-1}(zW^{L-r}) = 0, \quad 1 \leq r \leq L-1.$$

Rewriting Eqs. (1) and (3) in matrix form we get

$$\begin{bmatrix} G_0(z) & G_1(z) & \mathbf{L} & G_{L-1}(z) \\ G_0(zW) & G_1(zW) & \mathbf{L} & G_{L-1}(zW) \\ \vdots & \vdots & \mathbf{O} & \vdots \\ G_0(zW^{L-1}) & G_1(zW^{L-1}) & \mathbf{L} & G_{L-1}(zW^{L-1}) \end{bmatrix} \begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{L-1}(z) \end{bmatrix} = \begin{bmatrix} T(z) \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, if an L-channel filter bank is alias-free with a given set of analysis and synthesis filters, then the filter bank is still alias-free if the analysis and synthesis filters are interchanged.

### 10.49



From the above figure it follows that we can express the z-transforms of  $\{c_i[n]\}$  as

$$C_1(z) = \frac{1}{L} \sum_{k=0}^{L-1} (z^{1/L}W^k)^{-1} X(z^{1/L}W^k), \quad 0 \leq 1 \leq L-1, \quad \text{where } W = e^{-j2\pi/L}.$$

Likewise, the z-transforms of  $\{b_i[n]\}$  can be expressed as

$$B_s(z) = \sum_{l=0}^{L-1} P_{s,l}(z)C_l(z), \quad 0 \leq s \leq L-1, \quad \text{where } P_{s,l}(z) \text{ denotes the } (s,l)\text{-th element of } \mathbf{P}(z).$$

Finally, the z-transform of the output  $y[n]$  is given by

$$\begin{aligned} Y(z) &= \sum_{s=0}^{L-1} z^{-(L-1-s)} B_s(z^L) = \sum_{s=0}^{L-1} z^{-(L-1-s)} \sum_{l=0}^{L-1} P_{s,l}(z^L) C_l(z^L) \\ &= \frac{1}{L} \sum_{s=0}^{L-1} z^{-(L-1-s)} \sum_{l=0}^{L-1} P_{s,l}(z^L) \sum_{k=0}^{L-1} (zW^k)^{-1} X(zW^k) \\ &= \frac{1}{L} \sum_{k=0}^{L-1} X(zW^k) \sum_{l=0}^{L-1} W^{-kl} \sum_{s=0}^{L-1} z^{-1} z^{-(L-1-s)} P_{s,l}(z^L). \end{aligned}$$

In the above expression, terms of the form  $X(zW^k)$ ,  $k \neq 0$ , represent the contribution coming from aliasing. Hence, the expression for  $Y(z)$  is free from these aliasing terms for any arbitrary input  $x[n]$  if and only if

$$\sum_{l=0}^{L-1} W^{-kl} \sum_{s=0}^{L-1} z^{-1} z^{-(L-1-s)} P_{s,l}(z^L) \neq 0, \quad k \neq 0.$$



The above expression can be written in a matrix form as  $\mathbf{D}^\dagger \begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{L-1}(z) \end{bmatrix} = \begin{bmatrix} T(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ , where  $\mathbf{D}$  is the

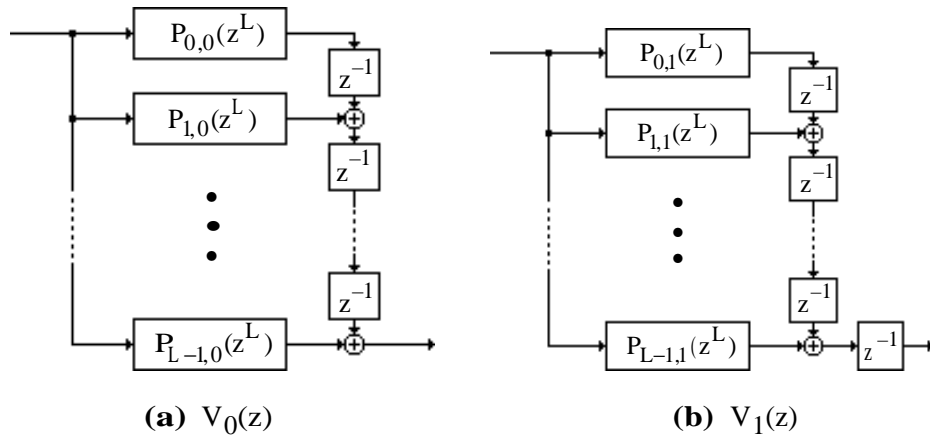
$L \times L$  DFT matrix,  $T(z)$  is the transfer function of the alias-free system, and

$V_1(z) = \sum_{s=0}^{L-1} z^{-1} z^{-(L-1-s)} P_{s,1}(z^L)$ . Since  $\mathbf{D}\mathbf{D}^\dagger = \mathbf{L}\mathbf{I}$ , the above matrix equation can be

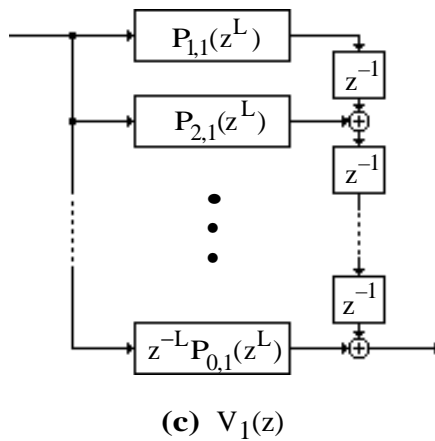
alternately written as  $\begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{L-1}(z) \end{bmatrix} = \mathbf{D} \begin{bmatrix} T(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ . This implies that  $V_1(z) = V(z)$ ,  $0 \leq 1 \leq L-1$ , as the

first column of  $\mathbf{D}$  has all elements equal to 1. As a result, the  $L$ -channel QMF bank is alias-free if and only if  $V_1(z)$  is the same for all  $l$ .

The two figures below show the polyphase realizations of  $V_0(z)$  and  $V_1(z)$ .



The realization of  $V_1(z)$  can be redrawn as indicated below.



Because of the constraint  $V_0(z) = V_1(z)$ , the polyphase components in Figures (a) and (c) should be the same. From these two figures it follows that the first column of  $\mathbf{P}(z)$  is an upwards-shifted version of the second column, with the topmost element appearing with a  $z^{-1}$

attached. this type of relation holds for the  $k$ -th column and the  $(k+1)$ -th column of  $\mathbf{P}(z)$ . As a result,  $\mathbf{P}(z)$  is a pseudo-circulant matrix of the form of Eq. (10.214).

**10.50**  $b_0y[3n]+b_1y[3n-1]+b_2y[3n-2]+b_3y[3n-3]=a_0x[3n]+a_1x[3n-1]+a_2x[3n-2]+a_3x[3n-3]$ ,  
 $b_0y[3n+1]+b_1y[3n]+b_2y[3n-1]+b_3y[3n-2]=a_0x[3n+1]+a_1x[3n]+a_2x[3n-1]+a_3x[3n-2]$ ,  
 $b_0y[3n+2]+b_1y[3n+1]+b_2y[3n]+b_3y[3n-1]=a_0x[3n+2]+a_1x[3n+1]+a_2x[3n]+a_3x[3n-1]$ .

In matrix form, the above set of equations can be written as

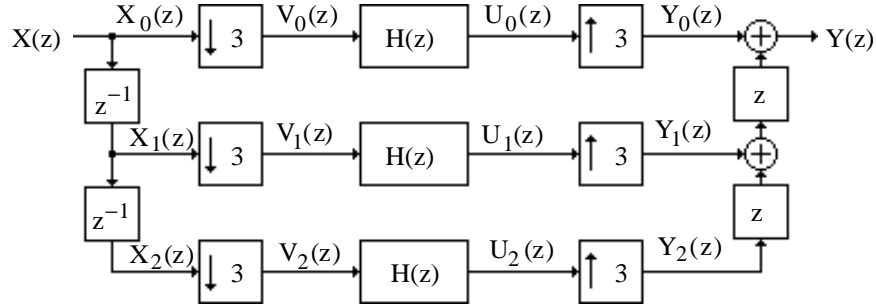
(1):  $\mathbf{B}_0\mathbf{Y}_k + \mathbf{B}_1\mathbf{Y}_{k-1} = \mathbf{A}_0\mathbf{X}_k + \mathbf{A}_1\mathbf{X}_{k-1}$ ,  
where  $\mathbf{B}_0 = \begin{bmatrix} b_0 & 0 & 0 \\ b_1 & b_0 & 0 \\ b_2 & b_1 & b_0 \end{bmatrix}$ ,  $\mathbf{B}_1 = \begin{bmatrix} b_3 & b_2 & b_1 \\ 0 & b_3 & b_2 \\ 0 & 0 & b_3 \end{bmatrix}$ ,  $\mathbf{Y}_k = \begin{bmatrix} y[3k] \\ y[3k+1] \\ y[3k+2] \end{bmatrix}$ ,  $\mathbf{Y}_{k-1} = \begin{bmatrix} y[3k-3] \\ y[3k-2] \\ y[3k-1] \end{bmatrix}$   
 $\mathbf{A}_0 = \begin{bmatrix} a_0 & 0 & 0 \\ a_1 & a_0 & 0 \\ a_2 & a_1 & a_0 \end{bmatrix}$ ,  $\mathbf{A}_1 = \begin{bmatrix} a_3 & a_2 & a_1 \\ 0 & a_3 & a_2 \\ 0 & 0 & a_3 \end{bmatrix}$ ,  $\mathbf{X}_k = \begin{bmatrix} x[3k] \\ x[3k+1] \\ x[3k+2] \end{bmatrix}$ ,  $\mathbf{X}_{k-1} = \begin{bmatrix} x[3k-3] \\ x[3k-2] \\ x[3k-1] \end{bmatrix}$ .

A multirate implementation of the block difference equation is shown below:

From Eq. (1) we have  $(\mathbf{B}_0 + z^{-1}\mathbf{B}_1)\mathbf{Y}_k = (\mathbf{A}_0 + z^{-1}\mathbf{A}_1)\mathbf{X}_k$ , or

$\mathbf{Y}_k = (\mathbf{B}_0 + z^{-1}\mathbf{B}_1)^{-1}(\mathbf{A}_0 + z^{-1}\mathbf{A}_1)\mathbf{X}_k$ . Hence,  $\mathbf{H}(z) = (\mathbf{B}_0 + z^{-1}\mathbf{B}_1)^{-1}(\mathbf{A}_0 + z^{-1}\mathbf{A}_1)\mathbf{X}_k$ .

**10.51**



Analysis of the 3-path filter yields

$V_i(z) = \frac{1}{3}[X_i(z^{1/3}W_3^0) + X_i(z^{1/3}W_3^{-1}) + X_i(z^{1/3}W_3^{-2})]$ ,  $U_i(z) = H(z)V_i(z)$ , and

$Y_i(z) = U_i(z^3)$ , for  $i = 0, 1, 2$ . Thus,  $Y_i(z) = \frac{1}{3}H(z^3)[X_i(zW_3^0) + X_i(zW_3^{-1}) + X_i(zW_3^{-2})]$ .

But  $X_i(z) = z^{-i}X(z)$ . Hence,

$Y_i(z) = \frac{1}{3}H(z^3)[(zW_3^0)^{-i}X_i(zW_3^0) + (zW_3^{-1})^{-i}X_i(zW_3^{-1}) + (zW_3^{-2})^{-i}X_i(zW_3^{-2})]$ . Therefore,

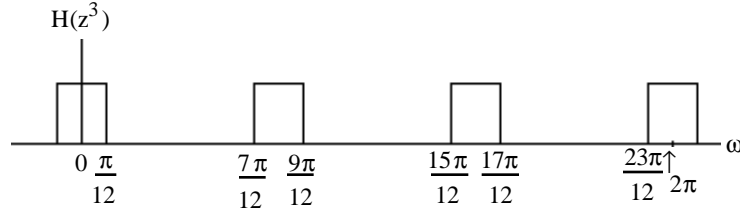
$Y(z) = Y_0(z) + zY_1(z) + z^2Y_2(z) = \frac{1}{3}H(z^3)[W_3^0X(zW_3^0) + W_3^0X(zW_3^{-1}) + W_3^0X(zW_3^{-2})]$   
 $+ \frac{1}{3}H(z^3)z[z^{-1}W_3^0X(zW_3^0) + z^{-1}W_3^1X(zW_3^{-1}) + z^{-1}W_3^2X(zW_3^{-2})]$   
 $+ \frac{1}{3}H(z^3)z^2[z^{-2}W_3^0X(zW_3^0) + z^{-2}W_3^2X(zW_3^{-1}) + z^{-2}W_3^4X(zW_3^{-2})]$

$$= \frac{1}{3}H(z^3) \left[ 3X(z) + (W_3^0 + W_3^1 + W_3^2)X(zW_3^{-1}) + (W_3^0 + W_3^2 + W_3^3)X(zW_3^{-2}) \right] = H(z^3)X(z) \text{ since}$$

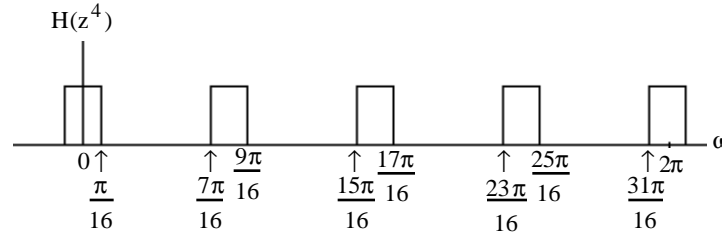
by Eq. (3.28),  $(W_3^0 + W_3^1 + W_3^2) = (W_3^0 + W_3^2 + W_3^3) = 0$ .

It follows from the above that in the general N-path case,  $Y(z) = H(z^N)X(z)$

**10.52** From the results of Problem 10.51 we observe that the transfer function of an N-path filter is  $H(z^N)$ . Hence the transfer function of the cascade of a 3-path filter and a 4-path filter is given by  $H(z^3)H(z^4)$ . For the magnitude response of  $H(z)$  shown in Figure P10.9, thus, the magnitude response of the transfer function of a 3-path filter is as indicated below



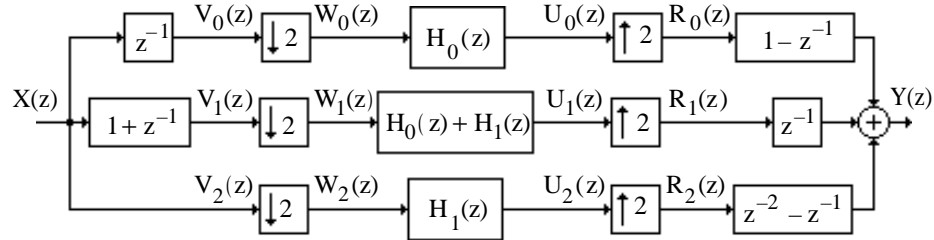
and the magnitude response of the transfer function of a 4-path filter is as indicated below



Hence, the magnitude response of their cascade is of the form shown below:



**10.53 (a)**



$$\mathbf{V}(z) = \begin{bmatrix} V_0(z) \\ V_1(z) \\ V_2(z) \end{bmatrix} = \begin{bmatrix} z^{-1} \\ 1+z^{-1} \\ 1 \end{bmatrix} X(z), \quad \mathbf{W}(z) = \begin{bmatrix} W_0(z) \\ W_1(z) \\ W_2(z) \end{bmatrix} = \begin{bmatrix} z^{-1/2} \\ 1+z^{-1/2} \\ 1 \end{bmatrix} X(z^{1/2}) + \begin{bmatrix} -z^{-1/2} \\ 1-z^{-1/2} \\ 1 \end{bmatrix} X(-z^{1/2}).$$

$$\mathbf{U}(z) = \begin{bmatrix} U_0(z) \\ U_1(z) \\ U_2(z) \end{bmatrix} = \begin{bmatrix} z^{-1/2} H_0(z) \\ (1+z^{-1/2})\{H_0(z)+H_1(z)\} \\ H_1(z) \end{bmatrix} X(z^{1/2}) + \begin{bmatrix} -z^{-1/2} H_0(z) \\ (1-z^{-1/2})\{H_0(z)+H_1(z)\} \\ H_1(z) \end{bmatrix} X(-z^{1/2}).$$

$$\mathbf{R}(z) = \begin{bmatrix} R_0(z) \\ R_1(z) \\ R_2(z) \end{bmatrix} = \begin{bmatrix} z^{-1} H_0(z^2) \\ (1+z^{-1})\{H_0(z^2)+H_1(z^2)\} \\ H_1(z^2) \end{bmatrix} X(z) + \begin{bmatrix} -z^{-1} H_0(z^2) \\ (1-z^{-1})\{H_0(z^2)+H_1(z^2)\} \\ H_1(z^2) \end{bmatrix} X(-z).$$

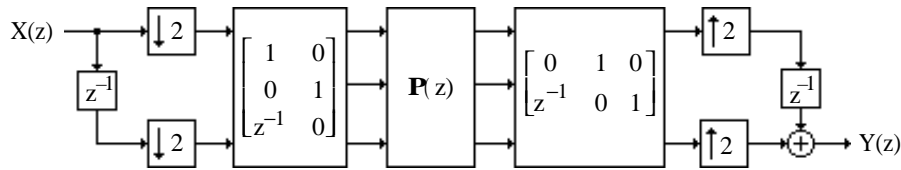
$$\begin{aligned} Y(z) &= (1-z^{-1})R_0(z) + z^{-1}R_1(z) + (z^{-2}-z^{-1})R_2(z) \\ &= \left[ (1-z^{-1})z^{-1}H_0(z^2) + (1+z^{-1})z^{-1}\{H_0(z^2)+H_1(z^2)\} + (z^{-2}-z^{-1})H_1(z^2) \right] X(z) \\ &\quad + \left[ -(1-z^{-1})z^{-1}H_0(z^2) + (1-z^{-1})z^{-1}\{H_0(z^2)+H_1(z^2)\} + (z^{-2}-z^{-1})H_1(z^2) \right] X(-z) \\ &= \left[ 2z^{-1}H_0(z^2) + 2z^{-2}H_1(z^2) \right] X(z) = 2z^{-1} \left[ H_0(z^2) + z^{-1}H_1(z^2) \right] X(z). \text{ Hence,} \\ T(z) &= 2z^{-1} \left[ H_0(z^2) + z^{-1}H_1(z^2) \right]. \end{aligned}$$

(b)  $T(z) = 2z^{-1} \left[ \frac{1}{2}H(z) + \frac{1}{2}H(-z) + \frac{1}{2}H(z) - \frac{1}{2}H(-z) \right] = 2z^{-1}H(z).$

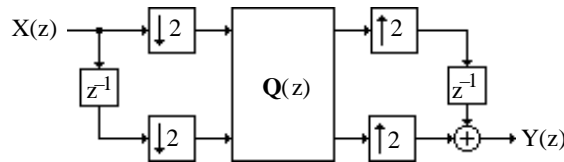
(c) Length of  $H_0(z) = K$  and length of  $H_1(z) = K$ .

(d) The number of multiplications in the above structure is given by  $3K \frac{F_T}{2}$  per second where  $F_T$  is the sampling frequency in Hz. On the other hand for a direct implementation of  $H(z)$  requires  $2KF_T$  multiplications per second.

**10.54 (a)** An equivalent representation of the structure of Figure P10.7 is as indicated below:



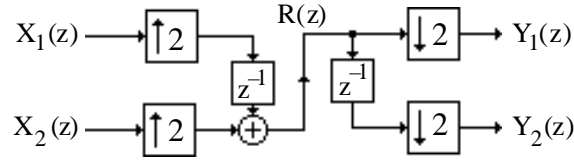
whose simplified equivalent representation is as shown below:



where  $\mathbf{Q}(z) = \begin{bmatrix} 0 & 1 & 0 \\ z^{-1} & 0 & 1 \end{bmatrix} \mathbf{P}(z) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ z^{-1} & 0 \end{bmatrix}$ . It thus follows from Problem 10.39 that the structure of Figure P10.7 is time-invariant (i.e. alias-free) if and only if  $\mathbf{Q}(z)$  is pseudo-circulant.

(b) A realization of the structure of Figure P10.7 based on critical down-sampling and critical up-sampling is shown above in Part (a).

10.55



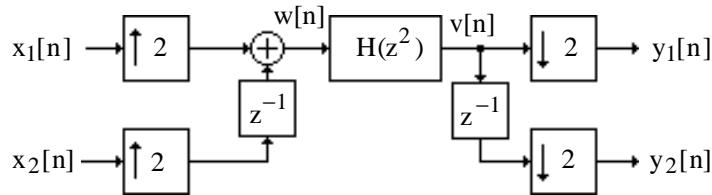
$$R(z) = z^{-1}X_1(z^2) + X_2(z^2).$$

$$Y_1(z) = R(z^{1/2}) + R(-z^{1/2}) = [z^{-1/2}X_1(z) + X_2(z)] + [-z^{-1/2}X_1(z) + X_2(z)] = 2X_2(z).$$

$$Y_2(z) = z^{-1/2}R(z^{1/2}) - z^{-1/2}R(-z^{1/2}) \\ = [z^{-1}X_1(z) + z^{-1/2}X_2(z)] + [z^{-1}X_1(z) - z^{-1/2}X_2(z)] = 2z^{-1}X_1(z).$$

Thus, the output  $y_1[n]$  is a scaled replica of the input  $x_2[n]$  while the output  $y_2[n]$  is a scaled replica of the delayed input  $x_1[n-1]$ .

10.56



$$W(z) = (X_1(z^2) + z^{-1}X_2(z^2)),$$

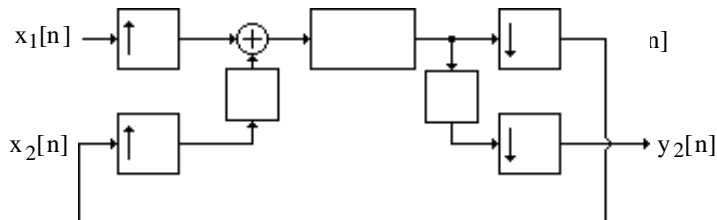
$$V(z) = H(z^2)X_1(z^2) + z^{-1}H(z^2)X_2(z^2),$$

$$Y_1(z) = \frac{1}{2}(V(z^{1/2}) + V(-z^{1/2})) = H(z)X_1(z),$$

$$Y_2(z) = \frac{1}{2}(z^{-1/2}V(z^{1/2}) - z^{-1/2}V(-z^{1/2})) = z^{-1}H(z)X_2(z).$$

Therefore,  $\frac{Y_1(z)}{X_1(z)} = H(z)$ , and  $\frac{Y_2(z)}{X_2(z)} = z^{-1}H(z)$ . Hence, the system is time-invariant.

10.57





**10.59 (a)**  $H_0(z) = \frac{1}{2} - z^{-1} + \frac{21}{2}z^{-2} - \frac{27}{2}z^{-3} - 5z^{-4} - \frac{5}{2}z^{-5}$ .

$$H_0(z^{-1}) = \frac{1}{2} - z + \frac{21}{2}z^2 - \frac{27}{2}z^3 - 5z^4 - \frac{5}{2}z^5.$$

$$H_0(z)H_0(z^{-1}) = 1.25z^5 - 28z^3 - 72.75z + 325 - 72.75z^{-1} - 28z^{-3} - 1.25z^{-5}. \text{ Thus,}$$

$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 650$ . Hence,  $H_0(z)$  is a power-symmetric function. The highpass analysis filter is given by

$$H_1(z) = z^{-5}H_0(-z^{-1}) = \frac{5}{2} - 5z^{-1} + \frac{27}{2}z^{-2} + \frac{21}{2}z^{-3} + z^{-4} + \frac{1}{2}z^{-5}.$$

The two synthesis filters are time-reversed versions of the analysis filter as per Eq. (10.146) and are given by

$$F_0(z) = 2z^{-5}H_0(z^{-1}) = -5 - 10z^{-1} - 27z^{-2} + 21z^{-3} - 2z^{-4} + z^{-5}, \text{ and}$$

$$F_1(z) = 2z^{-5}H_1(z^{-1}) = 1 + 2z^{-1} + 21z^{-2} + 27z^{-3} - 10z^{-4} + 5z^{-5}.$$

**(b)**  $H_0(z) = 1 + 3z^{-1} + 14z^{-2} + 27z^{-3} - 12z^{-4} + 4z^{-5}$ .

$$H_0(z^{-1}) = 1 + 3z + 14z^2 + 27z^3 - 12z^4 + 4z^5.$$

$$H_0(z)H_0(z^{-1}) = 4z^5 + 42z^3 + 41z + 850 + 41z^{-1} + 42z^{-3} + 4z^{-5}. \text{ Thus,}$$

$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 1700$ . Hence,  $H_0(z)$  is a power-symmetric function. The highpass analysis filter is given by

$$H_1(z) = z^{-5}H_0(-z^{-1}) = -4 - 12z^{-1} - 22z^{-2} + 14z^{-3} - 3z^{-4} + z^{-5}.$$

The two synthesis filters are time-reversed versions of the analysis filter as per Eq. (10.146) and are given by

$$F_0(z) = 2z^{-5}H_0(z^{-1}) = 8 - 24z^{-1} + 44z^{-2} + 28z^{-3} + 6z^{-4} + 2z^{-5},$$

$$F_1(z) = 2z^{-5}H_1(z^{-1}) = 2 - 3z^{-1} + 14z^{-2} - 22z^{-3} - 12z^{-4} - 4z^{-5}.$$

**10.60**  $H_0(z) = 1 + az^{-1} + z^{-2}$  and  $H_1(z) = 1 + az^{-1} + bz^{-2} + az^{-3} + z^{-4}$ . The corresponding synthesis filters are given by  $G_0(z) = H_1(-z) = 1 - az^{-1} + bz^{-2} - az^{-3} + z^{-4}$ , and  $G_1(z) = -H_0(-z) = -1 + az^{-1} - z^{-2}$ .

To show that the filter bank is alias-free and satisfies the perfect reconstruction property we

need to show that  $\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} cz^{-K} \\ 0 \end{bmatrix}$ , where  $c \neq 0$ . Now,

$$H_0(z)G_0(z) + H_1(z)G_1(z)$$

$$= (1 + az^{-1} + z^{-2})(1 - az^{-1} + bz^{-2} - az^{-3} + z^{-4}) + (-1 + az^{-1} - z^{-2})(1 + az^{-1} + bz^{-2} + az^{-3} + z^{-4})$$

$= 2a(b-2)z^{-3}$ , and  $H_0(-z)G_0(z) + H_1(z)G_1(-z) = H_0(-z)H_1(-z) - H_0(-z)H_1(-z) = 0$ . Thus, if  $a \neq 0$  and  $b \neq 2$ , the above filter bank is alias-free and also satisfies the perfect reconstruction property.

**10.61** If  $H_0(z)$  is required to have 2 zeros at  $z = -1$ , then it is of the form  $H_0(z) = (1 + z^{-1})^2 C(z)$ , where  $C(z)$  is a first-order polynomial.

Now  $P(z) = H_0(z)H_0(z^{-1}) = (1+z^{-1})^2(1+z)^2 R(z)$ , where  $R(z)$  is a zero-phase polynomial of the form  $R(z) = az + b + az^{-1}$ . For perfect reconstruction we require,  $P(z) + P(-z) = 2$ , i.e.,

$$(1+z^{-1})^2(1+z)^2(az + b + az^{-1}) + (1-z^{-1})^2(1-z)^2(-az + b - az^{-1}) = 2.$$

Since the above equation must hold for all values of  $z$ , we observe that at  $z = 1$ , we get  $2a + b = \frac{1}{8}$ . Likewise, at  $z = j$ , we get  $b = \frac{1}{4}$ . Hence,  $a = -\frac{1}{16}$ . Therefore,

$$P(z) = (1+z^{-1})^2(1+z)^2 \left( -\frac{1}{16}z + \frac{4}{16} - \frac{1}{16}z^{-1} \right).$$

The analysis filter  $H_0(z)$  is obtained by a spectral factorization of  $P(z)$ . Two choices of spectral factorization are given in Examples 10.17 and 10.20. The choice given in Example 10.20 results in linear-phase analysis filters with two zeros at  $z = -1$ .

### 10.62

The polyphase matrices are  $\mathbf{E}(z) = \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix}$  and  $\mathbf{R}(z) = \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix}$ . Therefore

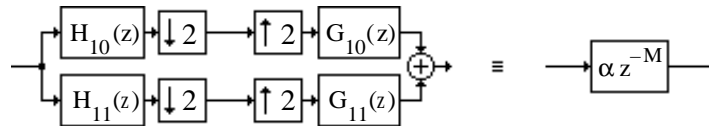
$\mathbf{R}(z)\mathbf{E}(z) = \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . Hence, the above structure is a perfect reconstruction filter bank.

### 10.63

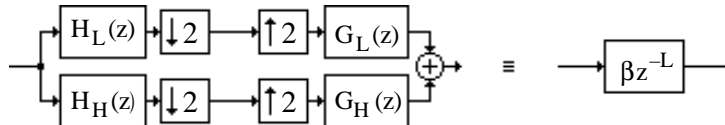
The polyphase matrices are  $\mathbf{E}(z) = \begin{bmatrix} 1 & Q(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix}$  and  $\mathbf{R}(z) = \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & -Q(z) \\ 0 & 1 \end{bmatrix}$ .

Therefore,  $\mathbf{R}(z)\mathbf{E}(z) = \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & -Q(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & Q(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix}$   
 $= \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . Hence, the above structure is a perfect reconstruction filter bank.

**10.64** If the 2-channel QMF banks in the middle of the structure of Figure 10.71 are of perfect reconstruction type, then each of these two filter banks have a distortion transfer function of the form  $\alpha z^{-M}$ , where  $M$  is a positive integer:

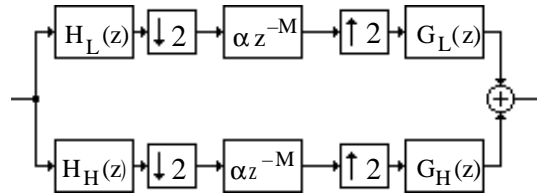


Likewise, the 2-channel analysis filter bank on the left with the 2-channel synthesis filter bank on the right form a perfect reconstruction QMF bank with a distortion transfer function  $\beta z^{-L}$ , where  $L$  is a positive integer:

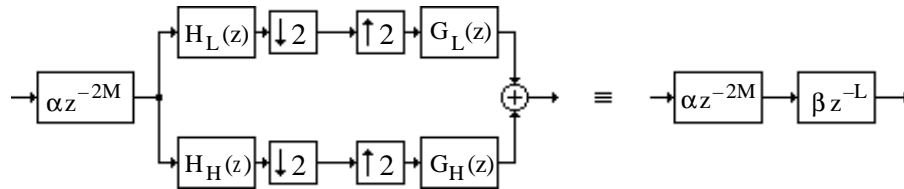




Hence an equivalent representation of Figure 10.71 is as indicated below

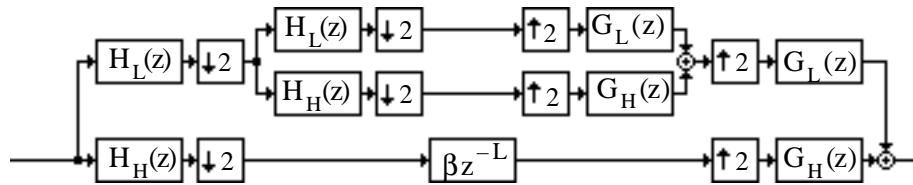


which reduces to

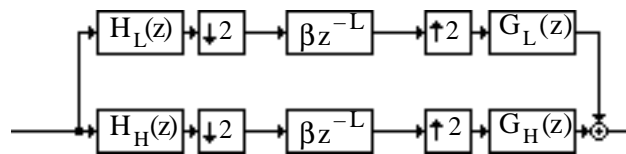


Thus, the overall structure is also of perfect reconstruction type with a distortion transfer function given by  $\alpha\beta z^{-(2M+L)}$ .

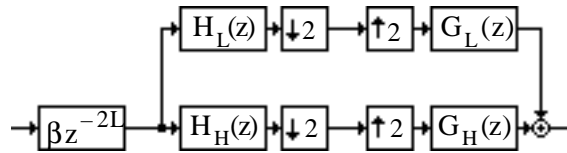
**10.65** We analyze the 3-channel filter bank of Figure 10.74(b). If the 2-channel QMF bank of Figure 10.74(a) is of perfect reconstruction type with a distortion transfer function  $\beta z^{-L}$ , the structure of Figure 10.74(b) should be implemented as indicated below to ensure perfect reconstruction:



An equivalent representation of the above structure is as shown below:



which reduces to



verifying the perfect reconstruction property of Figure 10.74(b).

In a similar manner the perfect reconstruction property of Figure 10.74(c) can be proved.

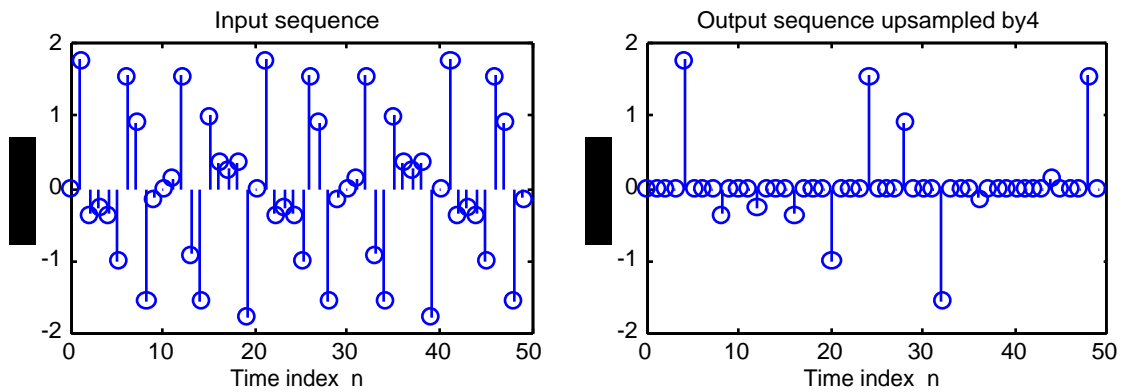
**M10.1 (a)** For Part (i) use the MATLAB statement

```
x = sin(2*pi*0.2*n) + sin(2*pi*0.35*n);
```

in Program 10\_1 with  $L = 4$  and  $N = 50$ , and remove the statement

```
wo = input('Input signal frequency = ');
```

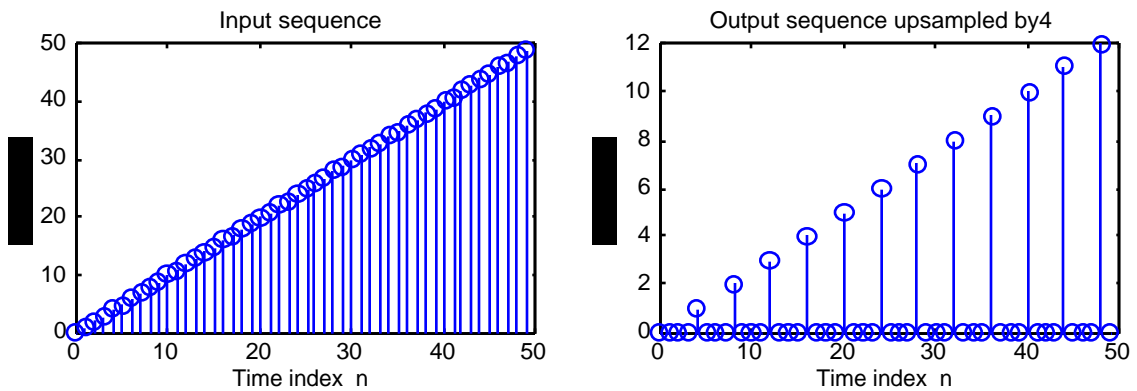
The plots generated by the modified program are shown below:



For Part (ii) use the MATLAB statement `x = n;` in Program 10\_1 with  $L = 4$  and  $N = 50$ , and remove the statement

```
wo = input('Input signal frequency = ');
```

The plots generated by the modified program are shown below:



For Part (iii) use `x = square(N, duty);` where `duty` can be chosen as 10, 20, 30, ...

**(b)** Use  $L = 5$  in the modified program as described above.

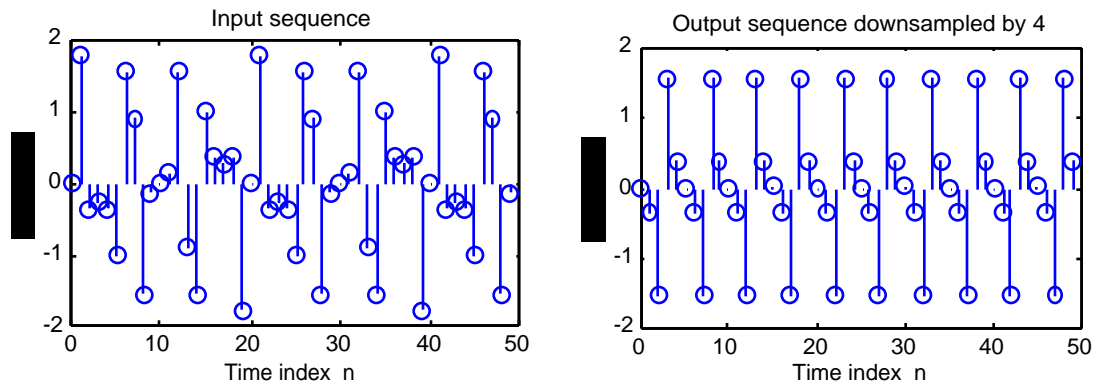
**M10.2 (a)** For Part (i) use the MATLAB statement

```
x = sin(2*pi*0.2*m) + sin(2*pi*0.35*m);
```

in Program 10\_2 with  $L = 4$  and  $N = 50$ , and remove the statement

```
wo = input('Input signal frequency = ');
```

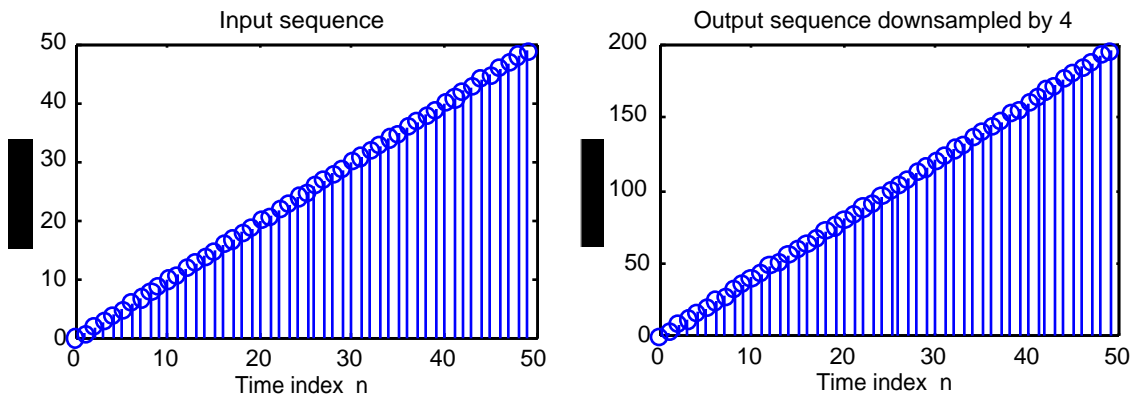
The plots generated by the modified program are shown below:



For Part (ii) use the MATLAB statement `x = m;` in Program 10\_1 with  $L = 4$  and  $N = 50$ , and remove the statement

`w0 = input('Input signal frequency = ');`

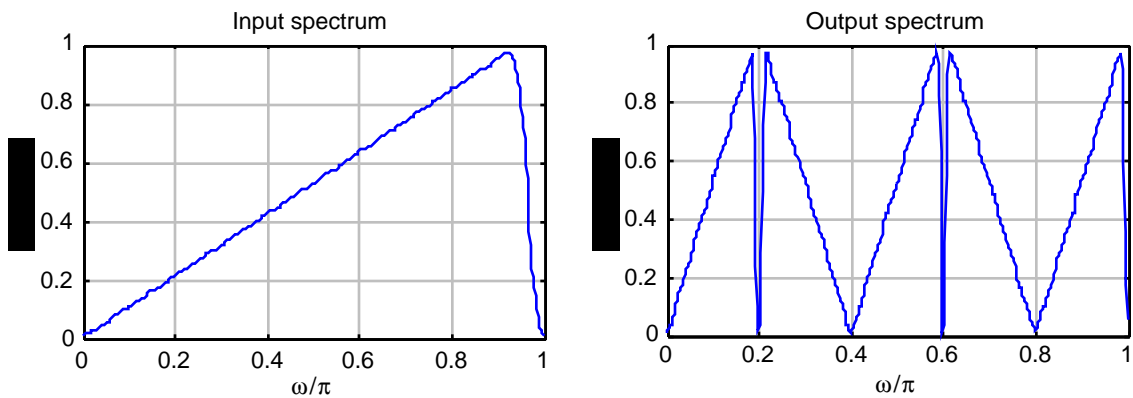
The plots generated by the modified program are shown below:



For Part (iii) use `x = square(N, duty);` where `duty` can be chosen as 10, 20, 30, ...

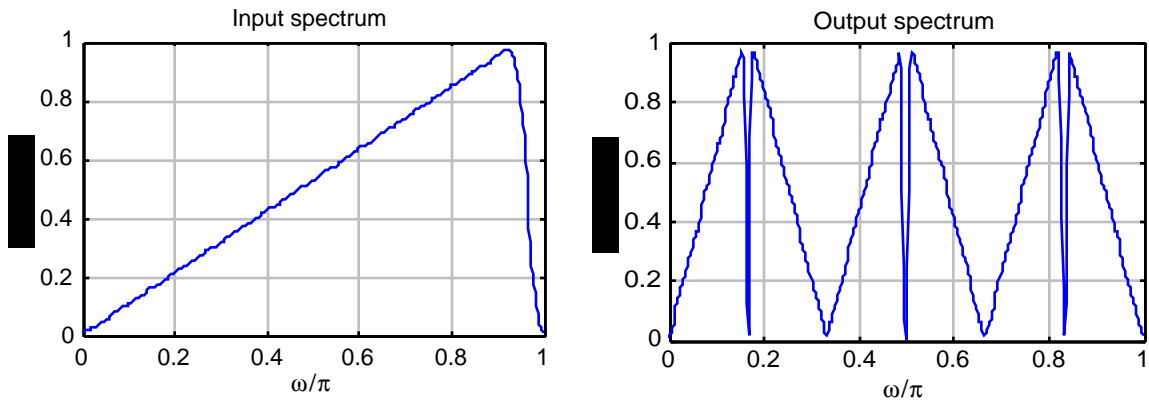
(b) Use  $L = 5$  in the modified program as described above.

**M10.3 (a)** Plots generated are shown below:



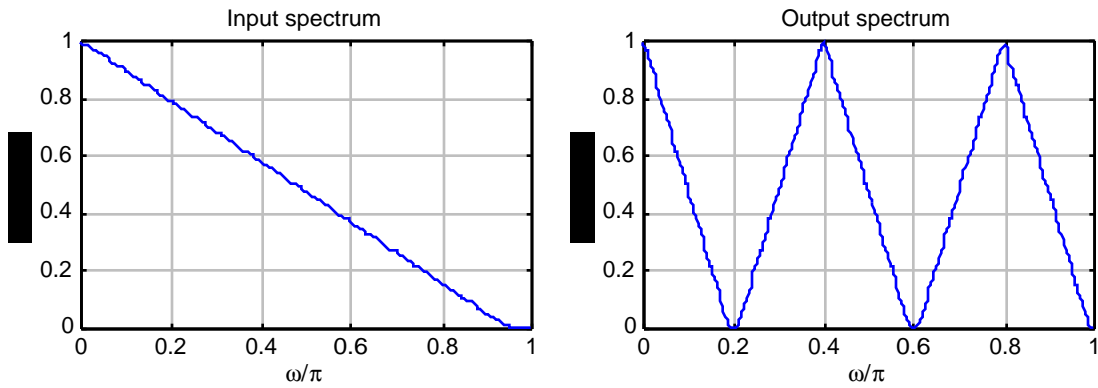
Since  $L = 5$ , the output spectrum consists of the input spectrum shrunk by a factor of 5 and there are  $L-1 = 4$  aliased spectra.

(b) Plots generated are shown below:

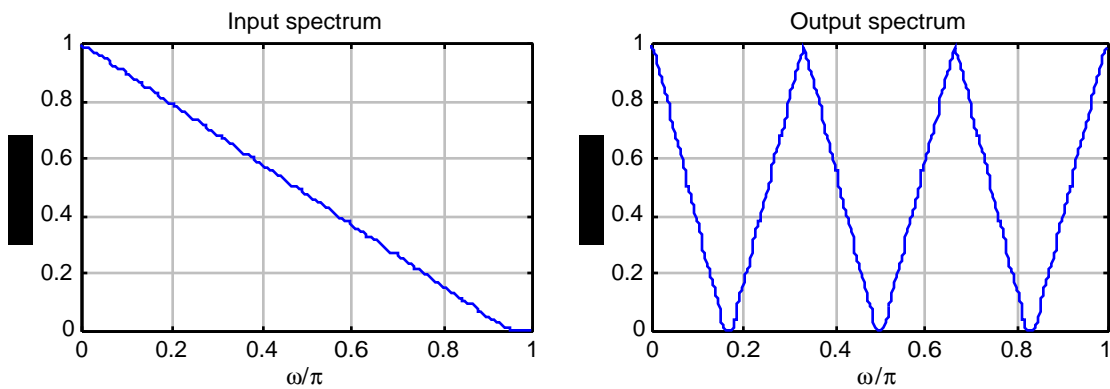


Since  $L = 6$ , the output spectrum consists of the input spectrum shrunk by a factor of 6 and there are  $L-1 = 5$  aliased spectra.

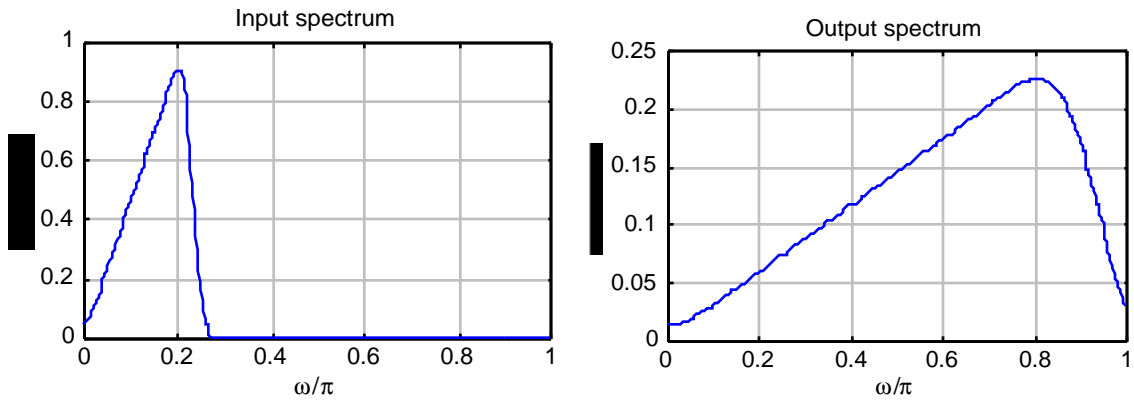
**M10.4 (a)** Plots generated are shown below:



(b) Plots generated are shown below:

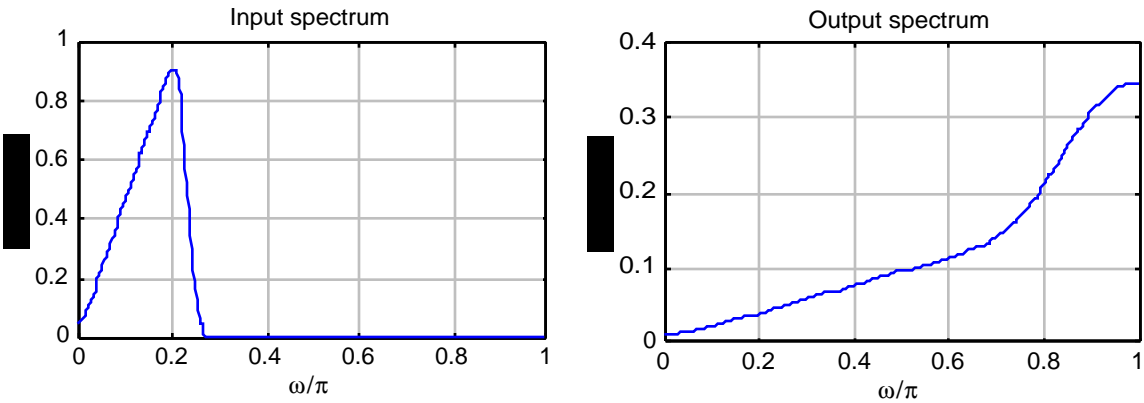


**M10.5 (a)** Plots generated are shown below:



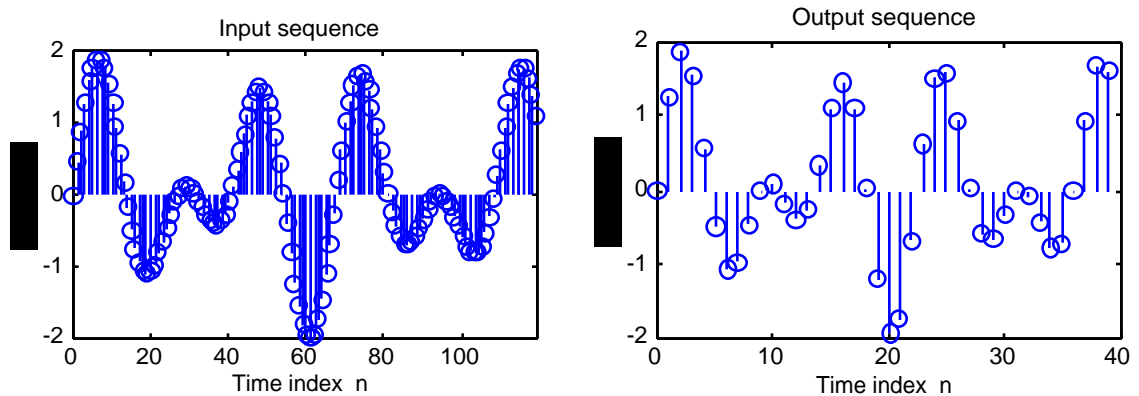
Since  $M = 4$  and the input is bandlimited to  $\pi/4$ , the output spectrum is a stretched version of the input spectrum stretched by a factor of 4 and there is no aliasing. Moreover, the output spectrum is scaled by a factor  $1/4$  as expected.

(b) Plots generated are shown below:

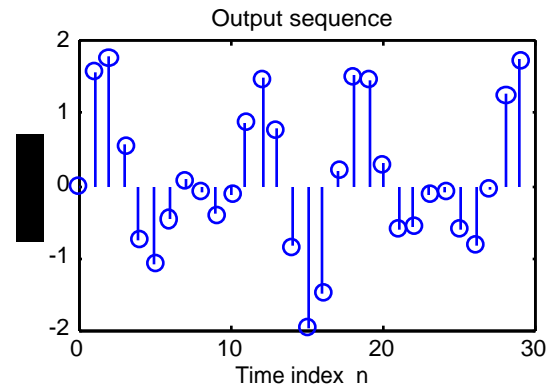
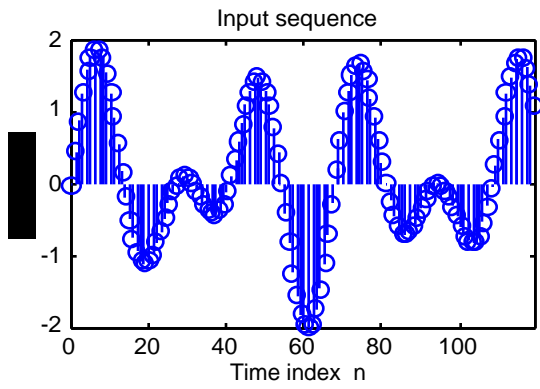


Since  $M = 5$  and the input is bandlimited to  $\pi/4$ , the output spectrum is a stretched version of the input spectrum stretched by a factor of 5 and there is some visible aliasing. Moreover, the output spectrum is scaled by a factor  $1/5$  as expected.

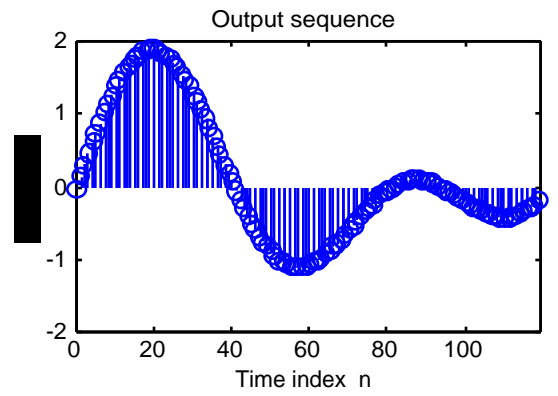
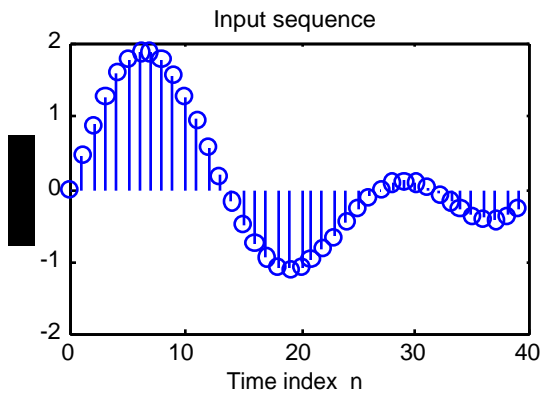
**M10.6 (a)** Plots generated are shown below:



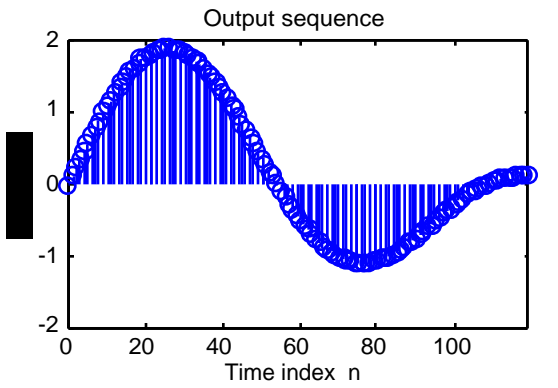
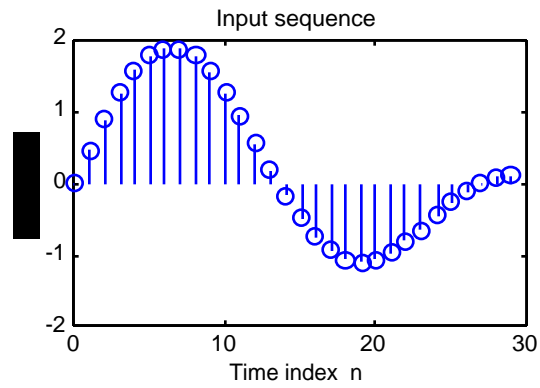
(b) Plots generated are shown below:



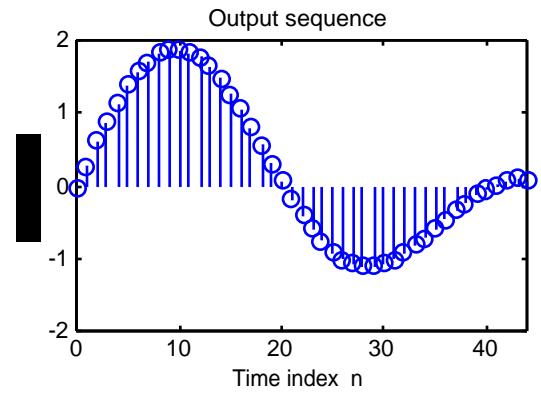
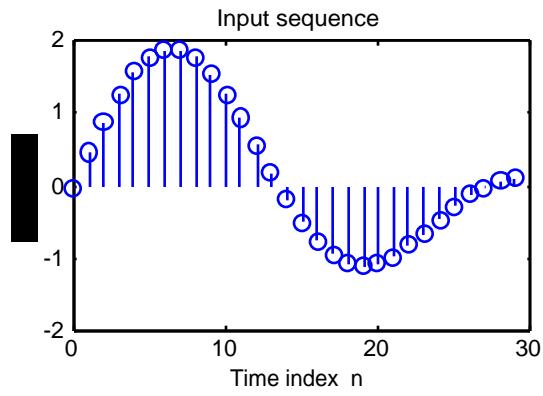
**M10.7 (a)** Plots generated are shown below:



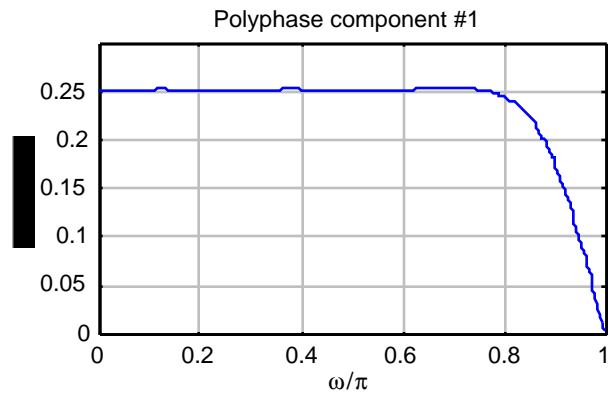
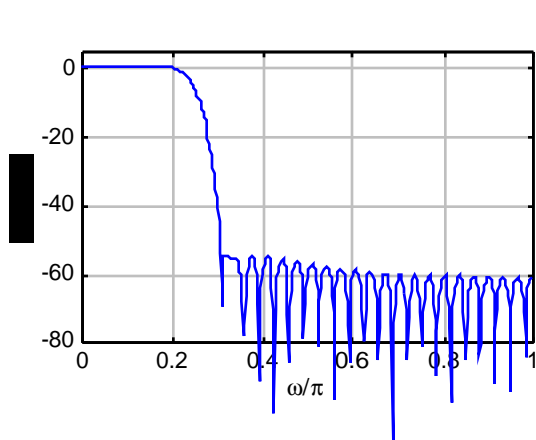
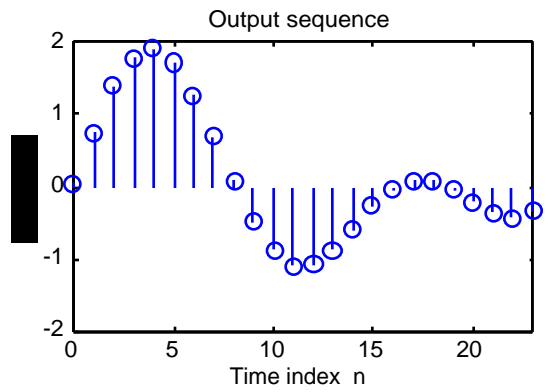
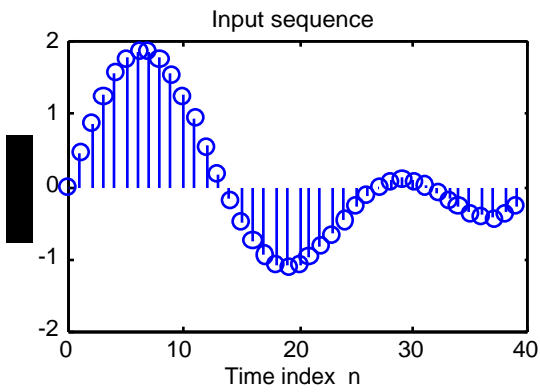
**(b)** Plots generated are shown below:

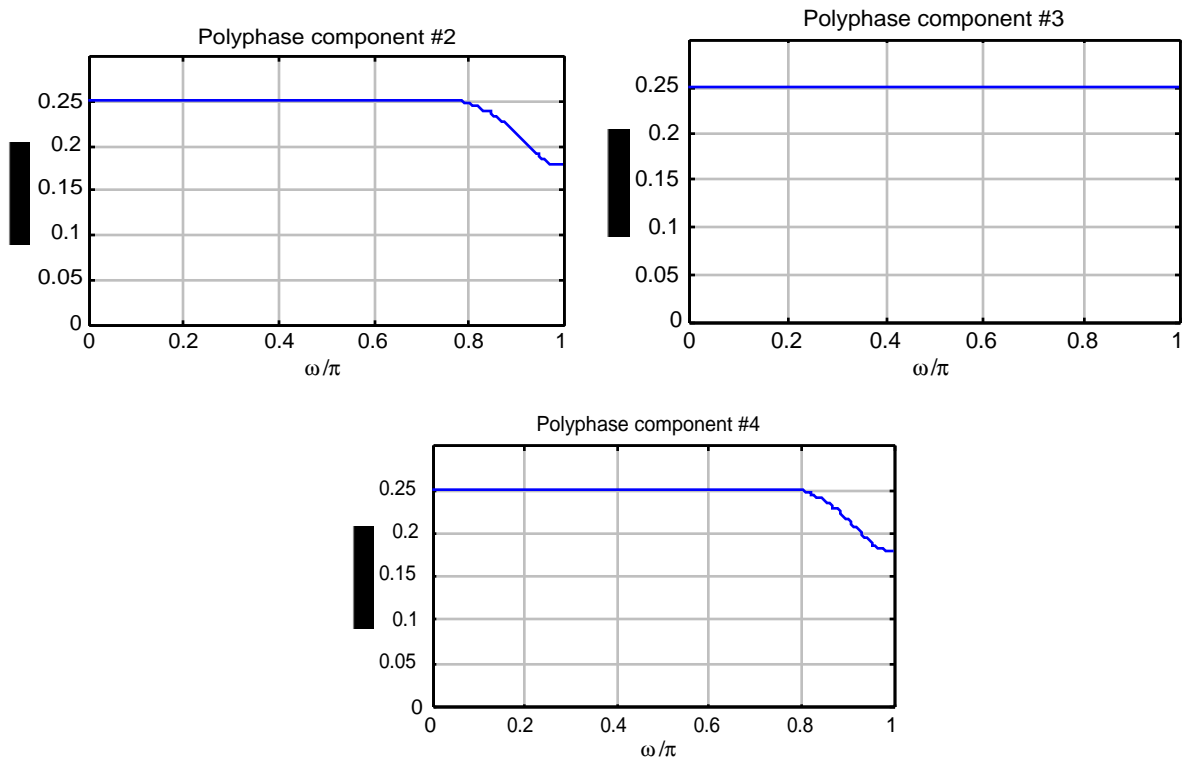


**M10.8 (a)** Plots generated are as shown below:



(b) Plots generated are as shown below:



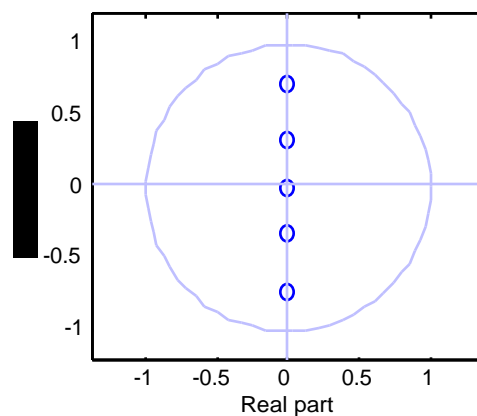


**M10.10** As shown in Problem 10.39, a digital lowpass Butterworth half-band filter can be designed by applying a bilinear transformation to an analog lowpass Butterworth transfer function with a 3-dB cutoff frequency at 1 rad/sec. The 3-dB cutoff frequency of the digital lowpass

Butterworth half-band filter is therefore at  $\omega_c = 2 \tan^{-1}(1)/\pi = 0.5$ . Thus to design a 5th-order digital lowpass Butterworth half-band filter we use the MATLAB statement

$$[\text{num}, \text{den}] = \text{butter}(5, 0.5); \text{ which yields } H(z) = \frac{0.052786(1+z^{-1})^5}{1+0.63344z^{-2}+0.055728z^{-4}}. \text{ As}$$

can be seen from the pole plot obtained using `zplane`, all poles are on the imaginary axis.



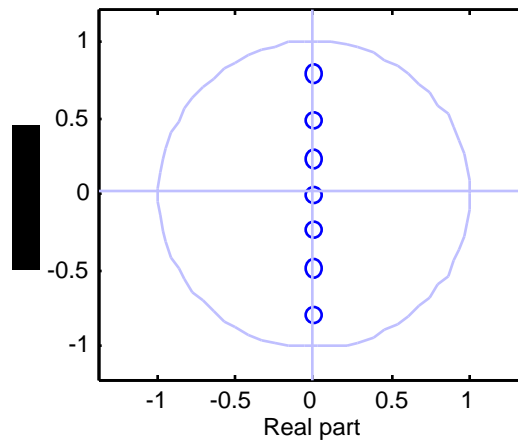


Using the pole interlacing property we can express  $H(z)$  as  $H(z) = \frac{1}{2}(A_0(z^2) + z^{-1}A_1(z^2))$ , where  $A_0(z) = \frac{0.52786 + z^{-1}}{1 + 0.52786z^{-1}}$  and  $A_1(z) = \frac{0.10557 + z^{-1}}{1 + 0.10557z^{-1}}$ . Hence,  $H(z)$  can be realized using only 2 multipliers.

**M10.11** To design a 7th-order digital lowpass Butterworth half-band filter we use the MATLAB statement `[num,den]= butter(5,0.5)`; which yields

$$H(z) = \frac{0.016565(1+z^{-1})^7}{1+0.91997z^{-2}+0.1927z^{-4}+0.0076835z^{-4}}$$

As can be seen from the pole plot obtained using `zplane`, all poles are on the imaginary axis.



Using the pole interlacing property we can express  $H(z)$  as  $H(z) = \frac{1}{2}(A_0(z^2) + z^{-1}A_1(z^2))$ , where  $A_0(z) = \frac{0.033131 + 0.68806z^{-1} + z^{-2}}{1 + 0.68806z^{-1} + 0.033131z^{-2}}$  and  $A_1(z) = \frac{0.23191 + z^{-1}}{1 + 0.23191z^{-1}}$ . Hence,  $H(z)$  can be realized using only 3 multipliers.

**M10.12** The specifications given are  $W_s = 0.55\pi$  and  $\delta_s = 0.01$ . Hence,  $W_p = 0.45\pi$  and

$$\delta_p = 1 - \sqrt{1 - \delta_s^2} = 0.0000005. \text{ Therefore, } R_p = -20 \log_{10}(1 - \delta_p) = 4.3429 \times 10^{-6} \text{ dB and}$$

$R_s = -20 \log_{10}(\delta_s) = 60 \text{ dB}$ . Using the commands

`[N,Wn] = ellipord(wp, ws, Rp, Rs)` and

`[b,a] = ellip(7, Rp, Rs, Wn)` we then determine the transfer function of an N-th order elliptic filter and plot its pole locations using `zplane(b)`. The poles of the transfer function generated are not on the imaginary axis. We next adjust the value of  $\delta_s$  and found that for  $\delta_s = 0.01$  the poles of the transfer function  $H_0(z)$  generated are on the imaginary axis.

Using the pole interlacing property we can express  $H_0(z)$  as  $H_0(z) = \frac{1}{2}(A_0(z^2) + z^{-1}A_1(z^2))$ ,

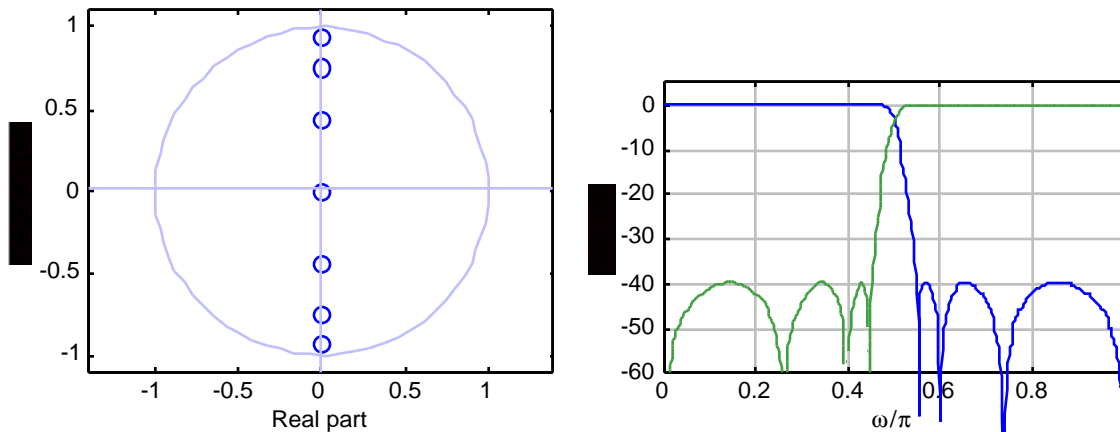
where  $A_0(z) = \frac{0.1657 + 1.0544z^{-1} + z^{-2}}{1 + 1.0544z^{-1} + 0.1657z^{-2}}$  and  $A_1(z) = \frac{0.5559 + z^{-1}}{1 + 0.5559z^{-1}}$ . Thus, the half-band

lowpass elliptic filter is given by  $H_0(z) = \frac{1}{2} \left( \frac{0.1657 + 1.0544z^{-2} + z^{-4}}{1 + 1.0544z^{-2} + 0.1657z^{-4}} + z^{-1} \frac{0.5559 + z^{-1}}{1 + 0.5559z^{-1}} \right)$ ,

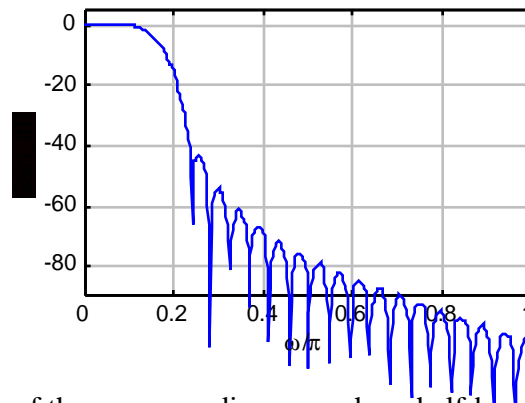
Its power-complementary highpass transfer function is then

$H_1(z) = \frac{1}{2} \left( \frac{0.1657 + 1.0544z^{-2} + z^{-4}}{1 + 1.0544z^{-2} + 0.1657z^{-4}} - z^{-1} \frac{0.5559 + z^{-1}}{1 + 0.5559z^{-1}} \right)$ , The gain responses of  $H_0(z)$  and

$H_1(z)$  are shown below:



**M10.13** Replace `win = hamming(N)` with `win = hanning(N)` in Program 10\_8. The gain response of the 6th-band filter of length 43 is shown below:



**M10.14** The specifications of the corresponding zero-phase half-band filter are as follows: stopband edge  $\omega_s = 0.65\pi$  and a minimum stopband attenuation of  $\alpha_s = 56$  dB. The desired stopband ripple is therefore  $\delta_s = 0.0016$ . The passband edge of the filter is at  $\omega_p = \pi - 0.65\pi = 0.35\pi$ . Using the function `remezord` we then estimate the order of  $F(z)$  and using the function `remez` we next design  $Q(z)$ . To this end the code fragments used are

```
[N, fpts, mag, wt] = remezord([0.35 0.65], [1 0], [0.0016 0.0016]);
```

The order of  $F(z)$  is found to be 19. The next number which is of the form  $4K+2$  is  $N = 22$ , therefore we use  $N = 22$  to determine  $H_0(z)$  which will be of order 11. The filter  $Q(z)$  is designed using the statement

```
[q, err] = remez(22, fpts, mag, wt);
```

To determine the coefficients of the filter  $F(z)$  we add  $err$  to the central coefficient  $q[12]$ . Next, using the function `roots`, we determine the roots of  $F(z)$  which should theoretically exhibit a mirror image symmetry with respect to the unit circle with double roots on the unit circle. Choosing the roots inside the unit circle along with the roots on the unit circle we get the minimum-phase spectral factor  $H_0(z)$ . The coefficients of  $H_0(z)$  are as follows:

The coefficients of are

Columns 1 through 6

```
0.19902    0.47583    0.42992    0.040154   -0.18235   -0.036342
```

Columns 7 through 12

```
0.095385   0.015368   -0.050690   -0.0011198   0.025506   -0.010676
```

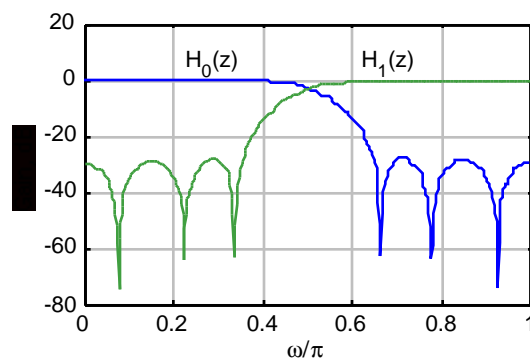
The highpass analysis filter  $H_1(z)$  can be obtained using the following MATLAB statements

```
k = 0:11;
h1 = ((-1).^k).*h0;
```

The synthesis filters can be obtained from the analysis filters using the commands

```
g0 = flipplr(h0);
g1 = flipplr(h1);
```

The gain responses of the two analysis filters are shown below:



**M10.15** The following MATLAB programs can be used to design the analysis filters corresponding to a two-channel QMF paraunitary lattice filter bank. (The program uses the function `fminu` from the MATLAB Optimization Toolbox, which in turn uses the functions `cubic12`, `cubic13`, `optint`, `searchq`, and `updhess`.)

```
% Program for the design of a two-channel QMF
% lattice filter bank.
Len = input('The length of the filter = ');
if (mod(Len,2) ~= 0)
    sprintf('Length has to be an even number')
```

```

    Len = Len+1;
end
ord = Len/2-1;
ws = 0.55*pi;
kinit = [1;zeros([ord,1])];
% set the parameters for the optimization routine
options = foptions;
options(1) = 1;
options(14) = 2500; % maximum number of iterations for the
optimization routine
kfin = fminu('filtopt',kinit,options,[],Len,ws);
e00old = 1;
e01old = kfin(1);
e10old = -kfin(1);
e11old = 1;
for k = 2:length(kfin)
    e00new = [e00old 0]-kfin(k)*[0 e01old];
    e01new = kfin(k)*[e00old 0]+[0 e01old];
    e10new = [e10old 0]-kfin(k)*[0 e11old];
    e11new = kfin(k)*[e10old 0]+[0 e11old];
    e00old = e00new;
    e01old = e01new;
    e10old = e10new;
    e11old = e11new;
end
E1 = [e00old;e01old];
h0 = E1(:);
scale_factor = abs(sum(h0));
h0 = h0/scale_factor;
E2 = [e10old;e11old];
h1 = E2(:);
h1 = h1/scale_factor;
[H0,W] = freqz(h0,1,1024);
[H1,W] = freqz(h1,1,1024);
plot(W/pi,20*log10(abs(H0)), W/pi, 20*log10(abs(H1)));
grid on
title('Gain response of the analysis filters');
xlabel('\omega/\pi'); ylabel('Gain, dB');

function val = filtopt(kval,Len,ws)
e00old = 1;
e01old = kval(1);
e10old = -kval(1);
e11old = 1;
for k = 2:length(kval)
    e00new = [e00old 0]-kval(k)*[0 e01old];
    e01new = kval(k)*[e00old 0]+[0 e01old];
    e10new = [e10old 0]-kval(k)*[0 e11old];
    e11new = kval(k)*[e10old 0]+[0 e11old];
    e00old = e00new;
    e01old = e01new;
    e10old = e10new;
    e11old = e11new;
end
E1 = [e00old;e01old];
h0 = E1(:);
[H0,W] = freqz(h0,1,1024);
val = 0;

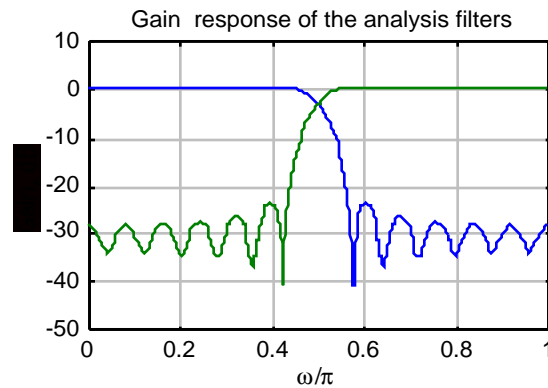
```

```

for k = 1:length(W)
    if (W(k) > ws)
        val = val+abs(H0(k))^2;
    end
end
end

```

Due to the non-linear nature of the function to be optimized, different values of `kinit` should be used to optimize the analysis filter's gain response. The gain responses of the two analysis filters is as shown below. From the gain response, the minimum stopband attenuation of the analysis filters is observed to be about 24 dB.



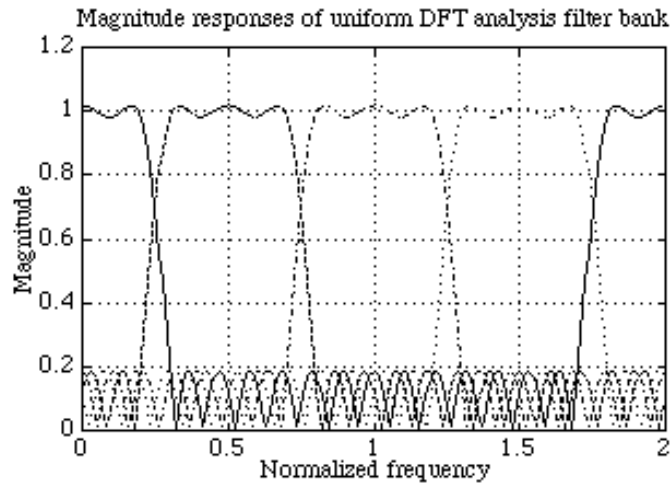
**M10.16** The MATLAB program used to generate the prototype lowpass filter and the analysis filters of the 4-channel uniform DFT filter bank is given below:

```

L = 21; f = [0 0.2 0.3 1]; m = [1 1 0 0]; w = [10 1];
N = 4; WN = exp(-2*pi*j/N);
plottag = ['- ' ; '--' ; '-.' ; ':' ];
h = zeros(N,L);
n = 0:L-1;
h(1,:) = remez(L-1, f, m, w);
for i = 1:N-1
    h(i+1,:) = h(1,:).*(WN.^(-i*n));
end;
clf;
for i = 1:N
    [H,w] = freqz(h(i,:), 1, 256, 'whole');
    plot(w/pi, abs(H), plottag(i,:));
    hold on;
end;
grid on;
hold off;
xlabel('Normalized frequency');ylabel('Magnitude');
title('Magnitude responses of uniform DFT analysis filter
bank');

```

The plots generated by the above program is given below:



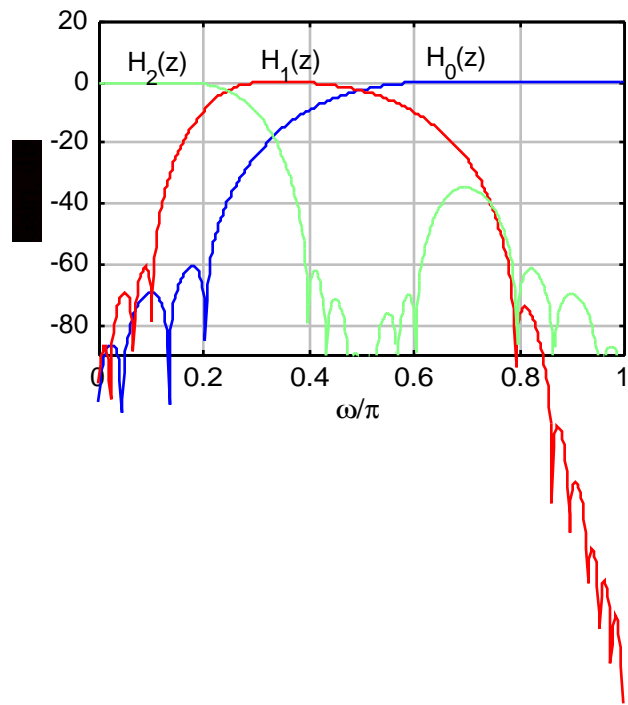
**M10.17** The first 8 impulse response coefficients of Johnston's 16A lowpass filter  $H_L(z)$  are given by

0.001050167,  $-0.005054526$ ,  $-0.002589756$ ,  $0.0276414$ ,  $-0.009666376$ ,  $-0.09039223$ ,  $0.09779817$ ,  $0.4810284$

The remaining 8 coefficients are given by flipping the coefficients left to right. From Eq. (10.157), the highpass filter in the tree-structured 3-channel filter bank is given by  $H_2(z) = z^{-15}H_L(z^{-1})$ . The two remaining filters are given by  $H_0(z) = H_L(z)H_L(z^2)$  and  $H_1(z) = H_L(z)H_H(z^2)$ . The MATLAB program used to generate the gain plots of the 3 analysis filters is given by:

```
G1 = [0.10501670e-2 -0.50545260e-2 -0.25897560e-2
0.27641400e-1 -0.96663760e-2 -0.90392230e-1 0.97798170e-1
0.48102840];
G = [G1 fliplr(G1)];
n = 0:15;
H0 = (-1).^n.*G;
Hsqar = zeros(1,31); Gsqar = zeros(1,31);
Hsqar(1:2:31) = H0; Gsqar(1:2:31) = G;
H1 = conv(Hsqar,G); H2 = conv(Gsqar,G);
[h0,w0] = freqz(H0,[1]); [h1,w1] = freqz(H1,[1]); [h2,w2] =
freqz(H2,[1]);
plot(w0/pi,20*log10(abs(h0)), 'b-',w1/pi,20*log10(abs(h1)), 'r-
',w2/pi,20*log10(abs(h2)), 'g-.');
axis([0 1 -120 20]);
grid on;
xlabel('Normalized Frequency');ylabel('Gain in dB');
```

The plots generated are given below:



## Chapter 11 (2e)

**11.1** If  $R = 1250$ , then the frequency resolution  $= \frac{7500}{1250} = 6$  Hz.

If frequency resolution  $= 4.5$  Hz, then  $R = \frac{7500}{4.5} = 1667$  points.

**11.2 (a)** resolution  $= \frac{8000}{256} = 31.25$  Hz.

**(b)** We need to take a  $\frac{8000}{16} = 500$ -point DFT.

**(c)** resolution  $= \frac{8000}{N}$ . Hence desired length  $N$  of the DFT is given by  $N = \frac{8000}{128} = 62.5$ .  
since  $N$  must be an integer we choose  $N = 63$  as the DFT size.

**11.3 (a)**  $F_{200} = \frac{kF_T}{R} = \frac{200 \times 10^4}{1000} = 2000$  Hz,  $F_{350} = \frac{kF_T}{R} = \frac{350 \times 10^4}{1000} = 3500$  Hz,  
 $F_{824} = \frac{kF_T}{R} = \frac{824 \times 10^4}{1000} = 8240$ .

**(b)**  $F_{195} = \frac{kF_T}{R} = \frac{195 \times 14 \times 10^3}{1010} = 2702.97$  Hz,  $F_{339} = \frac{kF_T}{R} = \frac{339 \times 14 \times 10^3}{1010} = 4699$  Hz,  
 $F_{917} = \frac{kF_T}{R} = \frac{917 \times 14 \times 10^3}{1010} = 12710.89$ .

**(c)**  $F_{97} = \frac{kF_T}{R} = \frac{97 \times 10^4}{517} = 1876.2$ ,  $F_{187} = \frac{kF_T}{R} = \frac{187 \times 10^4}{517} = 3617.02$ ,  
 $F_{301} = \frac{kF_T}{R} = \frac{301 \times 10^4}{517} = 5822.05$ .

**11.4** Let the samples sequence be represented as  $g[n] = \cos(\omega_0 n + \phi)$ . Then, its DTFT is given by

$$G(e^{j\omega}) = \pi \sum_{l=-\infty}^{\infty} \left( e^{j\phi} \delta(\omega - \omega_0 + 2\pi l) + e^{-j\phi} \delta(\omega + \omega_0 + 2\pi l) \right).$$

The windowed sequence is given by  $\gamma[n] = g[n] \cdot w[n]$  where  $w[n] = \begin{cases} 1, & 0 \leq n \leq N-1, \\ 0, & \text{otherwise.} \end{cases}$  The DTFT of the windowed sequence is then given by

$\Gamma(e^{j\omega}) = \frac{1}{2} e^{j\phi} \Psi_R(e^{j(\omega - \omega_0)}) + \frac{1}{2} e^{-j\phi} \Psi_R(e^{j(\omega + \omega_0)})$  where  $\Psi_R(e^{j\omega})$  is the DTFT of  $w[n]$  and is given by  $\Psi_R(e^{j\omega}) = e^{-j\omega(N-1)/2} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$ .

The DFT  $\Gamma[k]$  is a sampled version of  $\Gamma(e^{j\omega})$  sampled at  $\omega_k = \frac{2\pi k}{N}$ . To prevent leakage phenomenon we require that  $\omega_0 = \frac{2\pi l}{N}$  for some integer  $l$ . Now  $\omega_0 = \frac{\Omega}{F_T}$  where  $\Omega$  is the



angular frequency of the analog signal. If  $\Omega = \frac{2\pi l F_T}{N}$ , then the CTFT  $G_a(j\Omega)$  of the analog signal can be determined from  $\Gamma[k_o]$ .

**11.5**  $g_a(t) = \cos(200\pi t)$ . If we require  $\Gamma[k] = 0$  for all values of  $k$  except  $k = 64$  and  $k = 448$ , we

require  $\Omega_o = 200\pi$  to be of the form  $\Omega_o = \frac{2\pi l F_T}{N}$  for  $l = 64$ . Therefore

$$200\pi = \frac{2\pi \times 64 \times F_T}{512} \text{ or } F_T = 800 \text{ Hz.}$$

**11.6 (a)**  $F_m = 6$  kHz. Let  $F_T = 12$ . The resolution  $= F_T/R$ . Hence,  $R = \frac{12000}{2.6} = 4616$ .

**(b)**  $R = \frac{12000}{3} = 4000$ . The closest power-of-2 to 4000 is  $R = 4096$ . Hence, a 4096-point FFT should be used.

**11.7**  $x[n] = A \cos(2\pi f_1 n / 64) + B \cos(2\pi f_2 n / 64)$ . Since  $|X[k]| = 0$  for all values of  $k$  except  $k = 15, 27, 37$  and  $49$ , it follows then that either  $\frac{2\pi f_1}{64} = \frac{2\pi \times 15}{64}$  or  $\frac{2\pi f_1}{64} = \frac{2\pi \times 27}{64}$ . Therefore,  $f_1 = 15$  or  $27$ .

Case (i):  $f_1 = 15$ ,  $A = 64$ , and  $f_2 = 27$ ,  $B = 32$ . Therefore,  $x[n] = 64 \cos(30\pi n / 64) + 32 \cos(54\pi n / 64)$ .

Case (ii):  $f_1 = 27$ ,  $A = 32$ , and  $f_2 = 15$ ,  $B = 64$ . Therefore,  $x[n] = 32 \cos(54\pi n / 64) + 64 \cos(30\pi n / 64)$ .

Thus, there are two possible solutions as indicated above.

**11.8 (a)** A direct DFT evaluation (no FFT) requires  $N^2 = 250,000$  complex multiplications and  $N(N-1) = 249,500$  complex additions.

**(b)** Digital resolution  $= R = \frac{1}{500} = 0.002$ .

**(c)** Analog frequency resolution  $= \frac{100}{500} = 0.2$  Hz.

**(d)** The stopband edge of the filter should be at  $\frac{F_T}{2} = 50$  Hz.

**(e)** Digital frequency corresponding to the DFT sample  $X[31] = \frac{31}{500} = 0.062$  and that

corresponding to the DFT sample  $X[390] = \frac{390}{500} = 0.78$ .

Analog frequency corresponding to the DFT sample  $X[31] = 2\pi \times \frac{31}{500} \times 100 = 12.4\pi$  rad/s and

that corresponding to the DFT sample  $X[390] = 2\pi \times \frac{390}{500} \times 100 = 156\pi$  rad/s.

(f) length of FFT = 512, and therefore 12 zer-valued samples should be appended.

(g) The FFT sample index that is closest to the old DFT sample  $X[31]$  is  $k = 31 \times \frac{512}{500} = 32$  and the FFT sample index that is closest to the old DFT sample  $X[390]$  is  $k = 390 \times \frac{512}{500} \cong 399.36$  and hence  $k = 399$  is the closest.

**11.9** In order to distinguish two closely spaced sinusoids we require that the separation between the two frequencies be atleast half of the main-lobe width of the window being used. From Table 7.3, the main-lobe widths of the 4 windows are given by:

(a) Rectangular window:  $\Delta\omega = \frac{4\pi}{N} = \frac{4\pi}{60}$ . Therefore,  $f_2 > f_1 + \frac{2}{60} = 0.283$ .

(b) Hamming window:  $\Delta\omega = \frac{8\pi}{N} = \frac{8\pi}{60}$ . Therefore,  $f_2 > f_1 + \frac{4}{60} = 0.316$ .

(c) Hann window:  $\Delta\omega = \frac{8\pi}{N} = \frac{8\pi}{60}$ . Therefore,  $f_2 > f_1 + \frac{4}{60} = 0.316$ .

(d) Blackman window:  $\Delta\omega = \frac{12\pi}{N} = \frac{12\pi}{60}$ . Therefore,  $f_2 > f_1 + \frac{6}{60} = 0.35$ .

**11.10** (a)  $f_2 > f_1 + \frac{2}{110} = 0.268$ . (b)  $f_2 > f_1 + \frac{4}{110} = 0.286$ . (c)  $f_2 > f_1 + \frac{4}{110} = 0.286$ .

(d)  $f_2 > f_1 + \frac{6}{110} = 0.304$ .

**11.11** (a)  $F_T \geq 2F_m$ . Hence, the minimum sampling frequency is  $F_{T,\min} = 2F_m$ .

(b)  $R = \frac{F_T}{N} \leq \Delta F$ . Therefore  $F_T \leq N(\Delta F)$ . Hence the maximum sampling frequency is  $F_{T,\max} = N(\Delta F)$ .

(c) If  $F_m = 4$  kHz, then  $F_{T,\min} = 8$  kHz.

If  $\Delta F = 10$  Hz and  $N = 2^1$ , then  $F_{T,\max} = 2^1(10)$ .

**11.12** (a)  $X_{STFT}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} x[n-m]w[m]e^{-j\omega m}$ .

$$\begin{aligned} G_{STFT}(e^{j\omega}, n) &= \sum_{m=-\infty}^{\infty} g[n-m]w[m]e^{-j\omega m} = \sum_{m=-\infty}^{\infty} (\alpha x[n-m] + \beta y[n-m])w[m]e^{-j\omega m} \\ &= \alpha \sum_{m=-\infty}^{\infty} x[n-m]w[m]e^{-j\omega m} + \beta \sum_{m=-\infty}^{\infty} y[n-m]w[m]e^{-j\omega m} = \alpha X_{STFT}(e^{j\omega}, n) + \beta Y_{STFT}(e^{j\omega}, n). \end{aligned}$$

$$\begin{aligned}
\text{(b) } y[n] &= x[n - n_0]. \text{ Hence, } Y_{\text{STFT}}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} y[n - m]w[m]e^{-j\omega m} \\
&= \sum_{m=-\infty}^{\infty} x[n - n_0 - m]w[m]e^{-j\omega m} = X_{\text{STFT}}(e^{j\omega}, n - n_0).
\end{aligned}$$

$$\begin{aligned}
\text{(c) } y[n] &= e^{j\omega_0 n} x[n]. \text{ Hence, } Y_{\text{STFT}}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} y[n - m]w[m]e^{-j\omega m} \\
&= \sum_{m=-\infty}^{\infty} x[n - m]w[m]e^{-j(\omega - \omega_0)m} = X_{\text{STFT}}(e^{j(\omega - \omega_0)}, n).
\end{aligned}$$

**11.13**  $X_{\text{STFT}}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} x[n - m]w[m]e^{-j\omega m}$ . Replacing  $m$  in this expression with  $n - m$  we

arrive at  $X_{\text{STFT}}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} x[m]w[n - m]e^{-j\omega n} e^{-j\omega m} = e^{-j\omega n} \sum_{m=-\infty}^{\infty} x[m]w[n - m]e^{-j\omega m}$   
 $e^{-j\omega n} \bar{X}_{\text{STFT}}(e^{-j\omega}, n)$ . Hence,  $\bar{X}_{\text{STFT}}(e^{j\omega}, n) = e^{-j\omega n} X_{\text{STFT}}(e^{-j\omega}, n)$ . Thus, in computing  $X_{\text{STFT}}(e^{j\omega}, n)$  the input  $x[n]$  is shifted through the window  $w[n]$ , whereas, in computing  $\bar{X}_{\text{STFT}}(e^{j\omega}, n)$  the window  $w[n]$  is shifted through the input  $x[n]$ .

**11.14**  $\bar{X}_{\text{STFT}}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} x[m]w[n - m]e^{-j\omega m}$ . Hence, by inverse DTFT we obtain

$$x[m]w[n - m] = \frac{1}{2\pi} \int_0^{2\pi} \bar{X}_{\text{STFT}}(e^{j\omega}, n) e^{j\omega m} d\omega. \text{ Therefore,}$$

$$\sum_{n=-\infty}^{\infty} x[m]w[n - m] = \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=-\infty}^{\infty} \bar{X}_{\text{STFT}}(e^{j\omega}, n) e^{j\omega m} d\omega, \text{ which is equivalent to}$$

$$x[m] \sum_{n=-\infty}^{\infty} w[n - m] = x[m] W[0] = \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=-\infty}^{\infty} \bar{X}_{\text{STFT}}(e^{j\omega}, n) e^{j\omega m} d\omega, \text{ where}$$

$$W[0] = \sum_{n=-\infty}^{\infty} w[n - m] = \sum_{n=-\infty}^{\infty} w[n] \text{ or } x[m] = \frac{1}{2\pi W[0]} \int_0^{2\pi} \sum_{n=-\infty}^{\infty} \bar{X}_{\text{STFT}}(e^{j\omega}, n) e^{j\omega m} d\omega.$$

**11.15** From the alternate definition of the STFT given in Problem 11.13 we have

$$\bar{X}_{\text{STFT}}(e^{j\omega}, 1) = \sum_{m=-\infty}^{\infty} x[m]w[1 - m]e^{-j\omega m}. \text{ Therefore,}$$

$$\sum_{l=-\infty}^{\infty} \bar{X}_{\text{STFT}}(e^{j\omega}, l) e^{j\omega n} = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m] w[l-m] e^{-j\omega(m-n)}. \text{ Hence,}$$

$$\int_0^{2\pi} \sum_{l=-\infty}^{\infty} \bar{X}_{\text{STFT}}(e^{j\omega}, l) e^{j\omega n} d\omega = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m] w[l-m] \int_0^{2\pi} e^{-j\omega(m-n)} d\omega.$$

Now,  $\int_0^{2\pi} e^{j\omega k} d\omega = \begin{cases} 2\pi, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}$  Thus,,

$$\sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m] w[l-m] \cdot 2\pi \delta[m-n] = \int_0^{2\pi} \sum_{l=-\infty}^{\infty} \bar{X}_{\text{STFT}}(e^{j\omega}, l) e^{j\omega n} d\omega, \text{ or}$$

$$2\pi \sum_{l=-\infty}^{\infty} w[l-n] \cdot x[n] = \int_0^{2\pi} \sum_{l=-\infty}^{\infty} \bar{X}_{\text{STFT}}(e^{j\omega}, l) e^{j\omega n} d\omega.$$

Let  $W[0] = \sum_{n=-\infty}^{\infty} w[n]$ . Then  $x[n] = \frac{1}{2\pi W[0]} \int_0^{2\pi} \sum_{l=-\infty}^{\infty} \bar{X}_{\text{STFT}}(e^{j\omega}, l) e^{j\omega n} d\omega$ .

**11.16**  $X_{\text{STFT}}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} x[n-m] w[m] e^{-j\omega m}$ . Hence,

$$X_{\text{STFT}}[k, n] = \sum_{m=-\infty}^{\infty} x[n-m] w[m] e^{-j2\pi km/N} = x[n] \circledast w[n] e^{-j2\pi kn/N}. \text{ Or in other words,}$$

$X_{\text{STFT}}[k, n]$  can be obtained by filtering  $x[n]$  by an LTI dystem with an impulse response  $h_k[n] = w[n] e^{-j2\pi kn/N}$  as indicated in Figure P11.1.

**11.17**  $X_{\text{STFT}}(e^{j\omega}, n) = \sum_{m=-\infty}^{\infty} x[n-m] w[m] e^{-j\omega m}$ . Hence,

$$\left| X_{\text{STFT}}(e^{j\omega}, n) \right|^2 = \sum_{s=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[n-m] x[n-s] w[m] w[s] e^{-j\omega m} e^{-j\omega s}. \text{ Thus,}$$

$$r[k, n] = \frac{1}{2\pi} \int_0^{2\pi} \left| X_{\text{STFT}}(e^{j\omega}, n) \right|^2 e^{j\omega k} d\omega = \sum_{s=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[n-m] x[n-s] w[m] w[s] \delta[s+k-m]$$

$$= \sum_{m=-\infty}^{\infty} x[n-m] x[n-m+k] w[m] w[m-k].$$

**11.18 (a)** Length of the window  $N = F_T \tau$

(b) The number of complex multiplications is  $C = \frac{N}{2} \log_2 N = \frac{F_T \tau}{2} \log_2(F_T \tau)$ . An FFT is performed after every  $K$  samples, i.e. after every  $K/F_T$  seconds. Thus, the number of complex multiplications per second is  $\frac{C}{K/F_T} = \frac{F_T^2 \tau}{2K} \log_2(F_T \tau)$ .

**11.19** 
$$\varphi_{ST}[k, n] = \sum_{m=-\infty}^{\infty} x[m]w[n-m]x[m+k]w[n-k-m].$$

(a) 
$$\varphi_{ST}[-k, n] = \sum_{m=-\infty}^{\infty} x[m]w[n-m]x[m-k]w[n+k-m].$$

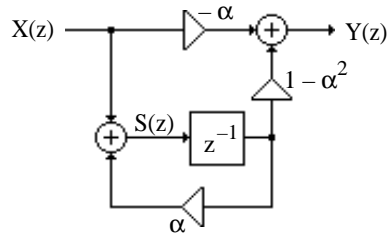
Substitute in the above expression  $m - k = s$ , i.e.  $m = k + s$ . This yields

$$\varphi_{ST}[-k, n] = \sum_{s=-\infty}^{\infty} x[s+k]w[n-k-s]x[s]w[n-s] = \varphi_{ST}[k, n].$$

(b) Let  $m + k = s$ . Then,  $\varphi_{ST}[k, n] = \sum_{s=-\infty}^{\infty} x[s-k]x[s]w[n-s+k]w[n-s]$ . It follows from

this expression that  $\varphi_{ST}[k, n]$  can be computed by a convolution of  $h_k[n] = w[n]w[n+k]$  with  $x[n]x[n-s]$  as indicated in Figure P11.2.

**11.20**

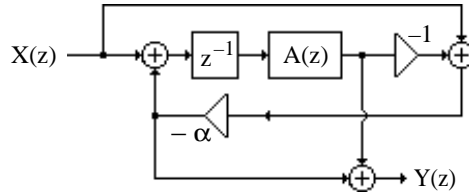


Analysis yields  $S(z) = \alpha z^{-1} S(z) + X(z)$ , and  $Y(z) = -\alpha X(z) + (1 - \alpha^2) z^{-1} S(z)$ . Solving the first equation we get  $S(z) = \frac{X(z)}{1 - \alpha z^{-1}}$ , which when substituted in the second equation yields

after some algebra  $\frac{Y(z)}{X(z)} = \frac{-\alpha + z^{-1}}{1 - \alpha z^{-1}}$ . The transfer function is thus seen to be a Type 1

allpass of the form of Eq. (6.58) and can thus be realized using any one of the single-multiplier structures of Figure 6.36.

**11.21** Analysis of the structure of Figure P11.4 yields  $\frac{Y(z)}{X(z)} = \frac{-\alpha + z^{-1} A(z)}{1 - \alpha z^{-1} A(z)}$ , where  $A(z)$  denotes the transfer function of the "allpass reverberator". Note that this expression is similar in form to that of Eq. (6.58) with "d" replaced by " $-\alpha$ " and " $z^{-1}$ " replaced by " $z^{-1} A(z)$ ". Hence an efficient realization of the structure of Figure P11.4 also is obtained readily from any one of the structures of Figure 6.36. One such realization is indicated below:



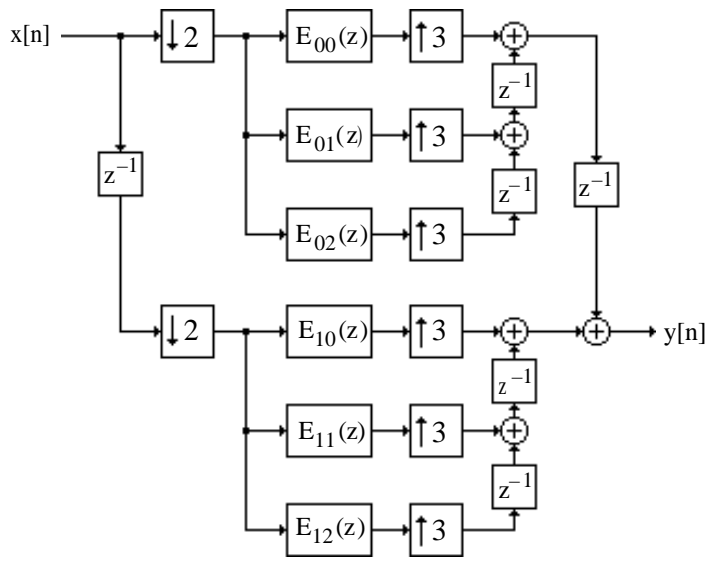
**11.22** 
$$G_2(z) = \frac{K_1}{2} \{1 - A_2(z)\} + \frac{K_2}{2} \{1 + A_2(z)\} = K_2 \left[ \frac{1}{2} \frac{K_1}{K_2} \{1 - A_2(z)\} + \frac{1}{2} \{1 + A_2(z)\} \right].$$

Hence in this case, the ratio  $K_1/K_2$  determines the amount of boost or cut at low frequencies,  $K_2$  determines the amount of dc gain or attenuation at all frequencies,  $\alpha$  determines the

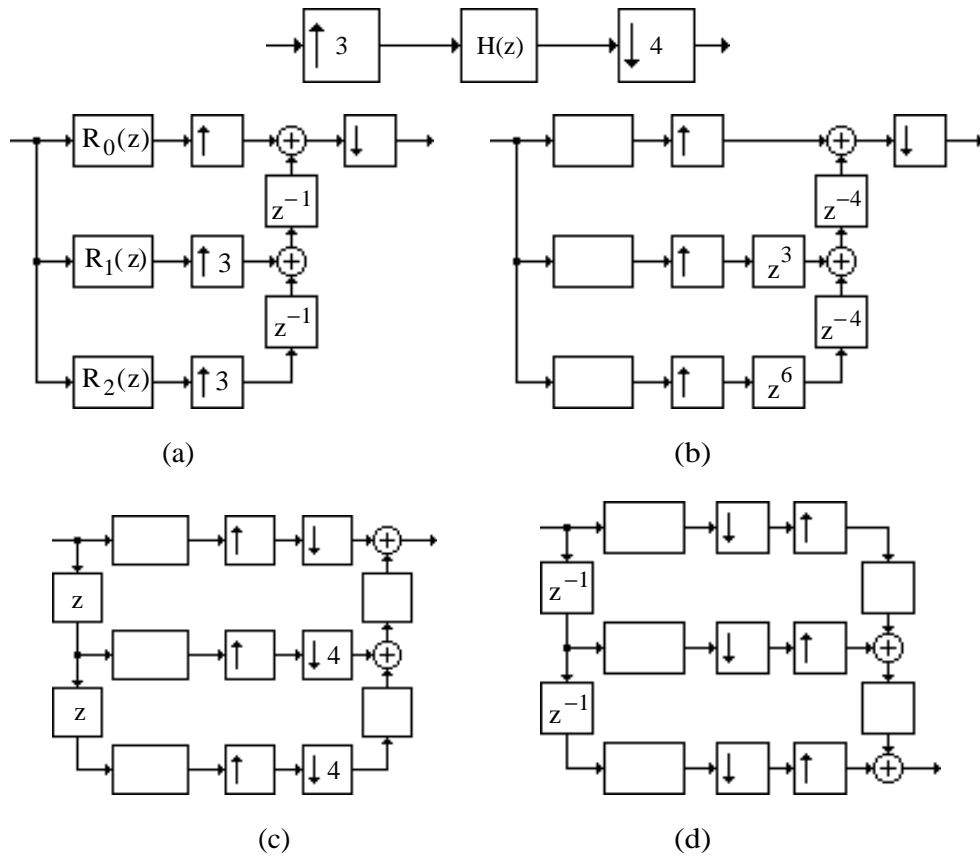
3-dB bandwidth  $\Delta\omega_{3\text{-dB}} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$ , and the center frequency  $\omega_o$  is related to  $\beta$  through  $\beta = \cos \omega_o$ .

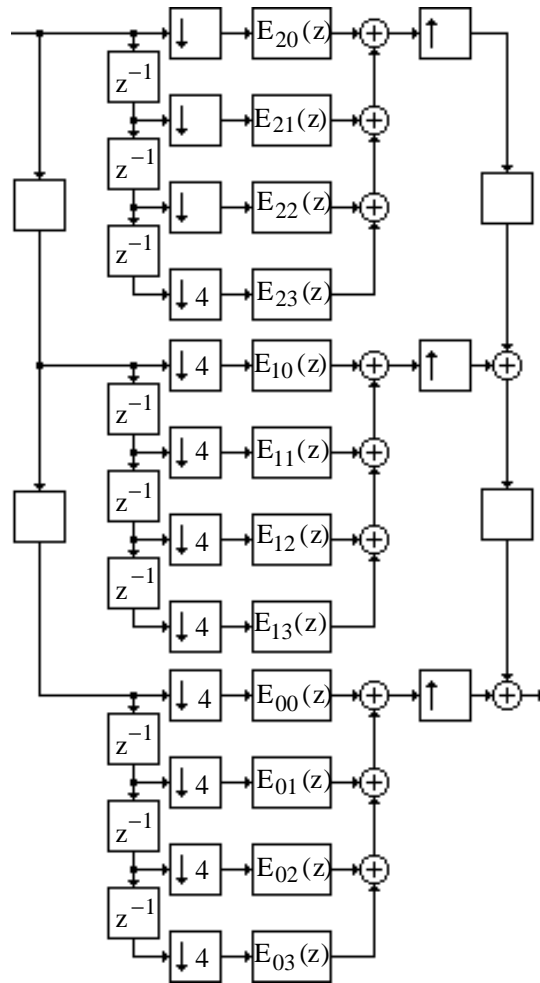
```
alpha = 0.8;
beta = 0.4;
K1 = [0.9];
K2 = [0.5 2];
nbp = ((1-alpha)/2)*[1 0 -1];
dbp = [1 -beta*(1+alpha) alpha];
nbs = ((1+alpha)/2)*[1 -2*beta 1];
dbs = dbp;
[Hlp,w] = freqz(nbp,dbp,512);
[Hhp,w] = freqz(nbs,dbs,512);
hold on
for k = 1:length(K1)
    for m = 1:length(K2)
        H = K1(k)*Hlp+K2(m)*Hhp;
        semilogx(w/pi,20*log10(abs(H)));
        xlabel('Gain, dB');
        ylabel('\omega/\pi');
        clear H;
        hold on;
    end
end
end
grid on
axis([.01 1 -8 8]);
```

**11.23** The transpose of the decimator structure of Figure 11.63 yields



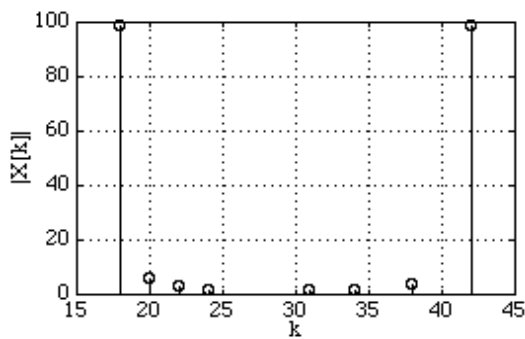
11.24



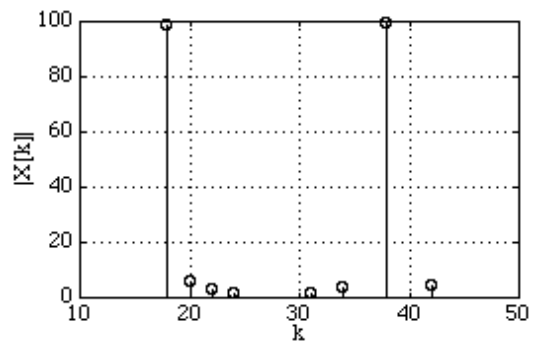


(e)

**M11.1** Figures below illustrate the application of Program 11\_1 in detecting the touch-tone digits A and 3:



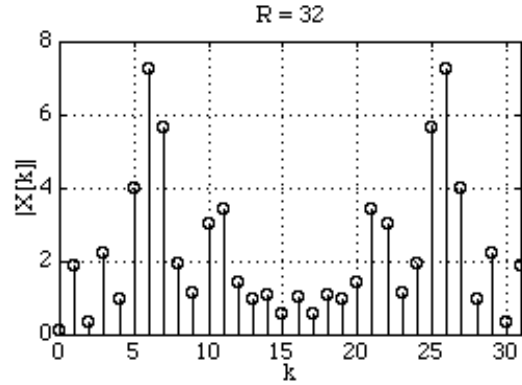
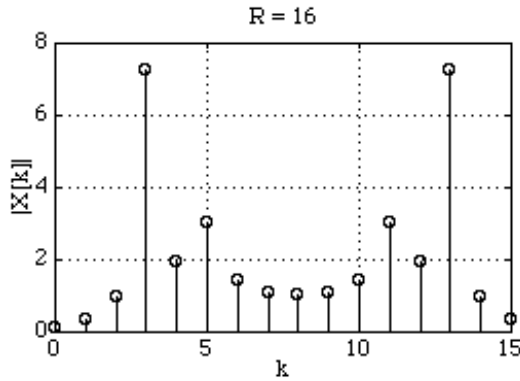
Touch-Tone Symbol = A



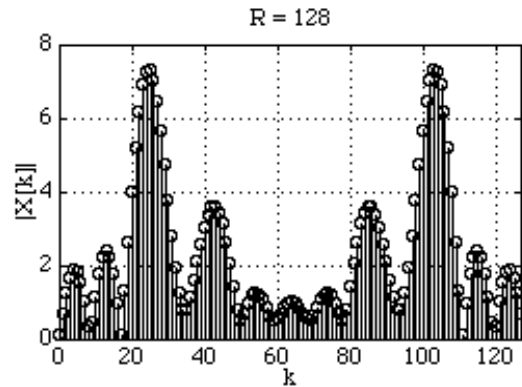
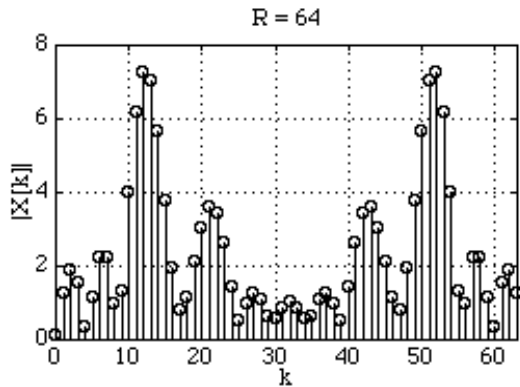
Touch-Tone Symbol = 3

**M11.2** For  $R = 16$ , the two strong peaks occur at  $k = 3$  and  $5$ . The associated frequencies are  $\omega_1 = \frac{2\pi \times 3}{16}$ , or  $f_1 = \frac{3}{16} = 0.1875$ , and  $f_2 = \frac{5}{16} = 0.3125$ . For  $R = 32$ , the two strong peaks occur at  $k = 5$  and  $10$ . The associated frequencies are  $f_1 = \frac{5}{32} = 0.15625$ , and  $f_2 = \frac{10}{32} = 0.3125$ .



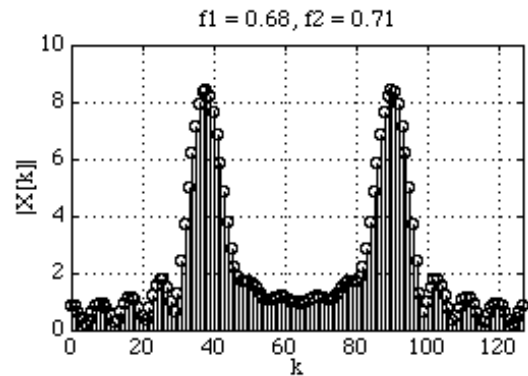
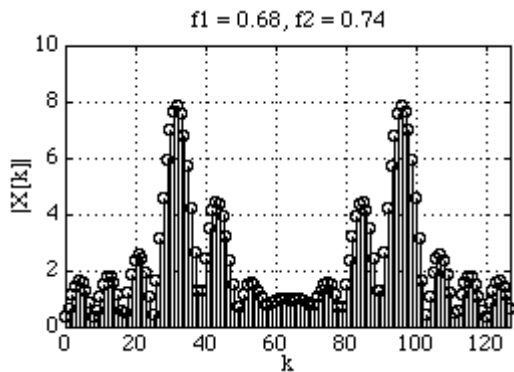
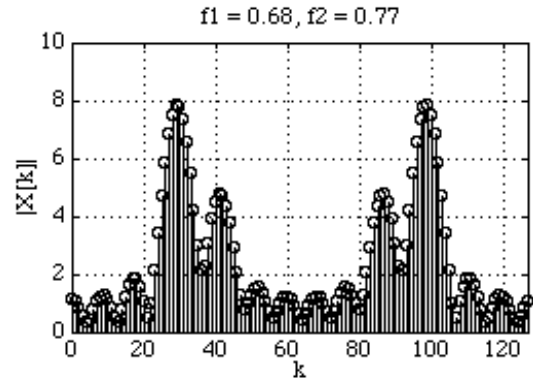
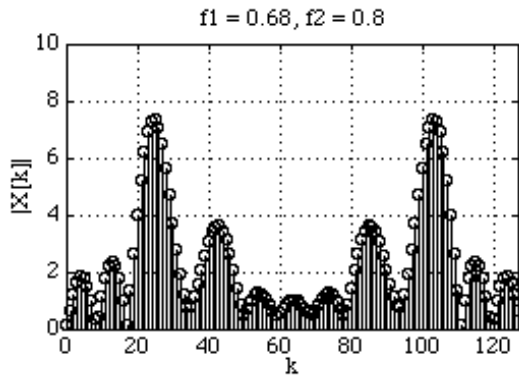


For  $R = 64$ , the two strong peaks occur at  $k = 11$  and  $22$ . The associated frequencies are  $f_1 = 11/64 = 0.1718$ , and  $f_2 = 22/64 = 0.34375$ . For  $R = 128$ , the two strong peaks occur at  $k = 21$  and  $39$ . The associated frequencies are  $f_1 = 21/128 = 0.1641$ , and  $f_2 = 39/128 = 0.3047$ . Moreover, the last two plots show a number of minor peaks and it is not clear by examining these plots whether or not there are other sinusoids of lesser strengths present in the sequence being analyzed.

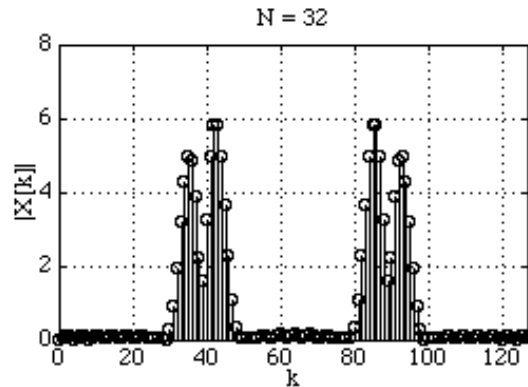
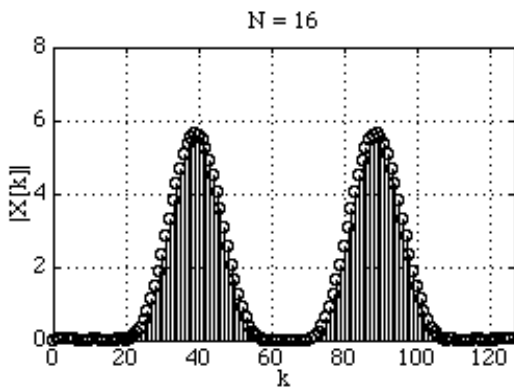


An increase in the size of the DFT increases the resolution of the spectral analysis by reducing the separation between adjacent DFT samples. Also the estimated values of the frequencies of the sinusoid get closer to the actual values of  $0.167$  and  $0.3076$  as the size of the DFT increases.

**M11.3** As the separation between the two frequencies decreases, the distance between the two maximas in the DFT of the sequence decreases, and when  $f_2 = 0.21$ , the second sinusoid cannot be determined from the DFT plot. This is due to the use of a length-16 rectangular window to truncate the original infinite-length sequence. For a length-16 rectangular window, two adjacent sinusoids can be distinguished if their angular frequencies are apart by half the mainlobe width of  $\frac{4\pi}{N}$  radians or equivalently, if their frequencies are apart by  $\frac{2}{N} = 0.0625$ . Note that the DFT length  $R = 128$  is all plots.

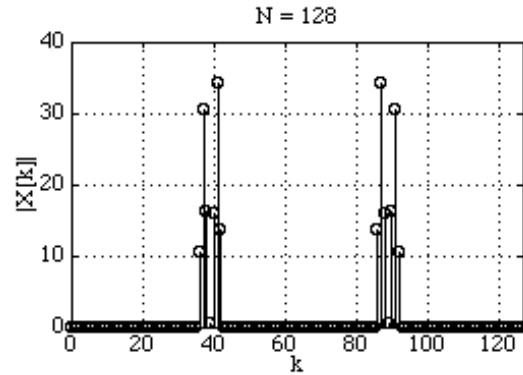
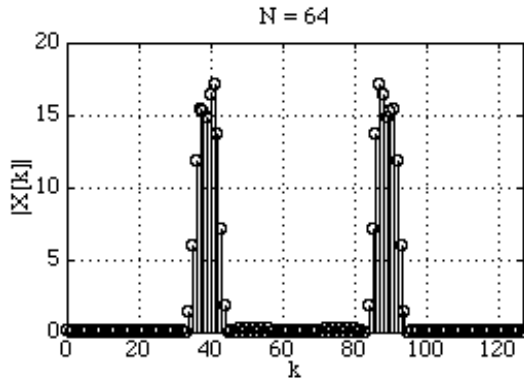


**M11.4**  $f_2 = 0.21, f_1 = 0.18$ . Hence,  $\Delta f = 0.03$ . For a Hamming window the mainlobe width  $\Delta_{ML} = \frac{8\pi}{N}$ . The DFT length  $R = 128$  in all plots.



(i)  $N = 16$ . Here it is not possible to distinguish the two sinusoids. This also can be seen from the value of  $\Delta_{ML} = \frac{8}{16} = 0.5$ , and hence, half of the mainlobe width is greater than  $\Delta f$ .

(ii) Increasing  $N$  to 32, makes the separation between the two peaks visible. However, it is difficult to identify the peaks accurately.

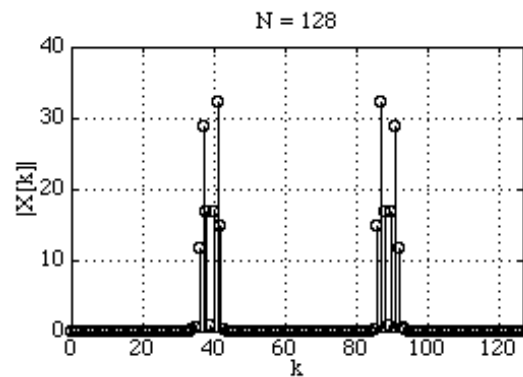
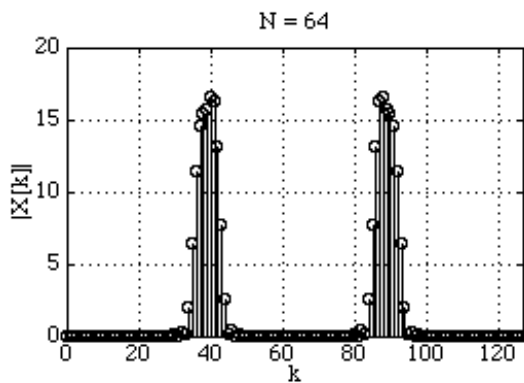
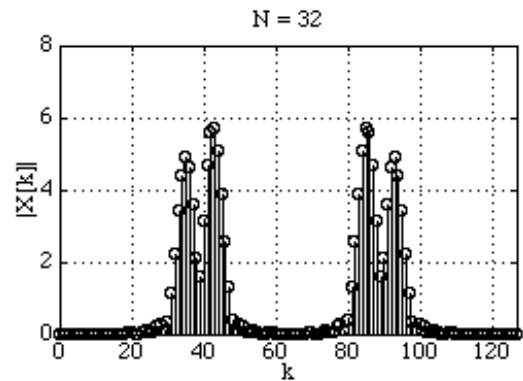
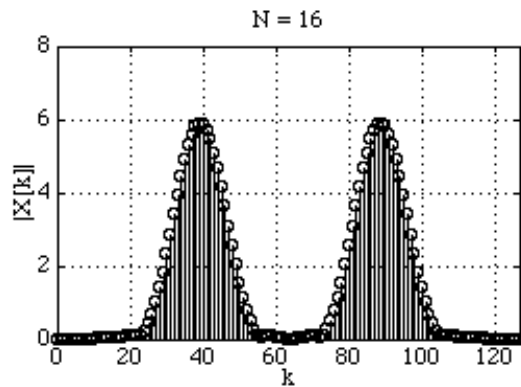


(iii) Increasing  $N$  to 64, makes the separation between the two peaks more visible. However, it is still difficult to identify the peaks accurately.

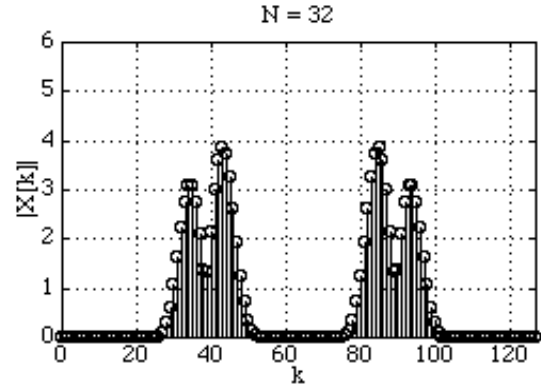
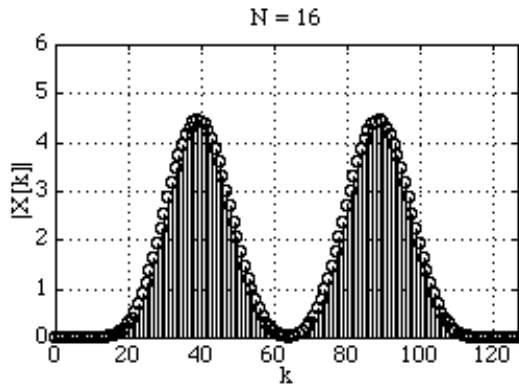
(iv) For  $N = 128$ , the separation between the two peaks clearly visible.

Note also the suppression of the minor peaks due to the use of a tapered window.

**M11.5** Results are similar to that in Problem M11.4.

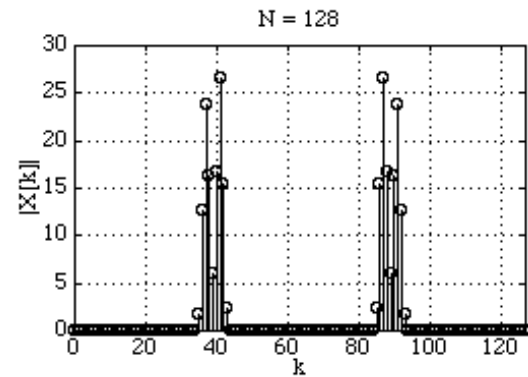
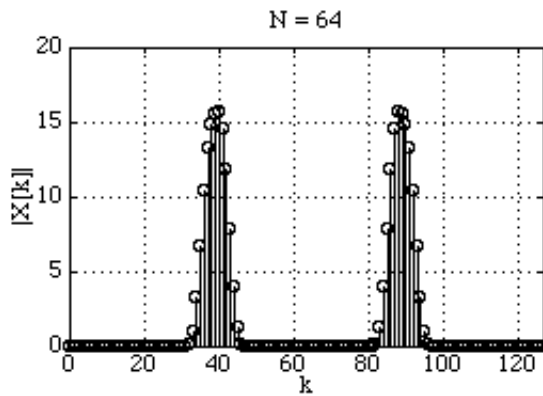


**M11.6**  $f_2 = 0.21$ ,  $f_1 = 0.18$ . Hence,  $\Delta f = 0.03$ . The DFT length  $R = 128$  in all plots.



(i) For  $N = 16$ , it is difficult to identify the two sinusoids.

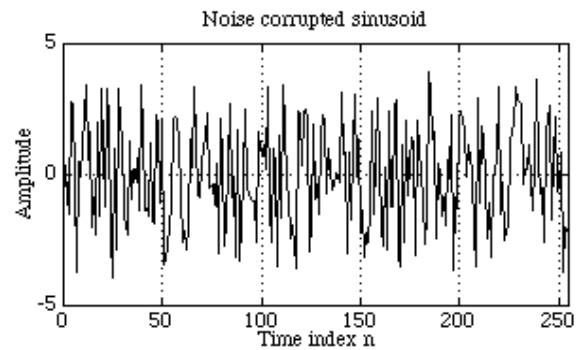
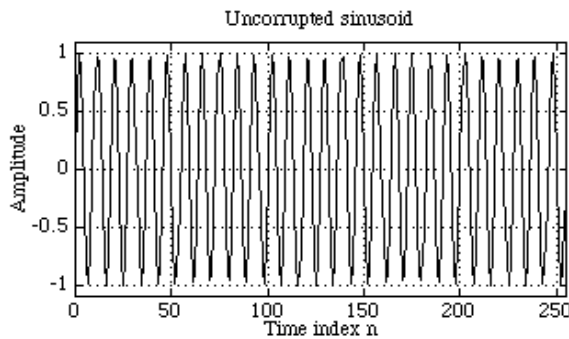
(ii) For  $N = 32$ , there are two peaks clearly visible at  $k = 36$  and  $43$ , respectively.

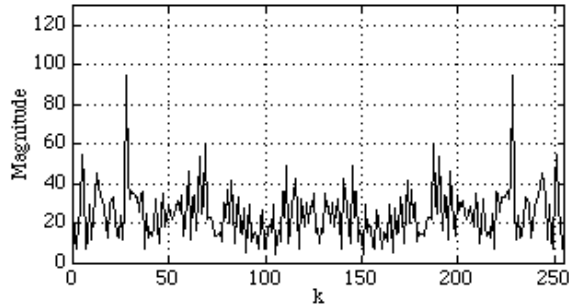


(iii) For  $N = 64$ , it is difficult to identify the two sinusoids.

(iv) For  $N = 128$ , there are two peaks clearly visible at  $k = 37$  and  $41$ , respectively.

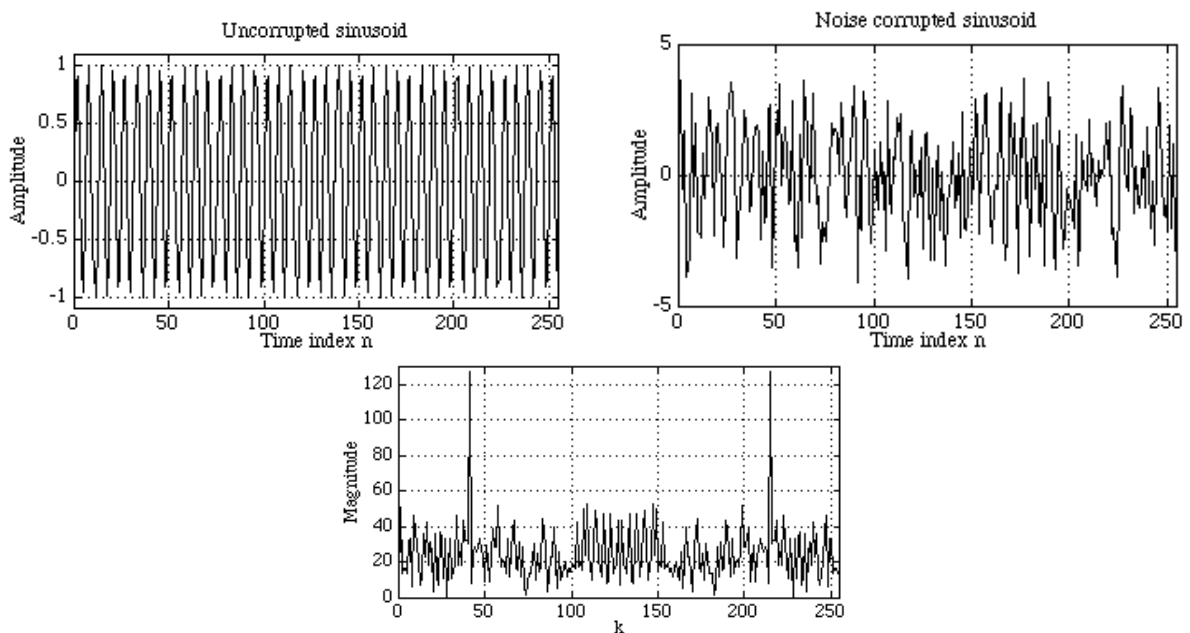
### M11.7





The SNR computed by the program is  $-7.4147$  dB. There is a peak at the frequency index 29 whose normalized frequency equivalent is equal to  $29/256 = 0.1133$ . Hence the DFT approach has correctly identified the frequency of the sinusoid corrupted by the noise.

### M11.8

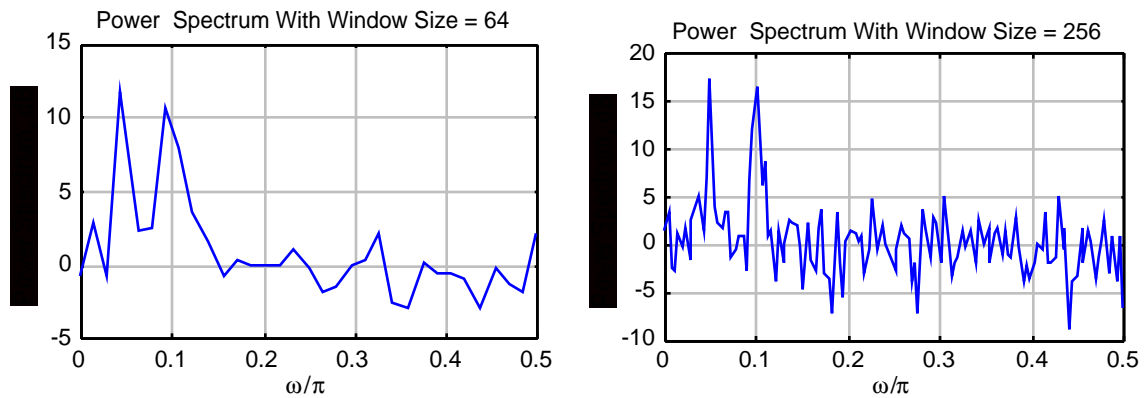


The SNR computed by the program is  $-7.7938$  dB. There is a peak at the frequency index 42 whose normalized frequency equivalent is equal to  $42/256 = 0.1641$ . Hence the DFT approach has correctly identified the frequency of the sinusoid corrupted by the noise.

**M11.9** The following program can be used to plot the power spectrum estimates of the noise corrupted signal windowed by a rectangular window.

```
% Power Spectrum Estimation %
nfft = input('Type in the fft size = ');
n = 0:1000;
g = sin(0.1*pi*n) + sin(0.2*pi*n) + randn(size(n));
window = boxcar(nfft);
[Pxx, f] = psd(g,nfft,2,window);
plot(f/2,10*log10(Pxx));grid
xlabel('\omega/\pi');ylabel('Power Spectrum, dB');
titletext=sprintf('Power Spectrum With Window Size =
%d',nfft);
title(titletext);
```

The power spectrum estimate with window size = 64 and 256 are shown below

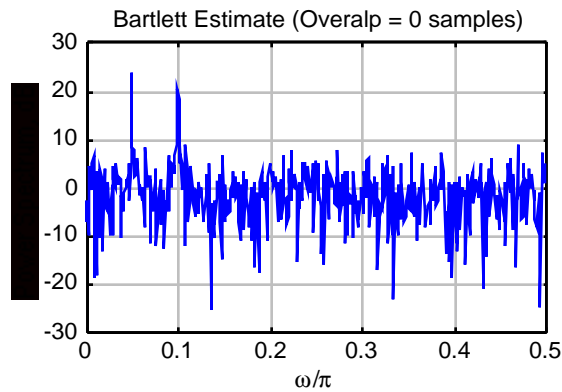


**M11.10** Program\_11\_4 can be used to evaluate the Bartlett and Welch estimate by changing

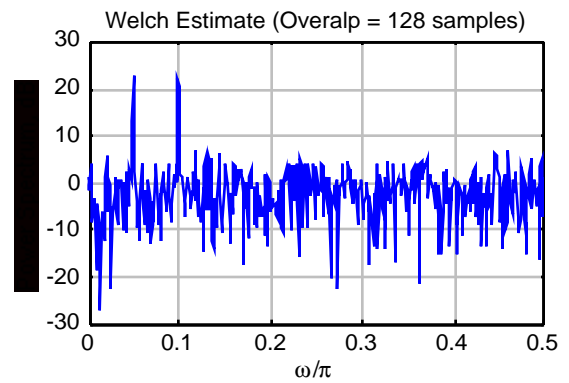
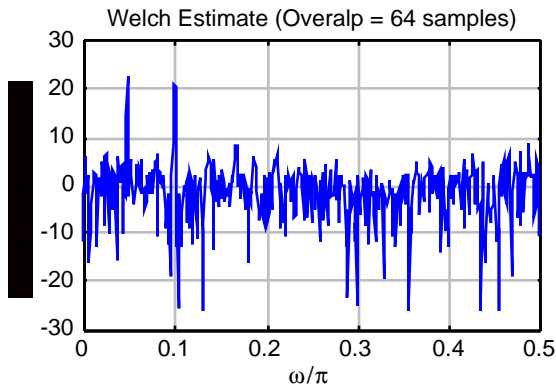
```
g = 2*sin(0.12*pi*n) + sin(0.28*pi*n) + randn(size(n));
to g = sin(0.1*pi*n) + sin(0.2*pi*n) + randn(size(n));
```

and the line window = hamming(256) with window = boxcar(1024) for the Bartlett estimate, and with window = hanning(1024) for the Welch estimate. Also set nfft=1024.

The plot of the Bartlett estimate of the power spectrum estimate for the noise corrupted signal is shown below



The Welch estimates of the power spectrum with an overlapping Hann window for overlaps of 64 and 128 samples are shown below



**M11.11** Use the statement `b = remez(11,[0 0.3 0.5 1],[1 1 0 0],[1 1]);` to design the FIR filter. Then use the statement `[d,p0] = lpc(b,order);` and run it for `order = 4,5` and `6` to determine an equivalent all-pole model. The magnitude response of the original FIR filter (solid line) and the all-pole equivalent (dotted line) are shown below for different values of order.

